# Perturbative treatment of the many-body problem in nuclear matter.

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- The essential goal of quantum many-body physics is to study the nature and the effects of interactions between particles as well as the observable properties of many-particle systems.
- The Schrödinger equation that describes the dynamics of a many-body system composed by $A$ nucleons is given by:

$$H\Psi = (T + V)\Psi = E\Psi,$$

where the Hamiltonian is written as the sum of a kinetic term $T$ and an interaction term $V$, that represents in principle a 2-body, 3-body, $\cdots$, up to a $A-$body force,

$$H = T + V.$$

- In other words, the Schrödinger equation may be written as:

$$H\Psi = \left\{ \sum_{i=1}^{A} -\frac{\hbar^2}{2m}\nabla_i^2 + \text{2-body} + \text{3-body} + \cdots + \sum_{i_1<\cdots<i_A} v(i_1, \cdots, i_A) \right\} = E\Psi,$$

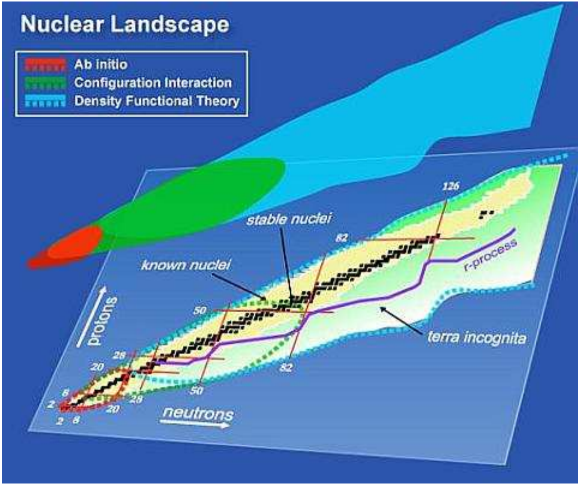
where $i$ represents all coordinates of the $i^{th}$ nucleon.

- From the phenomenological point of view, it turns out that, in most cases, the interaction is well enough described by the 2-body (and possibly the 3-body) terms, and therefore the Hamiltonian reduces to:

$$H \sim \sum_{i=1}^{A} -\frac{\hbar^2}{2m}\nabla_i^2 + \frac{1}{2}\sum_{i\neq j=1} V(i,j).$$

- A simple example of a two-body phenomenological interaction in nuclear physics is the Skyrme effective force (zero-range force) which is given by:

$$\begin{aligned} V_{12}(\vec{r},\vec{R}) \quad = \quad & t_0(1+x_0P^\sigma)\delta(\vec{r}) + \frac{1}{2}t_1(1+x_1P^\sigma)\left[\delta(\vec{r})\vec{k}^2 + \vec{k}'^2\delta(\vec{r})\right] \\ & + t_2(1+x_2P^\sigma)\vec{k}' \cdot \delta(\vec{r})\vec{k} + \frac{1}{6}t_3\left(1+x_3P^\sigma\right)\delta(\vec{r})\rho^\alpha(\vec{R}) \\ & + iW_0\vec{\sigma}\cdot\vec{k}' \times \delta(\vec{r})\vec{k}. \end{aligned}$$




- Which nuclei can be treated? From medium-mass to heavy nuclei.
- Blue region: Energy Density Functional (EDF) framework (for example Skyrme forces): Mean-field framework.
  - Mean field for ground-state nuclear structure (HF, HFB,..)
  - RPA and QRPA for small-amplitude oscillations
  - Beyond small amplitude oscillations: time-dependent mean field for dynamics (TDHF, TDHFB,. . . ).

- The Green's function, also known as the Feynman propagator is defined as:

$$G(r,t;r',t') = -i\langle\Phi_0|T\left[\Psi(r,t)\Psi^\dagger(r',t')\right]\Phi_0\rangle, \quad \text{by assuming} \quad \langle\Phi_0|\Phi_0\rangle = 1.$$

- When $t' > t$ , the function $G(r,t;r',t')$ creates a particle at time $t$ and position $r$ , then destroys it again at time $t'$ and position $r'$ , in other words, it measures the probability of a particle propagating from $(r,t)$ to $(r',t')$.
- The formula due to Gell-Mann and Low expresses the shift energy of the ground state with respect to the unperturbed system as:

$$E - \epsilon_k^{(0)} = \sum_{m=0}^{\infty} (-i)^m \frac{1}{m!} \int_{-\infty}^{0} dt_1 \cdots dt_m \, \langle\Phi_0|T\left[\hat{H}_1\hat{H}_1(t_1)\cdots\hat{H}_1(t_m)\right]|\Phi_0\rangle_{connected},$$

where the interacting Hamiltonian $\hat{H}_1$ can be written as:

$$H_1 = \frac{1}{2!}\sum_{\substack{\alpha\beta\alpha'\beta' \\ \gamma\mu\gamma'\mu'}} \int d^3\mathbf{r}\, d^3\mathbf{r}' \;\hat{\psi}^\dagger_{\alpha;\gamma}(\mathbf{r})\hat{\psi}^\dagger_{\beta;\mu}(\mathbf{r}')V(\mathbf{r},\mathbf{r}')_{\substack{\alpha\alpha',\beta\beta' \\ \gamma\gamma',\mu\mu'}}\hat{\psi}_{\beta';\mu'}(\mathbf{r}')\hat{\psi}_{\alpha';\gamma'}(\mathbf{r}).$$

- The first-order contributions to the total energy is given by:

$$E^{(1)} = \langle\Phi_0| T\left[\hat{H}_1\right]|\Phi_0\rangle_{connected}.$$

- The time-ordering can be expressed as:

$$\begin{aligned} T\left[\hat{\psi}^{\dagger}_{\alpha;\gamma}(\mathbf{r})\hat{\psi}^{\dagger}_{\beta;\mu}(\mathbf{r}^{'})\hat{\psi}_{\beta^{'};\mu^{'}}(\mathbf{r}^{'})\hat{\psi}_{\alpha^{'};\gamma^{'}}(\mathbf{r})\right] &= \hat{\psi}^{\dagger}_{\alpha;\gamma}(\mathbf{r}) :: \hat{\psi}_{\alpha^{'};\gamma^{'}}(\mathbf{r})\hat{\psi}^{\dagger}_{\beta;\mu}(\mathbf{r}^{'}) :: \hat{\psi}_{\beta^{'};\mu^{'}}(\mathbf{r}^{'}) \\ &\quad -\hat{\psi}^{\dagger}_{\alpha;\gamma}(\mathbf{r}) :: \hat{\psi}_{\beta^{'};\mu^{'}}(\mathbf{r}^{'})\hat{\psi}^{\dagger}_{\beta;\mu}(\mathbf{r}^{'}) :: \hat{\psi}_{\alpha^{'};\gamma^{'}}(\mathbf{r}). \end{aligned}$$

- By noting that: $\hat{\psi}_{a;b}(\mathbf{r})\hat{\psi}^{\dagger}_{a^{'};b^{'}}(\mathbf{r}^{'}) = iG^{0}_{ab;a^{'}b^{'}}(\mathbf{r},\mathbf{r}^{'})$, we get:

$$\begin{aligned} T\left[\hat{\psi}^{\dagger}_{\alpha;\gamma}(\mathbf{r})\hat{\psi}^{\dagger}_{\beta;\mu}(\mathbf{r}^{'})\hat{\psi}_{\beta^{'};\mu^{'}}(\mathbf{r}^{'})\hat{\psi}_{\alpha^{'};\gamma^{'}}(\mathbf{r})\right] &= iG^{0}_{\alpha^{'}\alpha;\gamma^{'}\gamma}(\mathbf{r},\mathbf{r})iG^{0}_{\beta^{'}\beta;\mu^{'}\mu}(\mathbf{r}^{'},\mathbf{r}^{'}) \\ &\quad -iG^{0}_{\beta^{'}\alpha;\mu^{'}\gamma}(\mathbf{r}^{'},\mathbf{r})iG^{0}_{\alpha^{'}\beta;\gamma^{'}\mu}(\mathbf{r},\mathbf{r}^{'}). \end{aligned}$$

- Therefore, we are able to write the first-order energy as a sum of a direct and exchange part:

$$E^{(1)} = E^{(1)}_{direct} + E^{(1)}_{exchange}.$$

- The expression for the first-order correction (direct term) is given by:

$$E^{(1)}_{direct} = \frac{1}{2!} \sum_{\substack{\alpha\beta\alpha'\beta' \\ \gamma\mu\gamma'\mu'}} \int d^4r\, d^4r'\, U(r,r')_{\substack{\alpha\alpha',\beta\beta' \\ \gamma\gamma',\mu\mu'}} \left[ iG^0_{\alpha'\alpha;\gamma'\gamma}(\mathbf{r},\mathbf{r}) iG^0_{\beta'\beta;\mu'\mu}(\mathbf{r}',\mathbf{r}') \right].$$

- The expression for the first-order correction (exchange term) is given by:

$$E^{(1)}_{exchange} = \frac{1}{2!} \sum_{\substack{\alpha\beta\alpha'\beta' \\ \gamma\mu\gamma'\mu'}} \int d^4r\, d^4r'\, U(r,r')_{\substack{\alpha\alpha',\beta\beta' \\ \gamma\gamma',\mu\mu'}} \left[ -iG^0_{\beta'\alpha;\mu'\gamma}(\mathbf{r}',\mathbf{r}) iG^0_{\alpha'\beta;\gamma'\mu}(\mathbf{r},\mathbf{r}') \right].$$

- Mean-field models are described by first-order contributions.

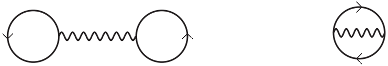


Figure : Left (Right): First-order energy correction direct (exchange) term.

### Advantages of mean-field approaches.

In nuclear physics, mean-field approaches lead to satisfactory results when applied to bulk properties of atomic nuclei.

- Masses, radii or ground state deformations.

### Mean-field approaches are not always very accurate.

- For example, in nuclear physics, mean-field approaches do not predict accurately the single-particle spectra.
- Spectroscopic factors and the fragmentation of the single-particle energies cannot be reproduced in a precise way.

- This why we want to formulate the nuclear many-body problem in a beyond mean-field framework.

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## Self-consistent mean-field models for nuclear structure

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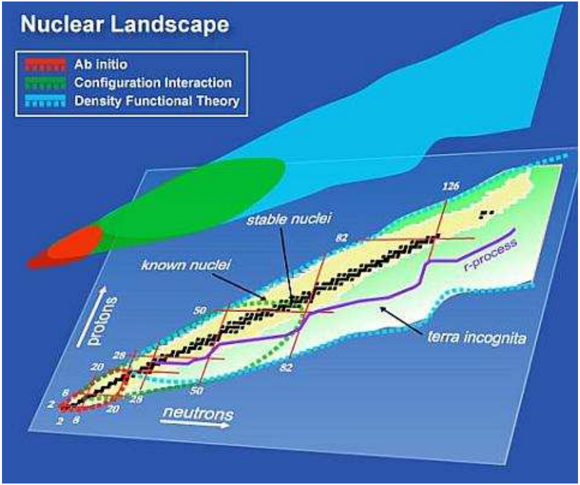



- Which nuclei can be treated?From medium-mass to heavy nuclei.
- Blue region: Energy Density Functional (EDF) framework (for example Skyrme forces): Mean-field framework.
- Beyond-mean-field models.
  - Particle-Vibration Coupling (PVC)
  - Multiparticle-Multihole Configuration Mixing.
  - $2^{nd}$ order energy correction.
  - Second RPA,....etc
- What are the problems?

- The double counting problem:
  1. The first question that one should address before adding correlations is: What are the many-body correlations that are effectively included implicitly in the mean-field approach?
  2. When one uses phenomenological interactions that are adjusted at the mean-field level, the problem of the double counting of correlations (that are implicitly contained to some extent in the parameters) should be addressed.
- The problem of ultraviolet divergences:
  1. UV divergences may appear for instance when one uses models beyond the mean-field level with zero-range interactions.
  2. For example, UV divergences appears in the so-called Bogoliubov-de Gennes or Hartree-Fock-Bogoliubov (HFB) theories if a zero-range interaction is employed in the pairing channel to treat a superfluid many-fermion system.

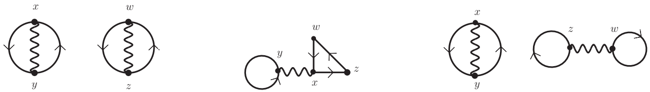
- The second-order energy correction is given by:

$$E^{(2)} = -i \int_{-\infty}^{0} dt_1 \langle \Phi_0 | T \left[ \hat{H}_1 \hat{H}_1(t_1) \right] | \Phi_0 \rangle_{connected}.$$

- The expectation value of all the terms containing normal-ordered products of operators vanishes in the non-interacting ground state $|\Phi_0\rangle$, leaving only the fully contracted products of field operators. We have:

$$T \left[ \hat{\psi}_a^\dagger(x) \hat{\psi}_b^\dagger(y) \hat{\psi}_d(y) \hat{\psi}_c(x) \hat{\psi}_\alpha^\dagger(z) \hat{\psi}_\beta^\dagger(w) \hat{\psi}_\mu(w) \hat{\psi}_\gamma(z) \right] = 24 \text{ terms}.$$

- Examples of second-order disconnected and anomalous diagrams are:




- Finally, according to Gold stone's theorem, the only contributions coming from the second-order terms are:

- The expression of the second-order correction due to the direct contribution is given by:

$$\boxed{E_d^{(2)} = \left(\frac{g^2}{2}\right)\frac{1}{(2\pi)^9}\int_{C_I} d^3p_1\ d^3p_2\ d^3q\ \frac{V^2(q)}{\epsilon^{(0)}_{p_1}+\epsilon^{(0)}_{p_2}-\epsilon^{(0)}_{p_1+q}-\epsilon^{(0)}_{p_2-q}}.}$$

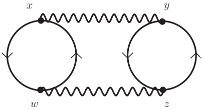



Figure : Direct diagram.

- Similarly, the expression of the second-order correction due to the exchange contribution is given by:

$$\boxed{E_{exch}^{(2)} = -\left(\frac{g}{2}\right)\frac{1}{(2\pi)^9}\int_{C_I} d^3p_1\ d^3p_2\ d^3q\ \frac{V(q)V(p_1-p_2-q)}{\epsilon^{(0)}_{p_1}+\epsilon^{(0)}_{p_2}-\epsilon^{(0)}_{p_1+q}-\epsilon^{(0)}_{p_2-q}}.}$$

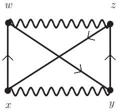



Figure : Exchange contribution.

- As a first attempt, we use a simple delta force which is spin-independent and we deal with nuclear matter, where all the calculations may be done analytically:

$$V(r_1, r_2) = t_0 \delta(r_1 - r_2).$$

- We calculate the equation of state for symmetric nuclear matter at the second-order in perturbation theory.

$$\frac{E}{A}(\rho) \quad = \quad \frac{E^{(0)}}{A}(\rho) + \frac{E^{(1)}}{A}(\rho) + \frac{E^{(2)}}{A}(\rho, \infty).$$

- Due to the short range character of the interaction, this equation of state diverges. We use the momentum cutoff $\Lambda$ procedure to regularize the divergent integrals:

$$\frac{E}{A}(\rho, \Lambda) \quad = \quad \frac{3}{10m} k_F^2 + \frac{t_0}{4\pi^2} k_F^3 - \left( \frac{9m}{8\pi^4} \right) t_0^2 \; k_F^4 \; I(k_F, \Lambda).$$

- $I(k_F, \Lambda)$ is an analytic function of $\rho$ and $\Lambda$ and it diverges linearly for large values of the cutoff $\Lambda$:

$$I(k_F, \Lambda >> M) = \frac{\Lambda}{9k_F} + \frac{-11 + 2\log 2}{105} + \mathcal{O}\left( \frac{k_F}{\Lambda} \right).$$

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- The asymptotic behaviour of the equation of state for large values of $\Lambda$ is given by:

$$\frac{E}{A}(\rho, \Lambda >> M) = \frac{3k_F^2}{10m} + \frac{t_0}{4\pi^2}k_F^3 - \left(\frac{9m}{8\pi^4}\right) t_0^2 \; k_F^4 \left[\frac{-11 + 2\log 2}{105} + \frac{\Lambda}{9k_F} + \mathcal{O}\left(\frac{k_F}{\Lambda}\right)\right].$$

- The terms that depend on the cutoff can be regrouped with the terms coming from the mean-field contribution:

$$\frac{E}{A}(\rho, \Lambda >> M) = \frac{3k_F^2}{10m} + \frac{1}{4\pi^2}k_F^3 \left[t_0 - \frac{mt_0^2}{2\pi^2}\Lambda\right] + \left(\frac{9m}{8\pi^4}\right)\left(\frac{11 - 2\log 2}{105}\right) t_0^2 \; k_F^4.$$

- The dependence on the cut-off will be eliminated by defining renormalized parameters $t_0^R$ from the bare parameter $t_0(\Lambda)$:

$$t_0^R = t_0(\Lambda) + C_2 \, \Lambda \, t_0^2(\Lambda) = t_0(\Lambda)\,[1 + C_2 \, \Lambda \, t_0(\Lambda)] \text{ such that: } \frac{d}{d\Lambda}t_0^R = 0.$$

- It is to be noted that:

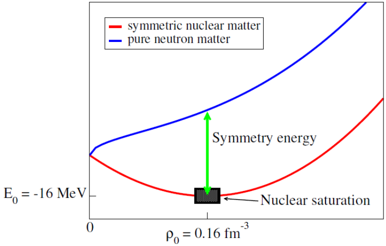
$$
\begin{aligned}
t_0(\Lambda) &= t_0^R - C_2\,\Lambda\, t_0^2(\Lambda) = t_0^R - C_2\,\Lambda \left(t_0^R\right)^2 + \mathcal{O}\left(t_0^R\right)^3 ; \\
t_0^2(\Lambda) &= \left[t_0^R - C_2\,\Lambda \left(t_0^R\right)^2 + \mathcal{O}\left(t_0^R\right)^3\right]^2 = \left(t_0^R\right)^2 + \mathcal{O}\left(t_0^R\right)^3 .
\end{aligned}
$$

- Therefore, the equation of state for symmetric matter evaluated at the second-order can be written as:

$$
\frac{E}{A}(\rho, \Lambda >> M) = \frac{3k_F^2}{10m} + \frac{t_0^R}{4\pi^2} k_F^3 + \left(\frac{9m}{8\pi^4}\right)\left(\frac{11 - 2\log 2}{105}\right)\left(t_0^R\right)^2 \; k_F^4 .
$$

- We conclude that the problem is renormalized in this case by redefining the parameter $t_0$.
- However, this simple model does not provide any saturation point for symmetric matter, even at the mean-field level.

- Calculations of the properties in nuclear matter show that the binding energy is $E_0 \approx -16$ MeV, and $\rho_0 \approx 0.16$ fm$^{-3}$.




- Three-body forces are considered as an indispensable ingredient in accurate calculations not only of few-nucleon systems and the structure of light nuclei but also for many-body systems in many cases.

- We have already concluded that in non-relativistic approaches the model of nucleons interacting only via a two-body force fails to reproduce the empirical saturation observables. Thus phenomenological three-body forces have been introduced with few adjustable parameters.
- For example, Skyrme introduced a zero-range force in order to achieve saturation in nuclear matter:

$$V_{123}(r_1, r_2, r_3) = t_3\delta(r_1 - r_2)\delta(r_2 - r_3).$$

- Later, Vautherin and Brink replaced the contact three-body force by a contact density-dependent two-body force.

$$t_3\delta(r_1 - r_2)\delta(r_2 - r_3) \longrightarrow \frac{t_3}{6}\rho^{\alpha}\ \delta(r_1 - r_2), \quad \text{where} \quad \alpha = 1.$$

- The next step is to add a zero-range density-dependent force $\frac{1}{6}t_3\rho^\alpha\delta(r)$ to the $t_0$ interaction.

$$V(r_1, r_2) = \left(t_0 + \frac{t_3}{6}\rho^\alpha\right)\delta(r_1 - r_2).$$

- It has to be noted that it corresponds to a three-body force at the mean-field level when $\alpha = 1$. The mean-field contribution to the density-dependent term is equal to:

$$e/A = \left(\frac{t_3}{36\pi^4}\right)k_F^6$$

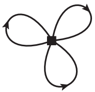


Figure : A contact three-body force at first-order.

- The density-dependent effective two-body interaction at second order is given by:

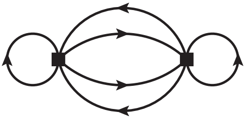


Figure : Density-dependent effective two-body interaction to second order.

- The equation of state for symmetric nuclear matter evaluated at the second-order is given by:

$$\frac{E}{A}(\rho,\Lambda)=\frac{3k_F^2}{10m}+\frac{t_0}{4\pi^2}k_F^3+\left(\frac{t_3}{36\pi^4}\right)k_F^6-\left(\frac{9m}{8\pi^4}\right)\left(t_0+\frac{t_3}{6}\rho^{\alpha}\right)^2 k_F^4\, I(k_F,\Lambda).$$

Where the expression $I(k_F,\Lambda)$ is an analytic function of $\rho$ and $\Lambda$ and it diverges linearly for large cutoff $\Lambda$:

$$I(k_F,\Lambda>>M)=\frac{-11+2\log 2}{105}+\frac{\Lambda}{9k_F}+\mathcal{O}\left(\frac{k_F}{\Lambda}\right).$$

- Then, the EoS for symmetric nuclear matter becomes (when $\alpha=1$):

$$\begin{aligned}\frac{E}{A}(\rho,\Lambda>>M) \quad = \quad & \frac{3k_F^2}{10m}+\frac{1}{4\pi^2}\left[t_0-\frac{m\Lambda}{2\pi^2}t_0^2\right]k_F^3+\frac{1}{36\pi^4}\left[t_3-\frac{m\Lambda}{\pi^2}t_0\,t_3\right]k_F^6\\ &+\left(\frac{9m}{8\pi^4}\right)\left(t_0+\frac{t_3}{6}\rho\right)^2\left[\frac{11-2\ln 2}{105}\right]k_F^4-\left(\frac{m\Lambda}{648\pi^8}t_3^2\right)k_F^9.\end{aligned}$$

- The last term can not be regrouped into the existing parameters unless we add a four-body term treated perturbatively at the mean-field level.

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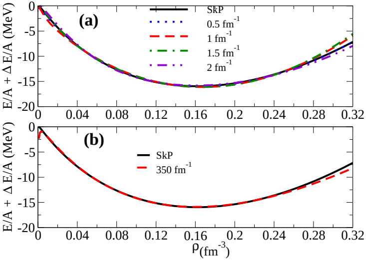
- Since the technique of adding counter-terms will complicate our problem, we will follow a phenomenological approach by which the divergence is absorbed by adjusting the parameters of the Skyrme interaction.
- In the case of effective interactions between point-nucleons, the cutoff $\Lambda$ must certainly be smaller than the momentum associated with the nucleon size, i.e., $\Lambda \leq 2$ fm$^{-1}$ .
- Moreover, the energy scale of low-energy nuclear phenomena in finite nuclei is much lower. See, for instance:
  J. Dobaczewski, K. Bennaceur, F. Raimondi, J. Phys. G39 (2012) 125103.
- In fact, to describe giant resonances or rotational bands of nuclei, the scale should be even smaller, perhaps around 0.5 fm$^{-1}$.
- For each value of $\Lambda$, we can perform a least-square fit to determine a new parameter set SkP$_\Lambda$ , such that the EoS including the second-order correction matches the one obtained with the original force SkP at the mean-field level.

- The SkP-equation of state is chosen as our reference on which we perform the fit.




K. Moghrabi, et al., Phys. Rev. Lett. 105, 262501 (2010).

- The quality of the fits demonstrates that the refitted interactions can describe satisfactorily the empirical equation of state for the case of symmetric nuclear matter treated with a simplified contact force.

Table : From the second line, columns 2, 3 and 4: parameter sets obtained in the fits associated with different values of the cutoff $\Lambda$ compared with the original set SkP (first line). In the fifth column the $\chi^2/N$-value ($\chi^2$ divided by the number of fitted points) associated to each fit is shown. In columns 6 and 7 the saturation point is shown.

| | $t_0$ (MeV fm$^3$) | $t_3$ (MeV fm$^{3+3\alpha}$) | $\alpha$ | $\chi^2/N$ | $\rho_0$ (fm$^{-3}$) | $E/A(\rho_0)$ (MeV) |
|---|---|---|---|---|---|---|
| SkP | -2931.70 | 18708.97 | 1/6 | | 0.16 | -15.95 |
| $\Lambda = 0.5$ fm$^{-1}$ | -2352.900 | 15379.861 | 0.217 | 0.00004 | 0.16 | -15.96 |
| $\Lambda = 1$ fm$^{-1}$ | -1155.580 | 9435.246 | 0.572 | 0.00142 | 0.17 | -16.11 |
| $\Lambda = 1.5$ fm$^{-1}$ | -754.131 | 8278.251 | 1.011 | 0.00106 | 0.17 | -16.09 |
| $\Lambda = 2$ fm$^{-1}$ | -632.653 | 5324.848 | 0.886 | 0.00192 | 0.16 | -15.82 |
| $\Lambda = 350$ fm$^{-1}$ | -64.904 | 360.039 | 0.425 | 0.00042 | 0.16 | -15.88 |

- For any value of the cutoff $\Lambda$ , it is possible to find a new refitted interaction that can be used to take into account the mean-field contribution plus the second-order corrections.

- Let us turn our attention to the full Skyrme interaction by including the velocity-dependent terms:

$$\begin{aligned} V_{12}(\vec{r},\vec{R}) &= t_0(1+x_0P^\sigma)\delta(\vec{r}) + \frac{1}{6}t_3\left(1+x_3P^\sigma\right)\delta(\vec{r})\rho^\alpha(\vec{R}) \\ &+\frac{1}{2}t_1(1+x_1P^\sigma)\left[\delta(\vec{r})\vec{k}^2 + \vec{k}'^2\delta(\vec{r})\right] + t_2(1+x_2P^\sigma)\vec{k}'\cdot\delta(\vec{r})\vec{k} \\ &+iW_0\vec{\sigma}\cdot\vec{k}'\times\delta(\vec{r})\vec{k}. \end{aligned}$$

- We calculate analytically the equation of state at second order in both symmetric and asymmetric nuclear matter with different neutron-to-proton ratios $\delta$.
- The equation of state at the mean-field level for asymmetric nuclear matter is given by:

$$\begin{aligned} \frac{E}{A}(\delta,\rho) &= \frac{3}{10m}G_{5/3}k_F^2 + \frac{t_0}{12\pi^2}[2(2+x_0)-(1+2x_0)G_2]k_F^3 \\ &+\frac{t_3}{48}\left(\frac{2}{3\pi^2}\right)^{1+\alpha}[2(2+x_3)-(1+2x_3)G_2]k_F^{3+3\alpha} \\ &+\frac{1}{40\pi^2}\left[\Theta_v G_{5/3} + \frac{1}{2}(\Theta_s - 2\Theta_v)G_{8/3}\right]k_F^5, \end{aligned}$$

- where $G_\beta = \frac{1}{2}[(1+\delta)^\beta + (1-\delta)^\beta]$, $\Theta_s = 3t_1 + t_2(5+4x_2)$, and $\Theta_v = t_1(2+x_1) + t_2(2+x_2)$.

- The asymptotic behavior of the second-order energy contribution ($\Lambda >> M$) is given by:

$$\frac{\Delta E^{(2)}}{A}(\delta,\rho) = k_F^3\left[a_0(\delta)\Lambda + a_1(\delta)\Lambda^3 + a_2(\delta)\Lambda^5\right] + k_F^5\left[b_0(\delta)\Lambda + b_1(\delta)\Lambda^3\right] + k_F^{3+3\alpha}\left[c_1(\delta)\Lambda + c_2(\delta)\Lambda^3\right] + k_F^7\left[c_0(\delta)\right] + k_F^{3+6\alpha}\left[c_3(\delta)\Lambda\right] + k_F^{5+3\alpha}\left[c_4(\delta)\right].$$

- What counter terms should be added?
  1. The first three terms can be regrouped with the mean-field contributions by redefining some existing parameters.
  2. The term $k_F^7\left[c_0(\delta)\right]$ can be absorbed by adding to the original Skyrme interaction the term: $\nabla^4\delta(r_1 - r_2)$.
  3. The term: $k_F^{3+6\alpha}\left[c_3(\delta)\Lambda\right] = k_F^9\left[c_3(\delta)\Lambda\right]$ ($\alpha = 1$), corresponds to a four-body force: $\delta(r_1 - r_2)\delta(r_2 - r_3)\delta(r_3 - r_4)$;
  4. Therm $k_F^{5+3\alpha}\left[c_4(\delta)\right] = k_F^8\left[c_4(\delta)\right]$ corresponds to $\nabla^2\delta(r_1 - r_2)\delta(r_2 - r_3)$.

- Possible directions:
  1. Dimensional regularization (DR) with minimal subtraction (MS) scheme.
  2. Absorbing the divergence through a fitting procedure.

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- The first + second–order EoS for symmetric matter is given by

$$\frac{E}{A}(k_F) = \frac{3}{10m}k_F^2 + \frac{t_0}{4\pi^2}k_F^3 + \frac{t_3}{16}\left(\frac{2}{3\pi^2}\right)^{1+\alpha} k_F^{3+3\alpha} + \frac{1}{40\pi^2}\Theta_s k_F^5 + \frac{\Delta E^S}{A}(k_F)$$

- The second–order correction is given by:

$$\begin{aligned}\frac{\Delta E^S}{A}(k_F) \quad = \quad & k_F^4\left(\frac{-11+2\ln 2}{105}\right)\chi_1^S(k_F) + k_F^6\left(\frac{-167+24\ln 2}{945}\right)\chi_2^S(k_F) \\ & + k_F^8\left(\frac{-2066+312\ln 2}{31185}\right)\chi_3^S(k_F) + k_F^8\left(\frac{-9997+1236\ln 2}{62370}\right)\chi_4^S(k_F).\end{aligned}$$

where all the $\chi$ terms depend on the parameters of Skyrme interaction and on the Fermi momentum $k_F$.

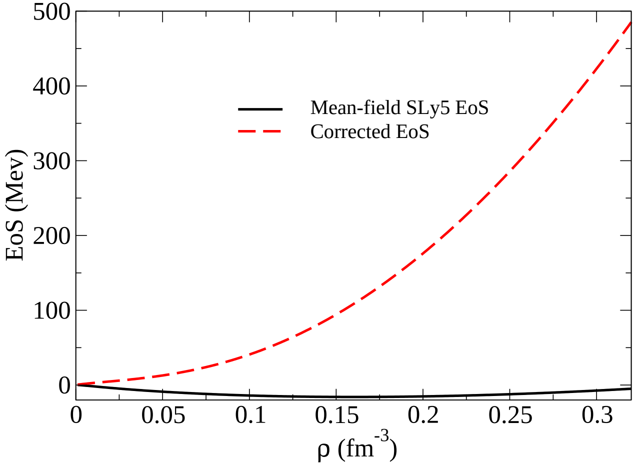



Figure : SLy5 mean–field EoS (full line) and mean–field + second–order EoS calculated with the SLy5 parameters (dashed line) for symmetric matter.

- The beyond mean–field EoS evaluated at second order for pure neutron matter is given by:

$$\begin{aligned}\frac{E}{A}(k_n) &= \frac{3}{10m}k_n^2 + \frac{1}{12\pi^2}t_0\,(1-x_0)\,k_n^3 + \frac{1}{24}\left(\frac{1}{3\pi^2}\right)^{1+\alpha}(1-x_3)\,t_3\,k_n^{3+3\alpha} \\ &+ \frac{1}{40\pi^2}(\Theta_s - \Theta_v)\,k_n^5 + \frac{\Delta E^N}{A}(k_n),\end{aligned}$$

- This time the second–order correction is given by:

$$\begin{aligned}\frac{\Delta E^N}{A}(k_n) &= k_n^4\left(\frac{-11+2\ln 2}{105}\right)\chi_1^N(k_n) + k_n^6\left(\frac{-167+24\ln 2}{2835}\right)\chi_2^N(k_n) \\ &+k_n^6\left(\frac{167-24\ln 2}{5670}\right)\chi_3^N(k_n) + k_F^8\left(\frac{461-24\ln 2}{31185}\right)\chi_4^N(k_n) \\ &+k_n^8\left(\frac{-4021+516\ln 2}{124740}\right)\chi_5^N(k_n)\end{aligned}$$

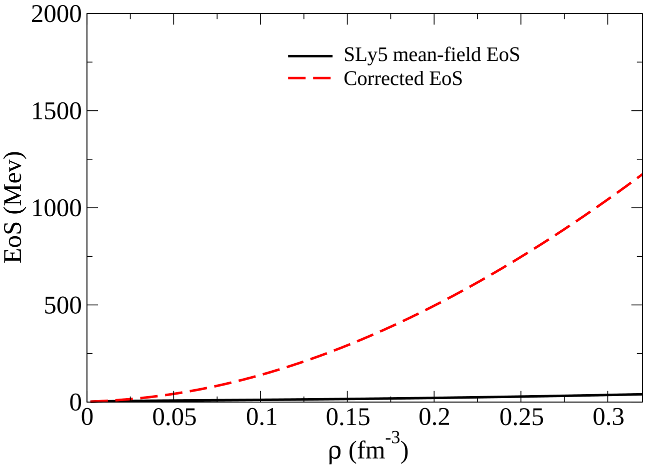
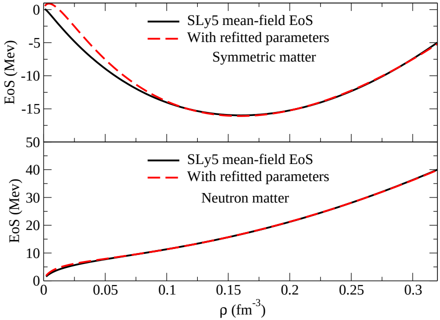



Figure : SLy5 mean–field EoS (full line) and mean–field + second–order EoS calculated with the SLy5 parameters (dashed line) for neutron matter.

- A problem of double counting! Need to refit the parameters.
- The following $\chi^2$ is minimized:

$$\chi^2 = \frac{1}{N-1} \sum_{i=1}^{N} \frac{(E_i - E_{i,ref})^2}{\Delta E_i^2}.$$

  1. The number $N$ of fitted points is 15 and the points are in the range of densities between 0.1 to 0.30 fm$^{-3}$.
  2. The errors $\Delta E_i$ are chosen equal to 1% of the reference SLy5 mean-field energies $E_{i,ref}$.
- We perform a global fit by considering together symmetric and pure neutron matter. In this case: $\chi^2 = \frac{1}{2}\left(\chi^2_{\delta=0} + \chi^2_{\delta=1}\right).$




K.M et al., Phys. Rev. C 86, 044319 (2012).

Table : Parameter sets obtained in the fit of the EoS of symmetric and pure neutron matter compared with the original set SLy5. In the last column the $\chi^2$ value is shown.

| | $t_0$ (MeV fm$^3$) | $t_1$ (MeV fm$^5$) | $t_2$ MeV fm$^5$) | $t_3$ (MeV fm$^{3+3\alpha}$) | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $\alpha$ | $\chi^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| SLy5 | -2484.88 | 483.13 | -549.40 | 13763.0 | 0.778 | -0.328 | -1.0 | 1.267 | 0.16667 | – |
| New | -460.73 | 10403.66 | -8485.73 | -141558.6 | 1.460 | -0.681 | -0.641 | -0.779 | 0.650 | 0.202 |

- It has to be noted that this set of parameters is cutoff-independent which is a good result.
- However, the values of the new parameter are far from their orignal values (SLy5) which may lead to problems of non-convergence at the Hartree-Fock level, when applied to finite nuclei.
- Let us go back to the asymptotic behavior of the second-order energy correction::

$$
\begin{aligned}
\frac{\Delta E^{(2)}}{A}(\delta,\rho) \quad = \quad & k_F^3\left[a_0(\delta)\Lambda + a_1(\delta)\Lambda^3 + a_2(\delta)\Lambda^5\right] + k_F^5\left[b_0(\delta)\Lambda + b_1(\delta)\Lambda^3\right] \\
& + k_F^{3+3\alpha}\left[c_1(\delta)\Lambda + c_2(\delta)\Lambda^3\right] \\
& + k_F^7\left[c_0(\delta)\right] + k_F^{3+6\alpha}\left[c_3(\delta)\Lambda\right] + k_F^{5+3\alpha}\left[c_4(\delta)\right].
\end{aligned}
$$

- Adding counter terms to the original Skyrme interaction to suppress the terms in red would strongly complicate the calculations, especially in the perspective of doing applications to finite nuclei.
- Therefore, we employ a simple procedure and absorb the UV divergences by adjusting the parameters of Skyrme in order to have a reasonable second-order EoS (as we have already done for the $t_0 - t_3$ model).

- We are dealing with phenomenological interactions where their corresponding parameters can be adjusted according to which diagrams are explicitly introduced.
- To have a reasonable second-order EoS, we have adjusted the nine parameters of the Skyrme interaction entering in the expression of the EoS to reproduce the reference SLy5 mean-field EoS.
- We have chosen 15 equidistant reference points ($N$) for densities ranging from 0.02 fm$^{-3}$ to 0.30 fm$^{-3}$. All the parameters are kept free in the adjustment procedure.
- The minimization has been performed using the following definition for the $\chi^2$,

$$\chi^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(E_i - E_{i,ref})^2}{\Delta E_i^2}.$$

- The errors or *adopted* standard deviations, $\Delta E_i$ are chosen equal to 1% of the reference SLy5 mean-field energies $E_{i,ref}$.

- K.M et al., Phys. Rev. C 85, 044323 (2012).

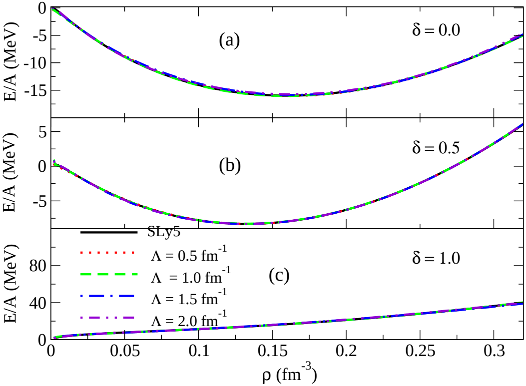



Figure : Refitted EoS (global fit) for symmetric (a), asymmetric (b) and pure neutron (c) matter. The reference SLy5 mean-field curves are also plotted in the 3 panels (solid lines).

Table : Parameter sets obtained in the fit of the EoS of symmetric, asymmetric and pure neutron matter for different values of the cutoff $\Lambda$ compared with the original set SLy5. In the last column the $\chi^2$ values are shown.

| | $t_0$ (MeV fm$^3$) | $t_1$ (MeV fm$^5$) | $t_2$ (MeV fm$^5$) | $t_3$ (MeV fm$^{3+3\alpha}$) | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $\alpha$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| SLy5 | -2484.88 | 483.13 | -549.40 | 13736.0 | 0.778 | -0.328 | -1.0 | 1.267 | 0.16667 | |
| $\Lambda$(fm$^{-1}$) | | | | | | | | | | $\chi^2$ |
| 0.5 | -2022.142 | 290.312 | 1499.483 | 12334.459 | 0.481 | -5.390 | -1.304 | 0.880 | 0.259 | 0.411 |
| 1.0 | -627.078 | 83.786 | -971.384 | 186.775 | 3.428 | -1.252 | -1.620 | 200.360 | 0.338 | 0.540 |
| 1.5 | -743.227 | 112.246 | -42.816 | 5269.849 | 1.013 | 3.478 | -2.114 | 0.189 | 0.814 | 1.733 |
| 2.0 | -718.397 | 573.884 | -497.766 | 6179.243 | 0.391 | -0.393 | -0.574 | 0.785 | 1.051 | 1.313 |

## Conclusions/Nuclear Matter

- Analytical study of the divergent terms at the second-order in the equation of state of nuclear matter with Skyrme interaction.
- We were not able to remove the divergences by their corresponding adding counter terms because the problem will become complicated.
    1. In the simple case of $t_0 - t_3$ model, the divergence was absorbed through a fitting procedure.
    2. In the case of the full interaction: the techniques of DR/MS were introduced. In this case, a unique set of parameters is obtained for the adjusted effective interaction.
    3. The UV divergences were absorbed so that the reference EOS (SLy5) of both symmetric and nuclear matter with different neutron-to-proton ratio are reproduced in the case of MC.

## Perspectives/ Finite Nuclei

- These encouraging results open new prospectives for future applications of this kind of interactions to treat finite nuclei in beyond mean-field models.
- Introduce the coupling between nucleon individual degrees of freedom and collective degrees of freedom by means of the particle-vibration coupling approach (pvc).

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Thanks for your attention!!

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- Motivation: Let us see what would happen if we replace the density-dependent term by the three-body term.
- At the first order, they give the same energy contribution. What about the second-order? Can we renormalize our problem?
- The $1^{st}$ order diagrams generated by a $3-$body contact interaction is given by:

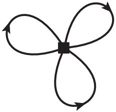

Figure : $1^{st}$ order diagrams generated by a $3-$body contact interaction.

- The energy contribution is given by:

$$e/A = \left(\frac{t_3}{36\pi^4}\right) k_F^6.$$

- The $2^{nd}$ order diagrams generated by a 3−body contact interaction is given by:

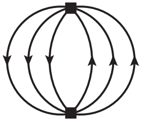

Figure : $2^{nd}$ order diagrams generated by a 3−body contact interaction.

- The second-order energy correction due to the three-body force is given by:

$$E = \frac{k_F^{13}}{(2\pi)^{15}} \, (t_3^2 M) \int_{C_I} d^3p_1 \; d^3p_2 \; d^3p_3 \; d^3x \; d^3y \; \frac{1}{\vec{x}^{\,2} + 3\vec{y}^{\,2}/4 - H/6 - i\epsilon},$$

where the expression for $C_I$ is given by:

$$C_I = \Big[1 - \theta(1 - |\vec{p} \pm \vec{x} - \vec{y}/2|)\Big] \Big[1 - \theta(1 - |\vec{p} + \vec{y}|)\Big] \theta(1 - |\vec{p}_1|) \; \theta(1 - |\vec{p}_2|) \; \theta(1 - |\vec{p}_3|).$$

- The chosen intermediate variables are: $\vec{p} + \vec{x} - \vec{y}/2$, $\vec{p} + \vec{y}$, $\vec{p} - \vec{x} - \vec{y}/2$, with $\vec{p} = (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)/3$.
- The other kinematical quantity $H$ appearing in the energy denominator is the Galilean invariant:

$$H = \frac{1}{k_F^2} \left[ (\vec{p}_1 - \vec{p}_2)^2 + (\vec{p}_1 - \vec{p}_3)^2 + (\vec{p}_2 - \vec{p}_3)^2 \right] < 9.$$

- We use the techniques of the dimensional regularization (DR) with a power-divergent subtraction (PDS) scheme because the second-order correction diverges at high momenta.
- After analyzing the asymptotic behavior of the scattering amplitude at high momenta, it becomes straightforward to analyze the divergent parts of the binding energy calculated at the second-order:

$$
\begin{aligned}
E^{div} \quad &\propto \quad k_F^{13}\,\delta t_3^a \left[\frac{1}{3-d}+\frac{3}{2}-\gamma\right] - \mu^2\, k_F^{11}\,\delta t_3^b \left[\frac{1}{2-d}-\gamma\right] + \mu\, k_F^{12}\,\delta t_3^c\, \left[\frac{1}{2-d}-\gamma\right] \\
&+ \quad \mu^4\, k_F^9\,\delta t_3^d \left[\frac{1}{1-d}-\gamma\right]
\end{aligned}
$$

- The terms $\delta t_3^{a,b,c,d}$ are proportional to $t_3^2$; $\mu$ is an auxiliary momentum parameter; $\gamma$ is the Euler's constant.
- We see that $E$ has poles:
  1. at $d=3$ which corresponds to a logarithmic divergence ($\propto k_F^{13}\ln\Lambda$);
  2. at $d=2$ which corresponds to a power-divergence ($\propto k_F^{12}\Lambda$ and $k_F^{11}\Lambda$);
  3. at $d=1$ corresponds to a power-divergence ($\propto k_F^{9}\Lambda^4$).

- The asymptotic behavior of the second-order energy correction can be written as:

$$E^{div} \propto \underbrace{k_F^{13}\,\delta t_3^a \left[\frac{1}{3-d}+\frac{3}{2}-\gamma\right]}_{I_1} - \underbrace{\mu^2\,k_F^{11}\,\delta t_3^b \left[\frac{1}{2-d}-\gamma\right]}_{I_2} + \underbrace{\mu\,k_F^{12}\,\delta t_3^d \left[\frac{1}{2-d}-\gamma\right]}_{I_3}$$

$$+ \underbrace{\mu^4\,k_F^{9}\,\delta t_3^d \left[\frac{1}{1-d}-\gamma\right]}_{I_4}$$

- The last term $I_4$ is proportional to $k_F^9$. It can be regrouped with the mean-field three-body contribution.

- Counter terms:
  1. $I_1$ is suppressed by adding a counter term: $\delta t_3^a \nabla_{12}^2 \delta(r_1 - r_2) \nabla_{23}^2 \delta(r_2 - r_3)$;
  2. $I_1$ is suppressed by adding a counter term: $\delta t_3^b \nabla_{12}^2 \delta(r_1 - r_2) \delta(r_2 - r_3)$;
  3. $I_1$ is suppressed by adding a counter term: $\delta t_3^c \delta(r_1 - r_2) \delta(r_2 - r_3) \delta(r_3 - r_4)$.