Can one sum all Feynman diagrams for the unitary Fermi gas?

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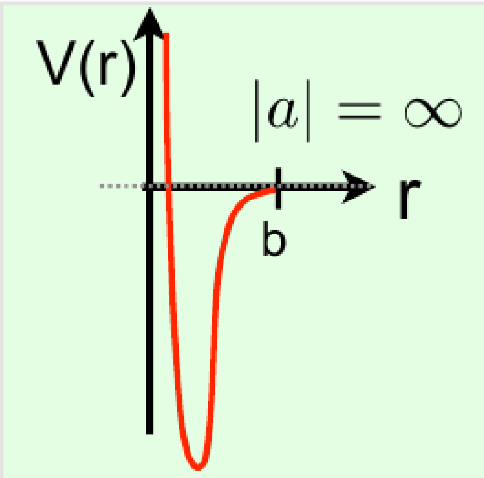
What is the unitary gas:

Spin−1/2 fermions in 3D, interactions have
- infinite scattering length
- zero range

Universality hypothesis:



Zero-range limit: $\begin{cases} n^{-1/3} \gg b \\ \lambda \gg b \end{cases}$ $\qquad \lambda \equiv \sqrt{\dfrac{2\pi\hbar^2}{mk_BT}}$

$\Longrightarrow$ $\exists$ (quasi) equilibrium homogeneous phase properties independent on V(r)

$(N_\uparrow = N_\downarrow)$ $n(T,\mu)\lambda^3 =$ universal function of $\beta\mu$ $\left[\beta \equiv \dfrac{1}{k_BT}\right]$

Cold atoms experiment:

^{6}Li atoms. 2 lowest hyperfine states denoted $\uparrow$ and $\downarrow$

Well in the universal regime:

Broad Feshbach resonance at $B_0 = 832.18 \pm 0.08$ G [Heidelberg 2013]

$$\begin{cases} |a| \gtrsim 300\ \mu\text{m} \\ \text{effective range } r_e \simeq 5\text{ nm} \\ \frac{1}{k_F} \sim 200 - 400\text{ nm} \end{cases} \Rightarrow \boxed{r_e \ll \frac{1}{k_F} \ll |a|}$$

FEYNMAN DIAGRAMS for the unitary gas

For ultraviolet regularisation: temporarily introduce a lattice (although continuum limit will be taken analytically)

$$H = \underbrace{\sum_\sigma \int_{[-\frac{\pi}{b},\frac{\pi}{b}]^3} \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} c^\dagger_\sigma(\mathbf{k}) c_\sigma(\mathbf{k})}_{\text{kinetic}} + \underbrace{g_0 \sum_{\mathbf{r}} b^3 \hat{n}_\uparrow(\mathbf{r}) \hat{n}_\downarrow(\mathbf{r})}_{\text{on-site interaction}}$$

$$g_0 = -\left[\int_{[-\frac{\pi}{b},\frac{\pi}{b}]} \frac{d^3k}{(2\pi)^3} \frac{m}{\hbar^2 k^2}\right]^{-1} = -\frac{\hbar^2}{m} \cdot b \cdot 5.14435\ldots \quad \Rightarrow \quad |a| = \infty$$

Continuum(=zero-range) limit: $b \to 0 \quad g_0(b) =$

Single-particle propagator: $G_\sigma(\vec{p},\tau) \equiv -\langle \mathrm{T}_\tau \; c_{\vec{p},\sigma}(\tau) c^\dagger_{\vec{p},\sigma}(0)\rangle$

$$c_{\mathbf{p},\sigma}(\tau) \equiv e^{\tau(H-\mu N)} c_{\mathbf{p},\sigma}\, e^{-\tau(H-\mu N)}$$

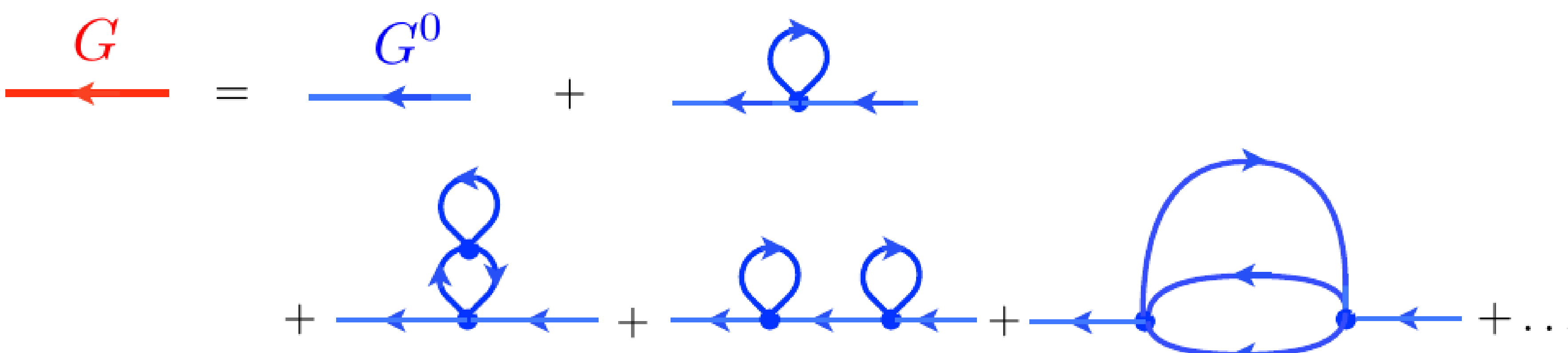
Momentum distribution:

$$G_\sigma(\mathbf{p}, \tau = 0^-) = \langle c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma}\rangle = n_\sigma(\mathbf{p})$$

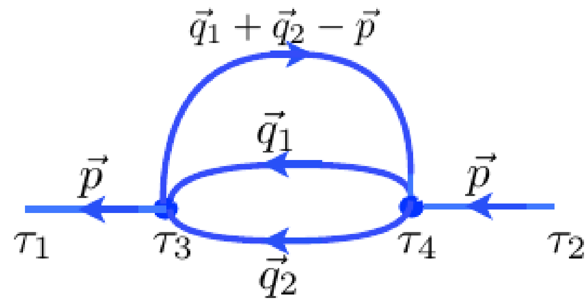
Density (equation of state): $\sum_{\sigma=\uparrow,\downarrow} \int \frac{d\mathbf{p}}{(2\pi)^3} n_\sigma(\mathbf{p}) = n$

Expansion of G in powers of g_0 :

$G^0 =$ ideal gas propagator



Feynman rules: example:

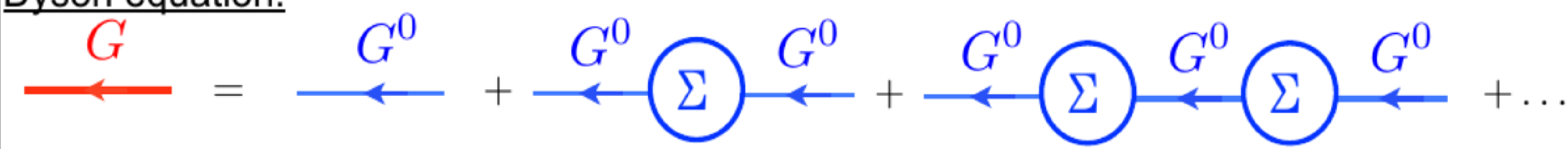


$$\{\text{contribution to } G(\vec{p}, \tau_1 - \tau_2)\} = (g_0)^2 (-1)^{2+1} \int \frac{d\vec{q}_1}{(2\pi)^3} \frac{d\vec{q}_2}{(2\pi)^3} \int_0^\beta d\tau_3 d\tau_4 \; G^0(\vec{p}, \tau_1 - \tau_3)$$
$$\times G^0(\vec{q}_2, \tau_3 - \tau_4) G^0(\vec{q}_1, \tau_3 - \tau_4) G^0(\vec{q}_1 + \vec{q}_2 - \vec{p}, \tau_4 - \tau_3) G^0(\vec{p}, \tau_4 - \tau_2)$$

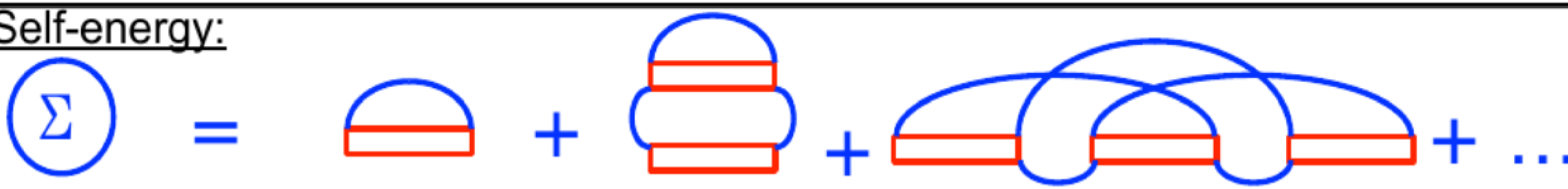
Ladder summation:

$\Rightarrow \Gamma^0$ is well-defined in the continuum limit, which can be taken analytically

Dyson equation:



Self-energy:



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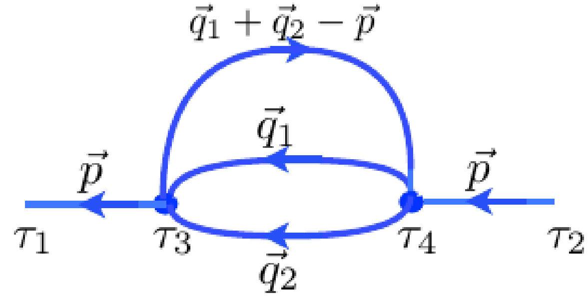

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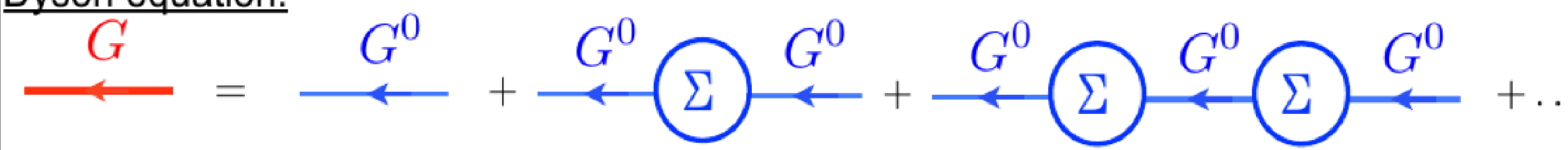

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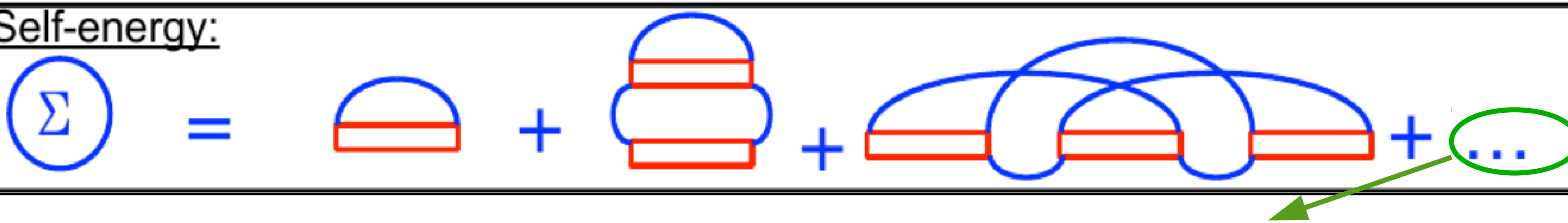
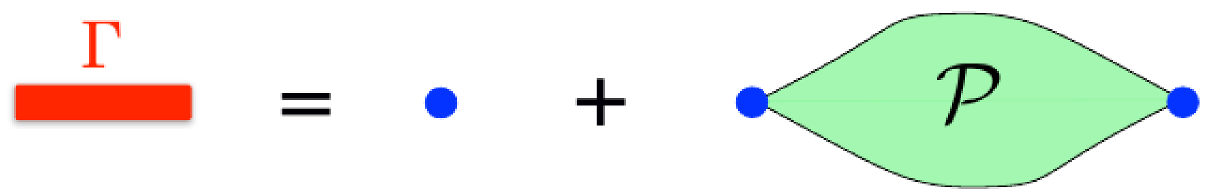
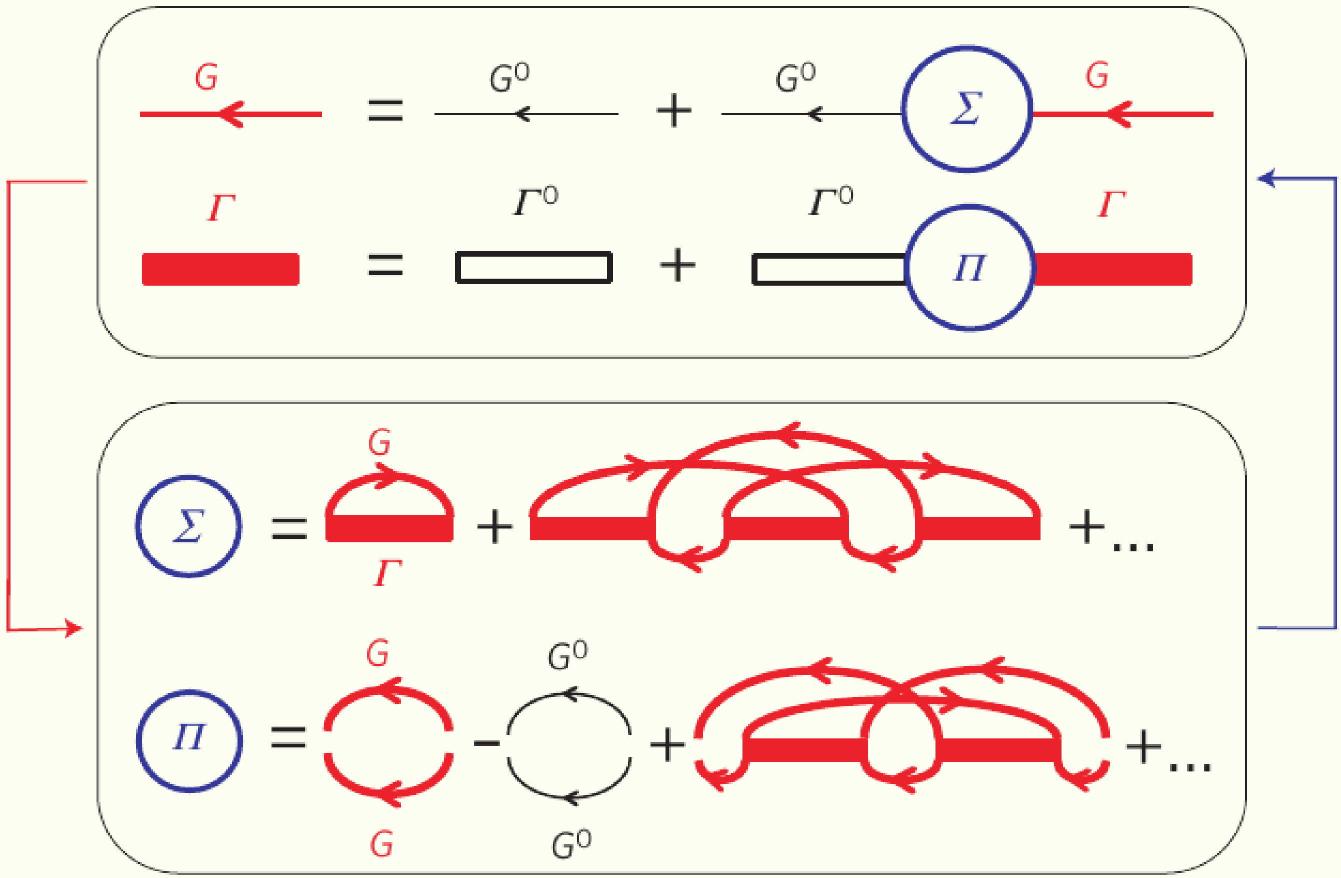


sum all diagrams up to order ~9
using Diagrammatic Monte Carlo

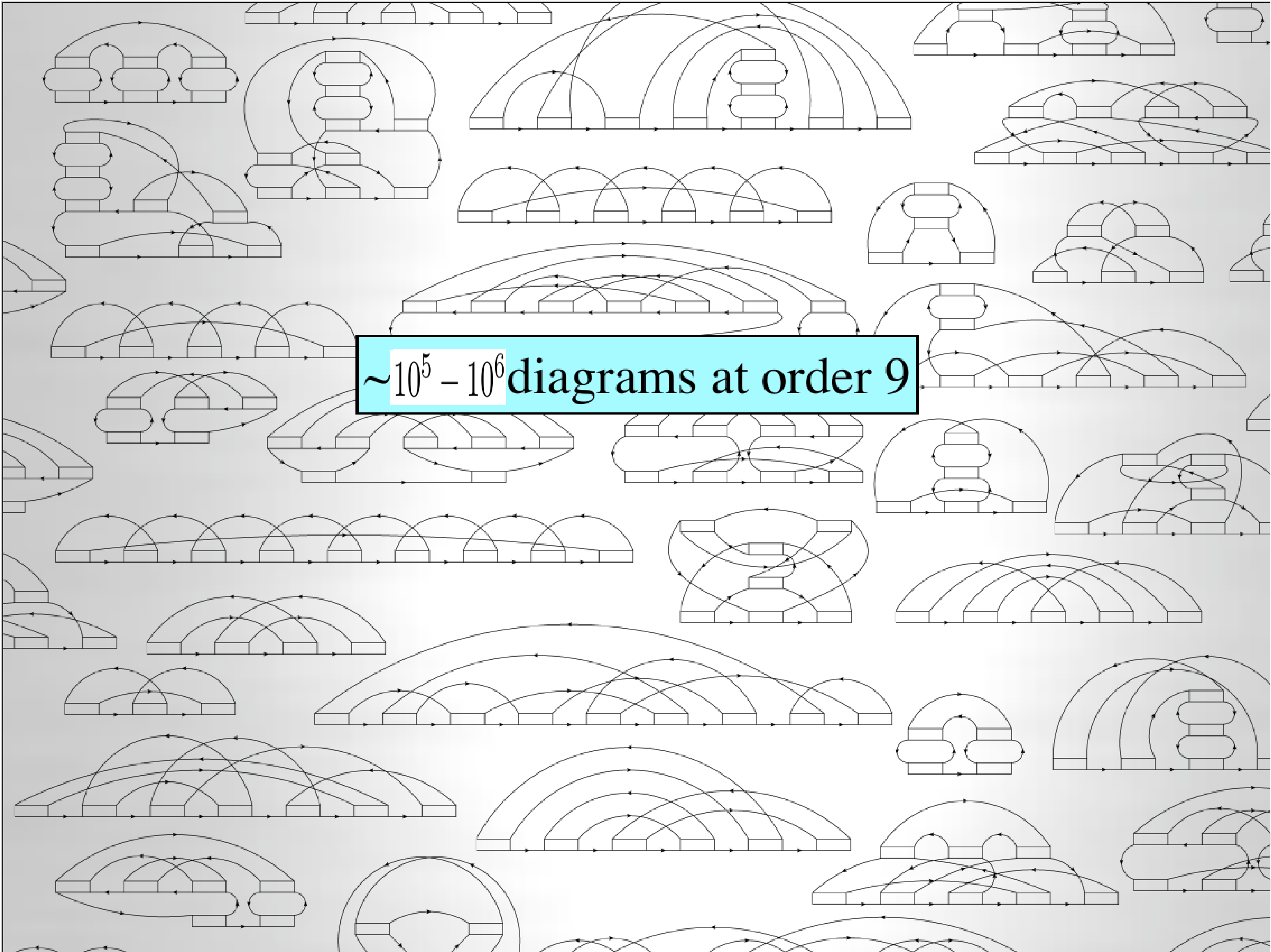
Fully dressed pair propagator:



$$\mathcal{P}(\vec{r},\tau) \equiv -\left\langle \mathrm{T}\ (\Psi_{\downarrow}\Psi_{\uparrow})(\vec{r},\tau)(\Psi_{\uparrow}^{\dagger}\Psi_{\downarrow}^{\dagger})(\vec{0},0) \right\rangle$$



Bold Diagrammatic Monte Carlo



$\sim 10^5 - 10^6$ diagrams at order 9

Resummation of divergent series

Diagrammatic series: $\sum_{n\geq 0} a_n$ (where a_n =sum of all diagrams of order n)

Lindelöf resummation: $\lim_{\epsilon\to 0} \sum_{n\geq 0} a_n e^{-\epsilon n \ln n}$ *assuming non-zero radius of convergence*

underlying logic: Euler's principle

$\exists f(z)$ such that

- The Taylor series of $f(z)$ at $z=0$ is $\sum_n a_n z^n$
- $f(z=1)$ gives the physical answer
- $f(z)$ is analytic in a domain containing $0 < z \leq 1$

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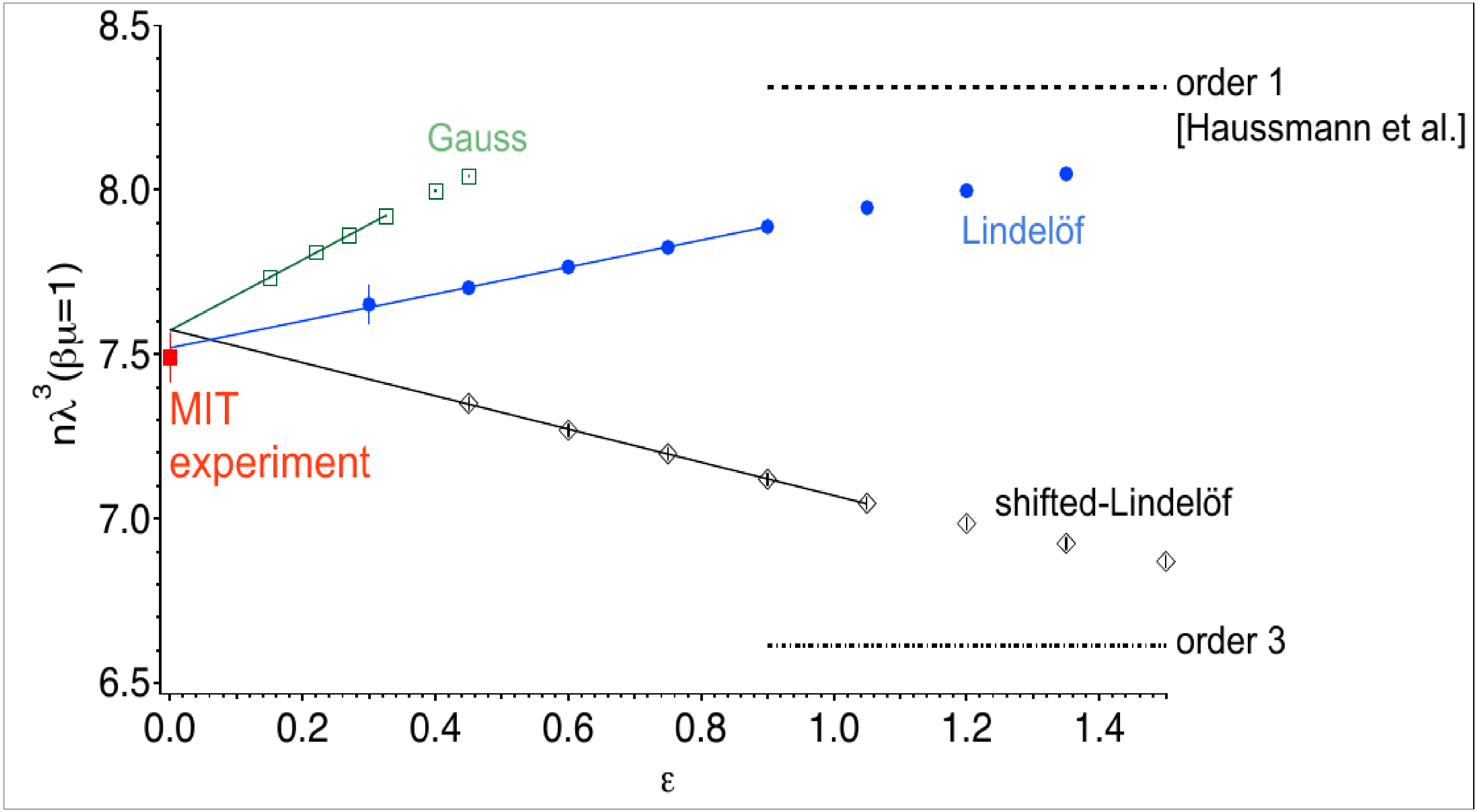


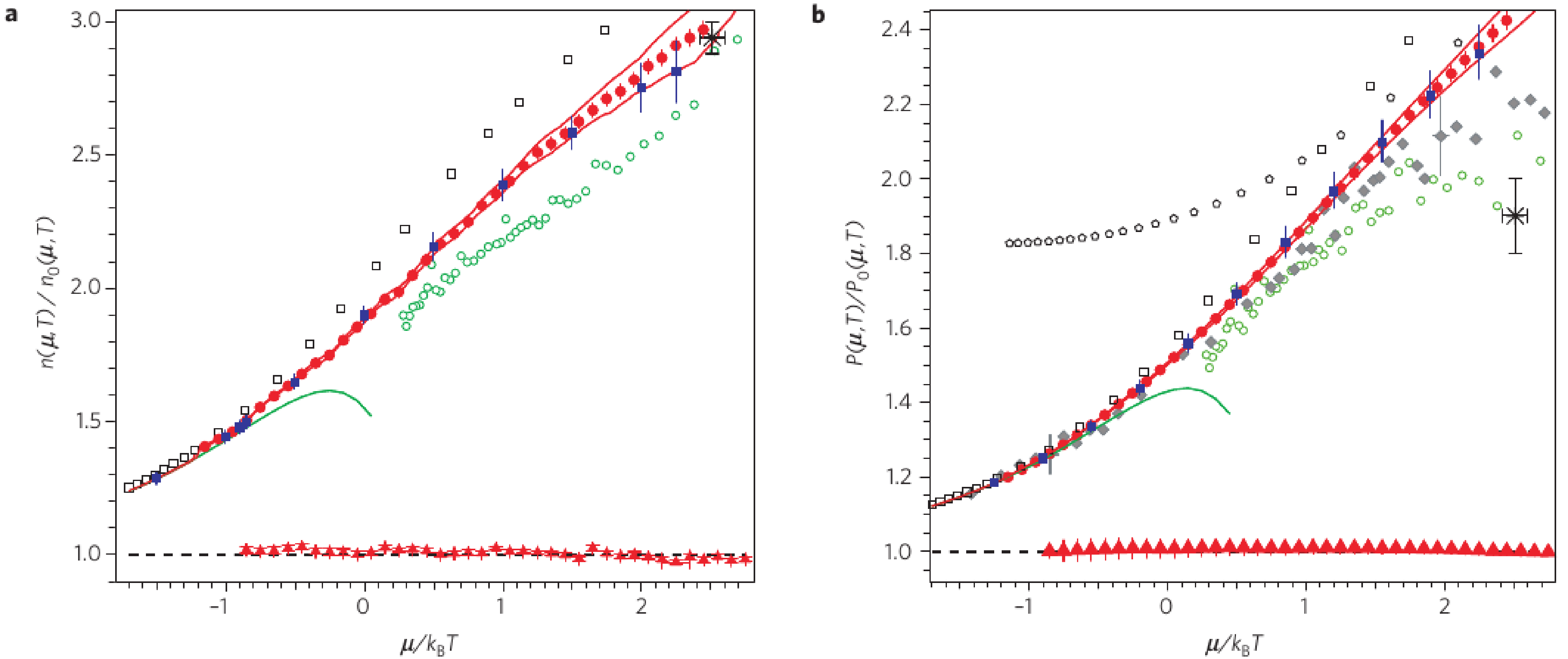


Figure 4 | Equation of state of the unitary Fermi gas in the normal phase. Density n (**a**) and pressure P (**b**) of a unitary Fermi gas, normalized by the density n_0 and the pressure P_0 of a non-interacting Fermi gas, versus the ratio of chemical potential μ to temperature T. Blue filled squares: BDMC (this work), red filled circles: experiment (this work). The BDMC error bars are estimated upper bounds on systematic errors. The error bars are one standard deviation systematic plus statistical errors, with the additional uncertainty from the Feshbach resonance position shown by the upper and lower margins as red solid lines. Black dashed line and red triangles: Theory and experiment (this work) for the ideal Fermi gas, used to assess the experimental systematic error. Green solid line: third order virial expansion.[31] Open squares: first order bold diagram[15,21]. Green open circles: Auxiliary Field QMC (ref. 11). Star: superfluid transition point from Determinental Diagrammatic Monte Carlo[13]. Filled diamonds: experimental pressure EOS (ref. 22). Open pentagons: pressure EOS (ref. 23).

[Van Houcke, Werner, Kozik, Prokof'ev, Svistunov, Ku, Sommer, Cheuk, Schirotzek, Zwierlein, Nature Phys. 2012] but ...

but ...

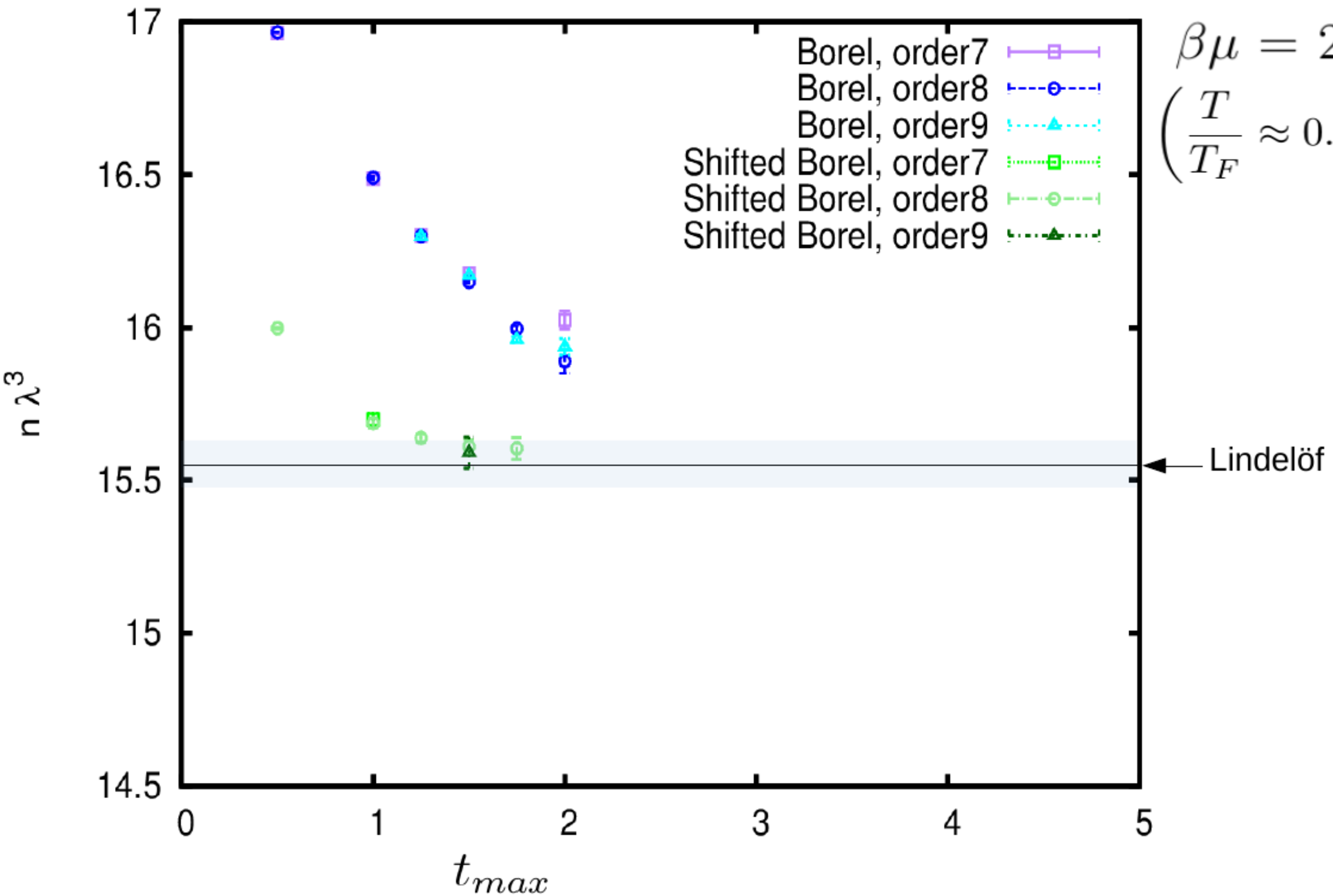
radius of convergence is zero

Borel resummation:

$$\int_0^\infty dt\, e^{-t} \sum_{n=0}^{\infty} \frac{a_n}{n!}\, t^n$$

can work even for zero radius of convergence

in practice: $$\lim_{t_{max}\to\infty} \lim_{n_{max}\to\infty} \int_0^{t_{max}} dt\, e^{-t} \sum_{n=0}^{n_{max}} \frac{a_n}{n!}\, t^n$$



$\beta\mu = 2$

$\left(\frac{T}{T_F} \approx 0.21\right)$

Can we design more powerful resummation methods?

How do we know that radius of convergence is zero?

Large-order asymptotics

(PhD of R. Rossi - work in progress)

Diagrammatic Monte Carlo algorithm for the unitary Fermi gas

[Van Houcke *et al.*, arXiv 2013]

see also: Prokof'ev – Svistunov 2008: Fermi–polaron

Van Houcke – Kozik – Prokof'ev – Svistunov 2008: doped Hubbard model

$Q = \Sigma$ or Π

$$\text{Feynman rules}: \quad Q(Y) = \sum_{\mathcal{T} \in \mathcal{S}_Q} \int dX \; \mathcal{D}(\mathcal{T}, X, Y)$$

external variables: $Y = (\vec{p}; \tau_1, \tau_2)$

internal variables: $X = (\vec{q}_1, \ldots, \vec{q}_N; \tau_3, \ldots, \tau_{2N})$

N = diagram order = [(number of internal lines) + 1]/3

$\mathcal{T}$ = diagram topology

$\mathcal{D} = (-1)^{N+N_{\text{loop}}} G(\ldots) \times \ldots \times G(\ldots) \times \Gamma(\ldots) \times \ldots \times \Gamma(\ldots)$

configuration: $\mathcal{C} \equiv (\mathcal{T}, X, Y)$ weight: $w(\mathcal{C}) \equiv |\mathcal{D}(\mathcal{C})| \, R(\mathcal{C})$

MC updates will generate random configurations $\mathcal{C}_1, \ldots, \mathcal{C}_n, \ldots$ of probability distribution $\propto w(\mathcal{C})$

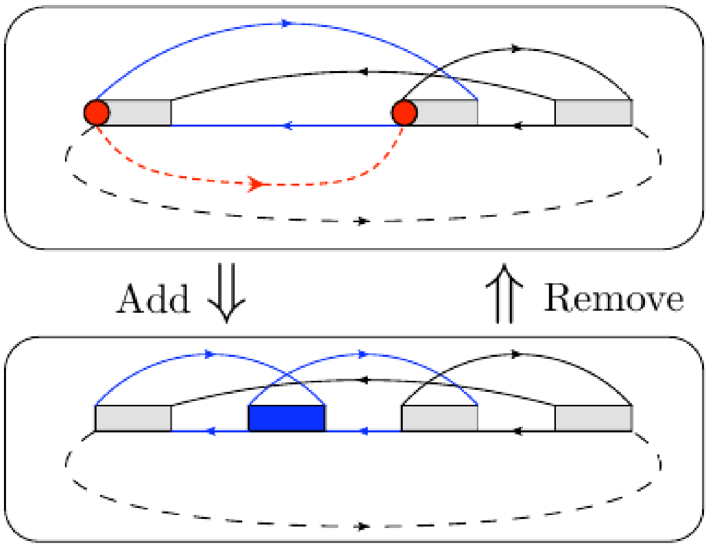
$$\frac{1}{n} \sum_{i=1}^{n} A(\mathcal{C}_i) \underset{n \to \infty}{\longrightarrow} \int d\mathcal{C} \, w(\mathcal{C}) \, A(\mathcal{C}) / \mathcal{Z}, \quad \mathcal{Z} \equiv \int d\mathcal{C} \, w(\mathcal{C})$$

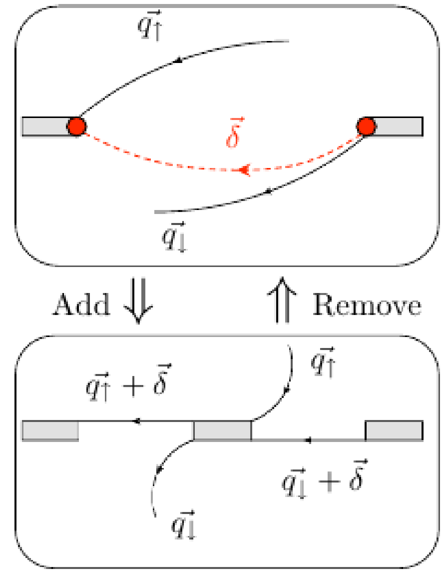
to determine $Q(Y)$, we compute its overlap with various functions $g(Y)$:

$$\begin{aligned} \int dY Q(Y) g(Y) &= \sum_{\mathcal{T} \in \mathcal{S}_Q} \int dX \, dY \; \mathcal{D}(\mathcal{T}, X, Y) \, g(Y) = \int d\mathcal{C} \, \mathcal{D}(\mathcal{C}) \, g(\mathcal{C}) \\ &= \int d\mathcal{C} \, w(\mathcal{C}) \underbrace{\frac{\text{sign}[\mathcal{D}(\mathcal{C})] \, g(\mathcal{C})}{R(\mathcal{C})} 1_{\mathcal{T} \in \mathcal{S}_Q}}_{A(\mathcal{C})} = \mathcal{Z} \lim_{n \to \infty} \frac{1}{n} A(\mathcal{C}_i) \end{aligned}$$

Pairs of complementary updates: (easy to ensure **detailed balance**)

example: add-remove





red line which carries excess momentum = "Worm"

other important updates: