Exercises for Quantum Field Theory II, WS 2015/16

H.-W. Hammer

Exercise Sheet 4   15.01.2016

A.4: Differential equation for n-point function

Let a general n-point vertex function $\Gamma^{(n)}$ depend on external momenta $\{p_i\}$, a (dimensionless) coupling constant g, plus a renormalization scale μ.

(a) Argue that, if all momenta are scaled according to $p_i \to \lambda p_i$, $\Gamma^{(n)}$ can be written as

$$\Gamma^{(n)}\big(\{\lambda p_i\}, g, \mu\big) \;=\; \mu^D f\big(\{\lambda^2 p_i \cdot p_j/\mu^2\}, g\big) \ ,$$

where D is the mass dimension of the vertex function $\Gamma^{(n)}$.

(b) Show that $\Gamma^{(n)}$ fulfils the differential equation

$$\left[\mu\frac{\partial}{\partial\mu} + \frac{\partial}{\partial t} - D\right]\Gamma^{(n)}\big(\{\lambda p_i\}, g, \mu\big) \;=\; 0 \ ,$$

where $e^t = \lambda$.

A.5: Behaviour of $\bar{g}$ near a simple fixed point

Derive the ultraviolet behaviour of the running coupling $\bar{g}(t)$ in the case that the β-function is given by

$$\beta(g) \;=\; g\left(a^2 - g^2\right) \ ,$$

with a being a known constant. To this end, proceed as follows:

(a) Sketch $\beta(g)$ versus g and read off the asymptotic behaviour of $\bar{g}(t)$ for $t \to \infty$, depending on the initial condition $g(t=0) = g_0$.

(b) Solve the differential equation for $\bar{g}$ (as given by the β-function) explicitly and confirm your findings of part (a).

A.6: Running coupling near a general fixed point

Show that, at an ultraviolet stable fixed point a,

(a) if $\beta(g)$ has s simple zero: $\beta(g) = -b(g-a)$ with $b > 0$, then $g(t)$ (subject to the initial condition $g(t=0) = g_0$) approaches a exponentially as $t \to \infty$;

(b) if $\beta(g)$ has a double or higher zero: $\beta(g) = -b(g-a)^n$ with $b > 0$ and $n > 1$, then $g(t)$ approaches a with some inverse power in t for $t \to \infty$.