Exercises for Quantum Field Theory II, WS 2015/16

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Exercise Sheet 2   18.11.2015

A.2: Gauß integrals and Grassmann variables

(a) First show that the following identity holds for multi-dimensional integrals over real variables x_i :

$$\frac{1}{(2\pi)^{n/2}}\int\prod_{i=1}^{n}dx_i\,\exp\Big(-\frac{1}{2}x_kA_{kl}x_l+J_kx_k\Big)\;=\;[\det A]^{-1/2}\exp\Big(\frac{1}{2}J_kA_{kl}^{-1}J_l\Big)\;,$$

where A is a real, symmetric, positive-definite matrix.

(b) Now convince yourself that the result from part (a) can be generalized in the following way for complex variables z_i :

$$\frac{1}{(2\pi i)^{n}}\int\prod_{i=1}^{n}dz_i^*dz_i\,\exp\Big(-z_k^*H_{kl}z_l+J_k^*z_k+J_kz_k^*\Big)\;=\;[\det H]^{-1}\exp\Big(J_k^*H_{kl}^{-1}J_l\Big)\;,$$

where H is now a hermitian positive-definite matrix.

(c) A Grassmann algebra is a collection $\theta_1,\ldots,\theta_n$ of anticommuting numbers, called Grassmann variables. They fulfill the relation

$$\theta_i\theta_j=-\theta_j\theta_i\ \forall\ i,j=1,\ldots,n\,. \tag{1}$$

Note that this directly implies $(\theta_i)^2=0\ \forall\ i$. In the lecture, the following rules for differentiation and intergration with respect to Grassmann variables will be derived:

$$\frac{\partial}{\partial\theta_i}\theta_j=\delta_{ij}\quad,\qquad\frac{\partial}{\partial\theta_i}(\text{any c-number})=0\,. \tag{2}$$

$$\int \mathrm{d}\,\theta_i\,\theta_j=\delta_{ij}\quad,\qquad\int\mathrm{d}\,\theta_i=0\,. \tag{3}$$

This means that intergration and differentiation are formally equivalent. In the following, you can use the above results without proof.

Show that integrals over Grassmann variables behave in the following way under linear transformations of the variables:

$$\int d\xi_1^*d\xi_1\ldots d\xi_n^*d\xi_nF(\xi^*,\xi)\;=\;\left|\frac{\partial(\eta^*,\eta)}{\partial(\xi^*,\xi)}\right|\int d\eta_1^*d\eta_1\ldots d\eta_n^*d\eta_nF\big(\xi^*(\eta^*,\eta),\xi(\eta^*,\eta)\big)\,,$$

where F is a smooth function.

For the proof, consider a linear transformation between two sets of Grassmann variables:

$$\begin{aligned} \left(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_{2n}\right) &\doteq \left(\xi_1^*, \xi_2^*, \ldots, \xi_n^*, \xi_1, \xi_2, \ldots, \xi_n\right) , \quad (4) \\ \left(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_{2n}\right) &\doteq \left(\eta_1^*, \eta_2^*, \ldots, \eta_n^*, \eta_1, \eta_2, \ldots, \eta_n\right) , \quad (5) \\ \tilde{\xi}_i &= M_{ij}\tilde{\eta}_j \ . \quad (6) \end{aligned}$$

Hint:

As a first step, show that $F(\xi^*, \xi) = F(\tilde{\xi})$ can be decomposed into a sum of monomials in $\tilde{\xi}_j$. Which of these terms contribute to the integral?

(d) Finally, use the result from part (c) to derive the following relation for multi-dimensional Gauß integrals over Grassmann variables:

$$\int \prod_{i=1}^{n} d\eta_i^* d\eta_i \exp\Big(-\eta_k^* H_{kl}\eta_l + \xi_k^*\eta_k + \xi_k\eta_k^*\Big) = \det H \exp\Big(\xi_k^* H_{kl}^{-1}\xi_l\Big) \ .$$

As in part (b), H is a hermitian positive-definite matrix.