Exercises for Quantum Field Theory II, WS 2015/16

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Exercise Sheet 5   03.02.2016

A.7: Ghosts and unitarity

Ghost fields have been introduced in the lectures as a rather abstract means to take care of the gauge field determinant arising in the quantisation procedure of gauge theories. It was however pointed out early by Feynman that ghosts are also required in order not to violate unitarity of the S-matrix at the one-loop level. In this exercise, we wish to demonstrate this connection.

A fundamental consequence of unitarity is the optical theorem, which for a scattering amplitude T_{ab} reads

$$\mathrm{Im}\, T_{ab} = \frac{1}{2} \sum_c T_{ac} T^*_{bc} (2\pi)^4 \delta^{(4)}(p_a - p_c)\,. \tag{1}$$

We want to consider the fermion-antifermion scattering amplitude $T(f\bar{f} \to f\bar{f})$ with massless gauge boson intermediate states. This is represented schematically in Fig. 1.

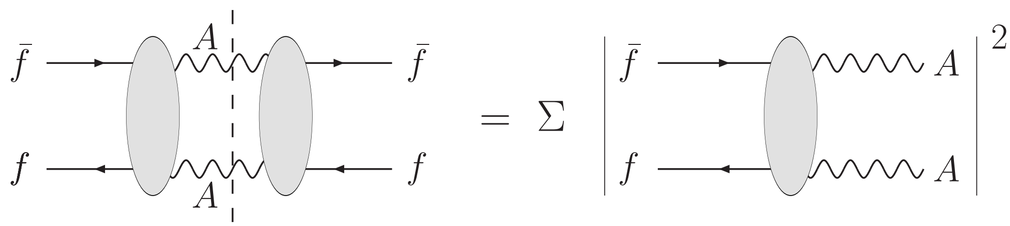

Abbildung 1: Unitarity condition relating the imaginary part of the $f\bar{f} \to f\bar{f}$ amplitude (left) to the sum of squared amplitudes $f\bar{f} \to AA$ (right).

The imaginary part of the "cut" diagram on the left-hand side can be calculated by taking the imaginary parts of the corresponding gauge boson and ghost propagators (we work in the Feynman gauge $\xi \equiv \beta = 1$),

$$\begin{aligned} \mathrm{Im}\, D^{ab}_{\mu\nu} &= \mathrm{Im} \frac{-\delta^{ab} g_{\mu\nu}}{k^2 + i\epsilon} = \pi \delta^{ab} g_{\mu\nu} \delta(k^2)\,, \\ \mathrm{Im}\, \Delta^{ab}_{\mu\nu} &= \mathrm{Im} \frac{-\delta^{ab}}{k^2 + i\epsilon} = \pi \delta^{ab} \delta(k^2)\,. \end{aligned}$$

The $\delta(k^2)$ functions put the intermediate gauge boson and ghost states on-shell ($k^2 = 0$). In this way, the unitarity condition Eq. (1) can be put into the form

$$\int d\rho_2 \left[\frac{1}{2} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} - S^{ab} S^{ab*} \right] = \frac{1}{2} \int d\rho_2 T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} P^{\mu\mu'}(k_1) P^{\nu\nu'}(k_2)\,, \tag{2}$$

where $T^{ab}_{\mu\nu}$ and S^{ab} are the amplitudes for $f\bar{f} \to A^a_\mu A^b_\nu$ and $f\bar{f} \to \eta^{a\dagger}\eta^b$, A^a_μ and η^a are gauge and ghost fields, respectively. $d\rho_2$ is the integration measure for the phase space of two massless particles; as it appears on both sides of Eq. (2), we will not need its specific form in the following.

The $P^{\mu\nu}$ are the polarisation sums of the gauge particles,

$$P^{\mu\mu'}(k_1) = \sum_{\sigma=1,2} \epsilon_1^{\mu}(k_1,\sigma)\epsilon_1^{\mu'}(k_1,\sigma) \ ,$$
$$P^{\nu\nu'}(k_2) = \sum_{\sigma=1,2} \epsilon_2^{\nu}(k_2,\sigma)\epsilon_2^{\nu'}(k_2,\sigma) \ .$$

They can be shown to obey

$$P^{\mu\nu}(k) = -g_{\mu\nu} + Q_{\mu\nu} \ , \quad Q_{\mu\nu} = \frac{k_\mu \eta_\nu + k_\nu \eta_\mu}{k \cdot \eta} \ , \tag{3}$$

where η_μ is a vector chosen such that k^μ, $\epsilon^\mu(k,1)$, $\epsilon^\mu(k,2)$, and η_μ span the whole four-dimensional space. These four vectors obey $\eta_\mu \epsilon^\mu(k,\sigma) = k_\mu \epsilon^\mu(k,\sigma) = 0$, $\epsilon^2(k,\sigma) = -1$ ($\sigma = 1,2$), $\epsilon_\mu(k,1)\epsilon^\mu(k,2) = 0$, $k^2 = \eta^2 = 0$, $k_\mu \eta^\mu \neq 0$.

A remarkable fact about the unitarity relation is that ghost contributions can contribute on the left-hand side of Eq. (2), but not on the right-hand side (as they cannot appear as physical, asymptotic states).

(a) Cut contributions Sketch all the one-loop diagrams (with cuts / imaginary parts) contributing to the left-hand side of Eq. (2). How many are there?

(b) $T^{ab}_{\mu\nu}$, S^{ab} to lowest order Sketch all the lowest-order (tree) diagrams contributing to $T^{ab}_{\mu\nu}$ and S^{ab} and write down the corresponding amplitudes, using the necessary Feynman rules. How many diagrams are there? Which of those are present only in a non-abelian theory?

(c) Ward identities Using the results from part b), calculate $k_1^\mu T^{ab}_{\mu\nu}$:

- **(i)** First, calculate this expression using only the diagrams that would also appear in a purely abelian theory; show that the expression does not vanish, but would in the abelian case. What does that mean for contributions on the right hand side of the unitarity relation Eq. (2) in the abelian case (considering the form of the polarisation sum Eq. (3))? Are ghost contributions required?
- **(ii)** Now complete the calculation and show that the following relation holds:

$$k_1^\mu T^{ab}_{\mu\nu} = -S^{ab} k_{2\nu} \ . \tag{4}$$

Similarly, one can show

$$T^{ab}_{\mu\nu} k_2^\nu = -S^{ab} k_{1\mu} \ . \tag{5}$$

An obvious consequence of Eqs. (4), (5) is $k_1^\mu T^{ab}_{\mu\nu} k_2^\nu = 0$ (it is obvious, isn't it?).

(d) Unitarity condition Finally, show the validity of the unitarity condition Eq. (2). Start with the right hand side and make repeated use of Eqs. (3), (4), (5).