Hard loop effective theory of the (anisotropic) quark-gluon plasma

Anton Rebhan

rebhana@hep.itp.tuwien.ac.at

Technische Universität Wien

work in collaboration with:

Paul Romatschke (INT Seattle), Mike Strickland(FIAS Frankfurt), Max Attems(TU Wien)

Quark-gluon plasma at RHIC

Surprises at RHIC (judging from hydro simulations):

- Very early thermalization/isotropization
- Very low shear viscosity

New paradigm: **sQGP**

(very successful toy model:

maximally supersymmetric large-N_c YM theory at infinite 't Hooft coupling from AdS/CFT)

But:

wQGP (thermal pQCD)

not yet fully understood, especially far from equilibrium!

furthermore: RHIC close to phase transition (reason for sQGP?)

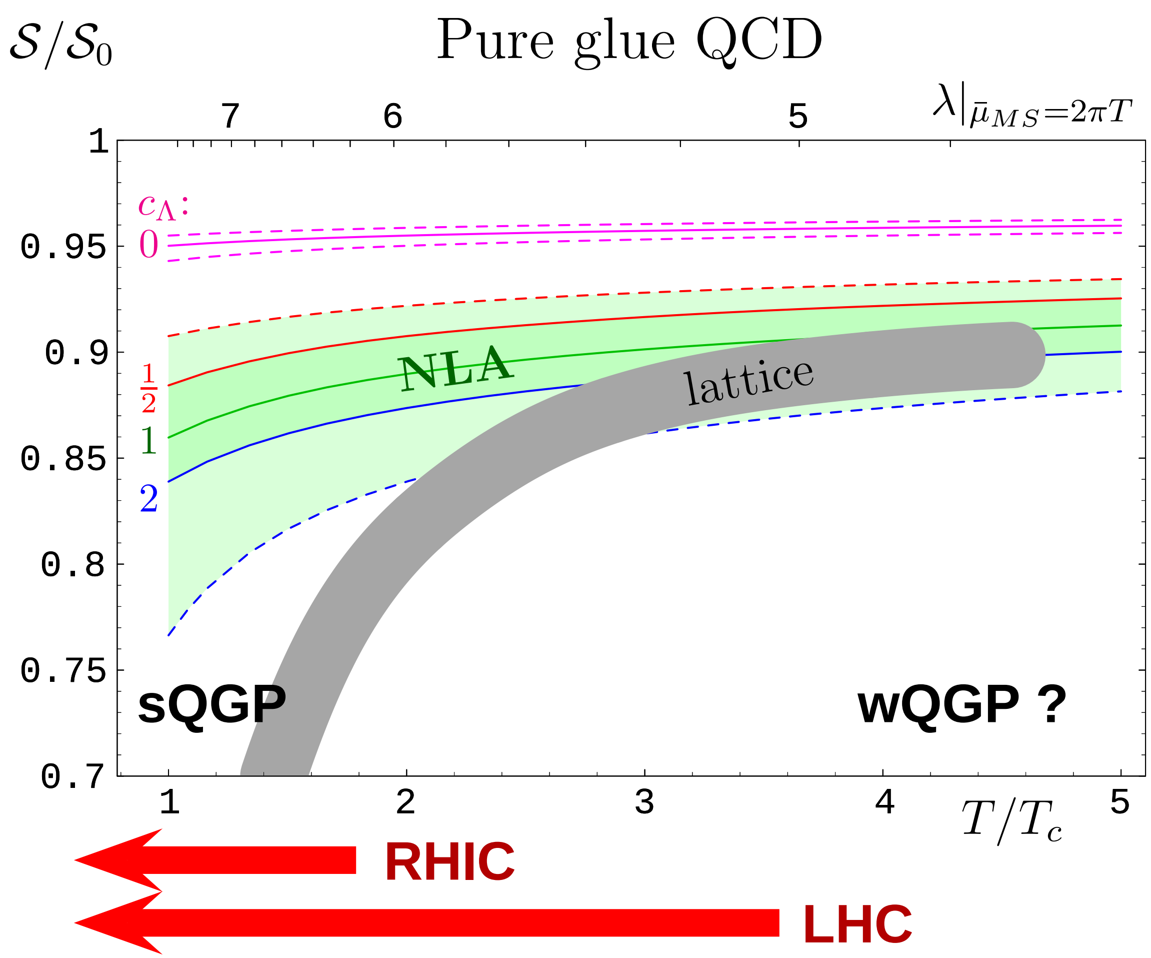
LHC will reach $\gtrsim 3T_c$ — wQGP only there?

wQGP or sQGP?

Entropy in pure-glue QCD: lattice vs. **H**ard-**T**hermal-**L**oop quasiparticle entropy with **N**ext-to-**L**eading **A**pproximations of asymptotic thermal masses

suggestive of dominance of weakly interacting (hard) quasiparticles for $T \gtrsim 3T_c$

[Blaizot, Iancu & AR, PRD63('01)065003]



$\alpha_s(\bar{\mu} = \pi T \ldots 4\pi T)$

from *standard* 2-loop pQCD !

(no fitting here)

NLA: agrees with

perturbative result to order g^3

(in contrast to DQPM of Peshier)

cp. talks by

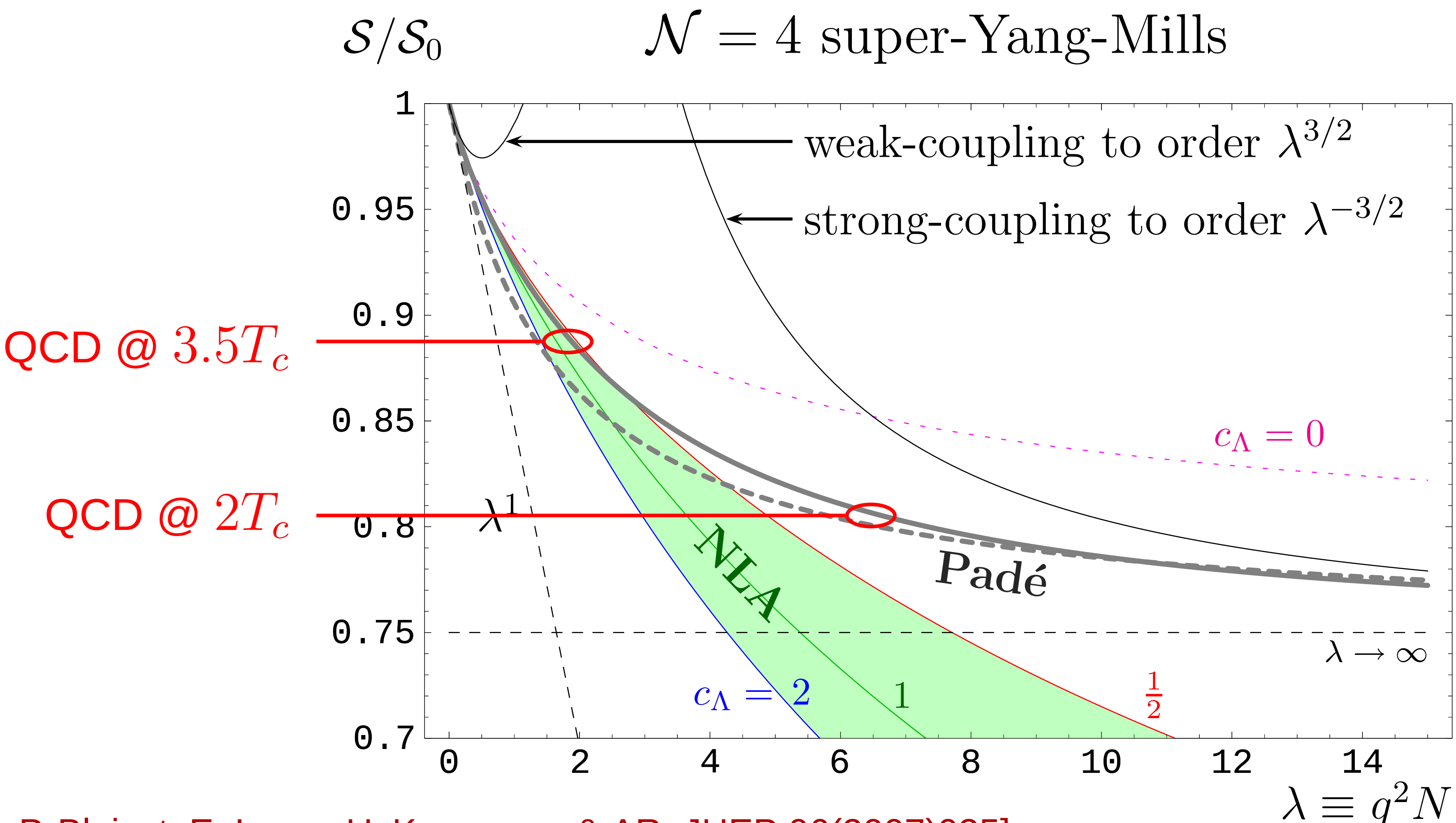
Kämpfer, Schulze, Cassing

wQGP or sQGP?

AdS/CFT: No lattice EOS results for $\mathcal{N} = 4$ SYM,

but essentially unique Padé approximant $R_{[4,4]} = \frac{1+\alpha\lambda^{1/2}+\beta\lambda+\gamma\lambda^{3/2}+\delta\lambda^2}{1+\bar{\alpha}\lambda^{1/2}+\bar{\beta}\lambda+\bar{\gamma}\lambda^{3/2}+\bar{\delta}\lambda^2}$

for known weak and strong coupling results



[J.-P. Blaizot, E. Iancu, U. Kraemmer & AR, JHEP 06(2007)035]

wQGP or sQGP?

- Truth probably in between wQGP and sQGP
- Need to understand wQGP in systematic limit $g \ll 1$

  (*really* weakly coupled QGP)

  — theoretical challenge (sQGP in many ways simpler)

  — use for bald extrapolation to $g \sim 1$

Scales of wQGP

- T: energy of hard particles
- gT: thermal masses, Debye screening mass,
  Landau damping, **plasma instabilities** [Mrówczyński 1988, 1993, . . .]
- g^2T: magnetic confinement, color relaxation, rate for small angle scattering
- g^4T: rate for large angle scattering; inverse shear viscosity $\eta^{-1}T^4$

Effective theory at scale gT: Hard-~~(Thermal-)~~Loop Effective Action

[Frenkel, Taylor & Wong; Braaten & Pisarski 1991]

equivalent to: gauge-covariant Boltzmann-Vlasov

[Blaizot & Iancu 1993, Kelly, Liu, Lucchesi & Manuel 1994]

in particular required (to leading order!) for:

- Bottom-up thermalization [Baier, Mueller, Schiff & Son 2000]

  $t_{eq} \propto g^{-13/5} \longrightarrow g^{-?}$ [Arnold, Lenaghan, Moore, JHEP 08 ('03) 002]
- Shear viscosity [Arnold, Moore & Yaffe]

  $(\eta/s)^{-1} = g^4 \ln(1/g) f(\ln(1/g))$ add $(\eta/s)^{-1}_{\text{anomalous}}$!

  [Asakawa, Bass & Müller, PRL 96 ('06) 252301]

Hard (Thermal) Loops = gauge covariant Boltzmann-Vlasov

With color-neutral background distribution $v \cdot \partial\, f_0(\mathbf{p}, \mathbf{x}, t) = 0$, $v^\mu = p^\mu / p^0$

$$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = g v_\mu F_a^{\mu\nu} \partial_\nu^{(p)} f_0(\mathbf{p}, \mathbf{x}, t) = -g(\mathbf{E}_a + \mathbf{v} \times \mathbf{B}_a) \cdot \nabla_{\mathbf{p}} f_0,$$

$$D_\mu F_a^{\mu\nu} = j_a^\nu = g \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{2p^0} \delta f_a(\mathbf{p}, \mathbf{x}, t).$$

- isotropic: $f_0(\mathbf{p}) = f_0(|\mathbf{p}|)$, $\nabla_{\mathbf{p}} f_0 \propto \mathbf{v}$

  $$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = -g \mathbf{E}_a \cdot \nabla_{\mathbf{p}} f_0$$

- anisotropic $f_0(\mathbf{p})$, $\nabla_{\mathbf{p}} f_0 \not\propto \mathbf{v}$

  $$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = -g(\mathbf{E}_a + \mathbf{v} \times \mathbf{B}_a) \cdot \nabla_{\mathbf{p}} f_0$$

- anisotropic expansion: $f_0(\mathbf{p}; \mathbf{x}, t)$

Hard loop gauge boson self energy

Linearize in A^μ and Fourier transform

$$j^\mu(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \underbrace{\partial^\beta_{(p)} f(\mathbf{p})}_{0 \text{ for } \beta=0} \left(g_{\gamma\beta} - \frac{v_\gamma k_\beta}{k\cdot v + i\epsilon} \right) A^\gamma(k) = \Pi^{\mu\nu}(k) A_\nu(k)$$

$i\epsilon \leftrightarrow$ retarded boundary condition

Isotropic case: $\partial^\beta_{(p)} f(|\mathbf{p}|) = f'(|\mathbf{p}|)\left(0, p^b/|\mathbf{p}|\right)$

$\longrightarrow$ 2 functions $\Pi_T(k_0/|\mathbf{k}|),\ \Pi_L(k_0/|\mathbf{k}|) \propto m^2 = g^2 p^2_{\text{hard}}$

Generic case:

$10 - 4 = 6$ functions, each depending on 3 variables k_i/k_0

Isotropic gauge boson self energy

Gauge invariance of HTL/HDL effective action $\longrightarrow$ transverse gauge boson self energy

2 tensors transverse w.r.t. 4-momentum in a thermal medium (rest frame velocity $u^\mu = \delta_0^\mu$)

$$A_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - B_{\mu\nu},$$

$$B_{\mu\nu} = \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2} \text{ with } \tilde{n}_\mu = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})u^\nu$$

$$\Pi_A \equiv \Pi_T = \tfrac{1}{2} A_{\mu\nu}\Pi^{\mu\nu} = \tfrac{1}{2}\left(\Pi^\mu{}_\mu - \Pi_B\right)$$

$$\Pi_B \equiv \Pi_L = -\frac{k^2}{\mathbf{k}^2}\Pi_{00}$$

$$\Pi^\mu{}_\mu = m_D^2, \quad \Pi_{00} = m_D^2 \left(1 - \frac{k^0}{2|\mathbf{k}|} \ln \frac{k^0 + |\mathbf{k}|}{k^0 - |\mathbf{k}|}\right)$$

Gauge boson propagator (Landau gauge)

$$-G_{\mu\nu} = \Delta_T A_{\mu\nu} + \Delta_L B_{\mu\nu}$$

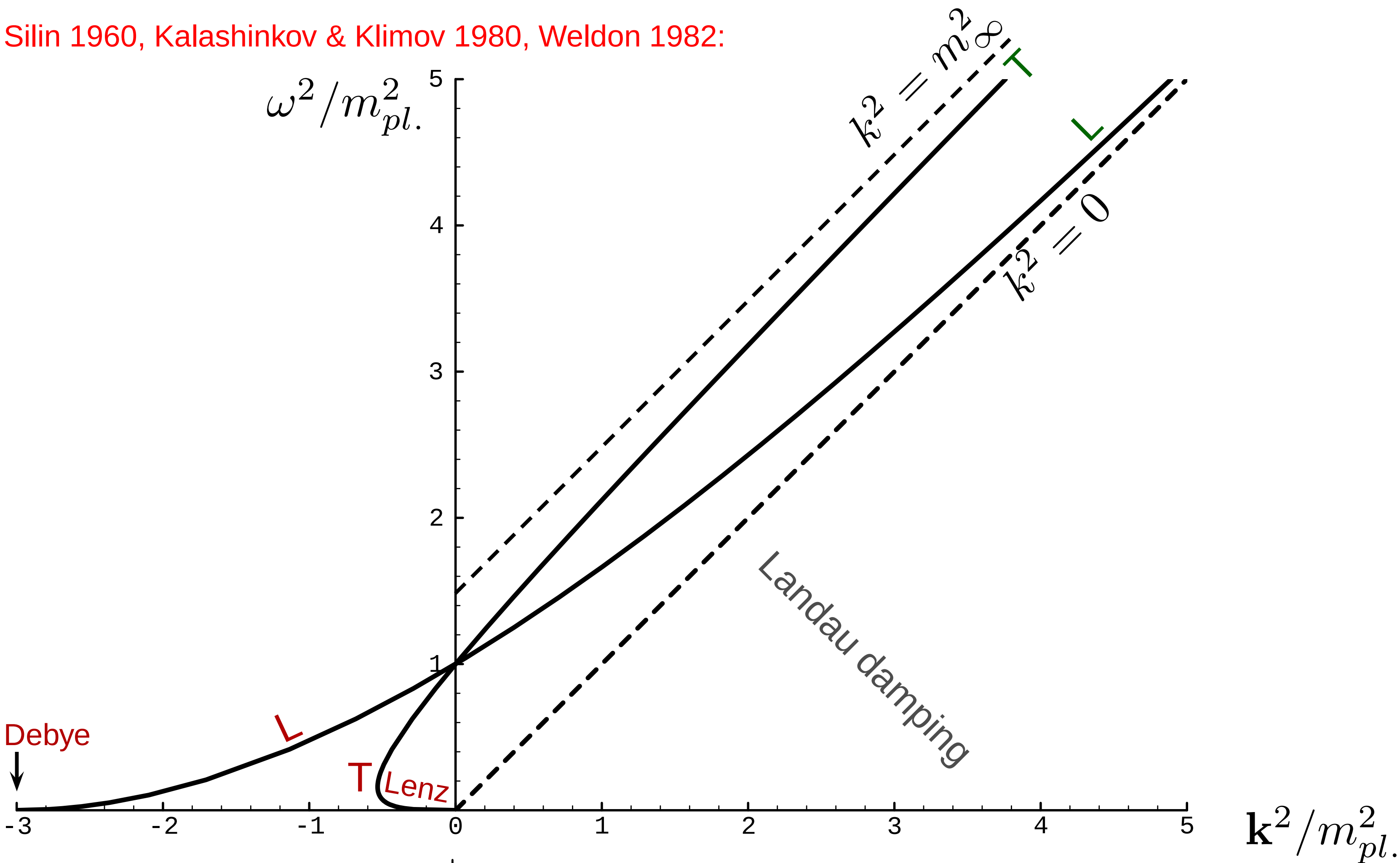
$$\Delta_T = [k^2 - \Pi_T]^{-1}, \quad \Delta_L = [k^2 - \Pi_L]^{-1}$$

$\longrightarrow$ 2 branches with different dispersion laws

Dispersion laws of HTL/HDL gauge bosons

Isotropic case (not necessarily thermal)

Silin 1960, Kalashinkov & Klimov 1980, Weldon 1982:



(dynamical) screening | propagating modes

Hard anisotropic loop gauge boson self energy

$$\Pi^{\mu\nu}(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \partial_\beta^{(p)} f(\mathbf{p}) \left(g^{\nu\beta} - \frac{v^\nu k^\beta}{k\cdot v + i\epsilon} \right), \quad v^\mu \equiv \frac{p^\mu}{p^0}, \quad p^0 = |\mathbf{p}|$$

$\Pi^{\mu\nu}$ symmetric, $\Pi^{0\nu}$ fixed by transversality $k_\mu \Pi^{\mu\nu} = 0 \longrightarrow$ 6 structure functions in general

Assume just one direction of anisotropy (axisymmetry): $\mathbf{n} = (0,0,1)$

$\longrightarrow$ 4 symmetric tensors for Π^{ij}, 4 independent structure functions

$$A^{ij} = \delta^{ij} - k^i k^j / k^2, \quad B^{ij} = k^i k^j / k^2,$$

$$C^{ij} = \tilde{n}^i \tilde{n}^j / \tilde{n}^2, \qquad D^{ij} = k^i \tilde{n}^j + k^j \tilde{n}^i, \qquad \tilde{n}^i = A^{ij} n^j$$

$$\Pi^{ij} = \alpha A^{ij} + \beta B^{ij} + \gamma C^{ij} + \delta D^{ij}$$

Propagator (temporal axial gauge $A^0 = 0$ for simplicity)

$$\mathbf{\Delta}(K) = \Delta_T \mathbf{A} + \sout{(k^2 - \omega^2 + \alpha + \gamma)} \Delta_{\mathcal{L}} \mathbf{B} + [(\beta - \omega^2)\Delta_{\mathcal{L}} - \Delta_T] \mathbf{C} - \delta \Delta_{\mathcal{L}} \mathbf{D}$$

$$\Delta_T(k) = [k^2 - \omega^2 + \alpha]^{-1}$$

$$\Delta_{\mathcal{L}}(k) = [\sout{(k^2 - \omega^2 + \alpha + \gamma)}(\beta - \omega^2) \sout{- k^2 \tilde{n}^2 \delta^2}]^{-1}$$

generally: 2 branches from $\Delta_{\mathcal{L}}$; only 1 from $\Delta_{\mathcal{L}}$ when $\mathbf{k} \parallel \mathbf{n} \Rightarrow \tilde{n} = 0$

Hard anisotropic loop gauge boson self energy

$$\Pi^{\mu\nu}(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \partial_\beta^{(p)} f(\mathbf{p}) \left(g^{\nu\beta} - \frac{v^\nu k^\beta}{k \cdot v + i\epsilon} \right), \quad v^\mu \equiv \frac{p^\mu}{p^0}, \quad p^0 = |\mathbf{p}|$$

Special important case: $f(\mathbf{p}) = f_{\rm iso}\left(\mathbf{p}^2 + \xi(\mathbf{p}\cdot\mathbf{n})^2\right)$

$\xi = 0$: isotropic; $-1 < \xi < 0$: prolate (cigar-shaped); $0 < \xi < \infty$: oblate (squashed)

Can be evaluated in closed form: [Romatschke & Strickland 2003]

Change variables $\mathbf{p}^2 + \xi(\mathbf{p}\cdot\mathbf{n})^2 = \bar{\mathbf{p}}^2$

$$\Pi^{ij}(k) = m^2 \int \frac{d\Omega}{4\pi} v^i \frac{v^l + \xi(\mathbf{v}.\mathbf{n})n^l}{(1+\xi(\mathbf{v}.\mathbf{n})^2)^2} \left(\delta^{jl} + \frac{v^j k^l}{k \cdot v + i\epsilon} \right)$$

$$m^2 \equiv -\frac{g^2}{2\pi^2} \int_0^\infty d\bar{p}\, \bar{p}^2 \frac{df_{\rm iso}(\bar{p}^2)}{d\bar{p}}$$

Magnetostatic polarization function for $\mathrm{k}||\mathrm{n}$

Static limit: $\alpha(k) \equiv \Pi_T \to \frac{1}{2}\Pi^{ii}(\omega = 0, \mathbf{k}.\mathbf{n}/k)$ because then $k^i \Pi^{ij} \to 0$

Easy exercise: calculate $\Pi^{ii}(\omega = 0)$ for $\mathbf{k} \parallel \mathbf{n}$!

Solution:

$$\alpha/m^2 = \frac{1}{4}[(1-\xi)\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}} - 1] \qquad \text{for } \xi > 0$$

$$= \tfrac{1}{4}[(1-\xi)\tfrac{\mathrm{atanh}\sqrt{-\xi}}{\sqrt{-\xi}} - 1] \qquad \text{for } \xi < 0$$

$\alpha = \Pi_T$ is magnetic screening mass

- $\xi = 0$ (isotropic):

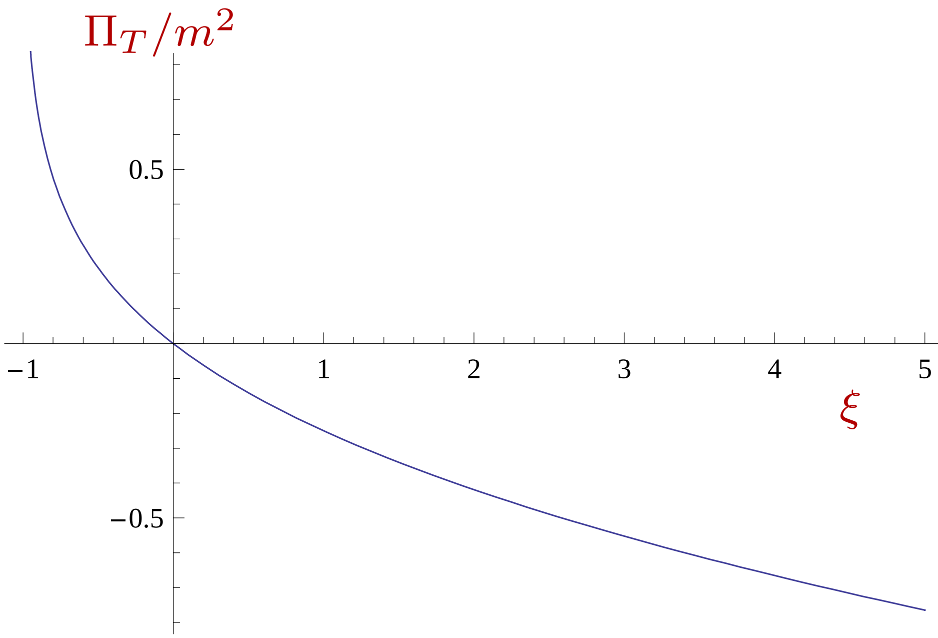
  no magnetic screening mass

- $\xi < 0$ (prolate):

  magnetostatic screening!

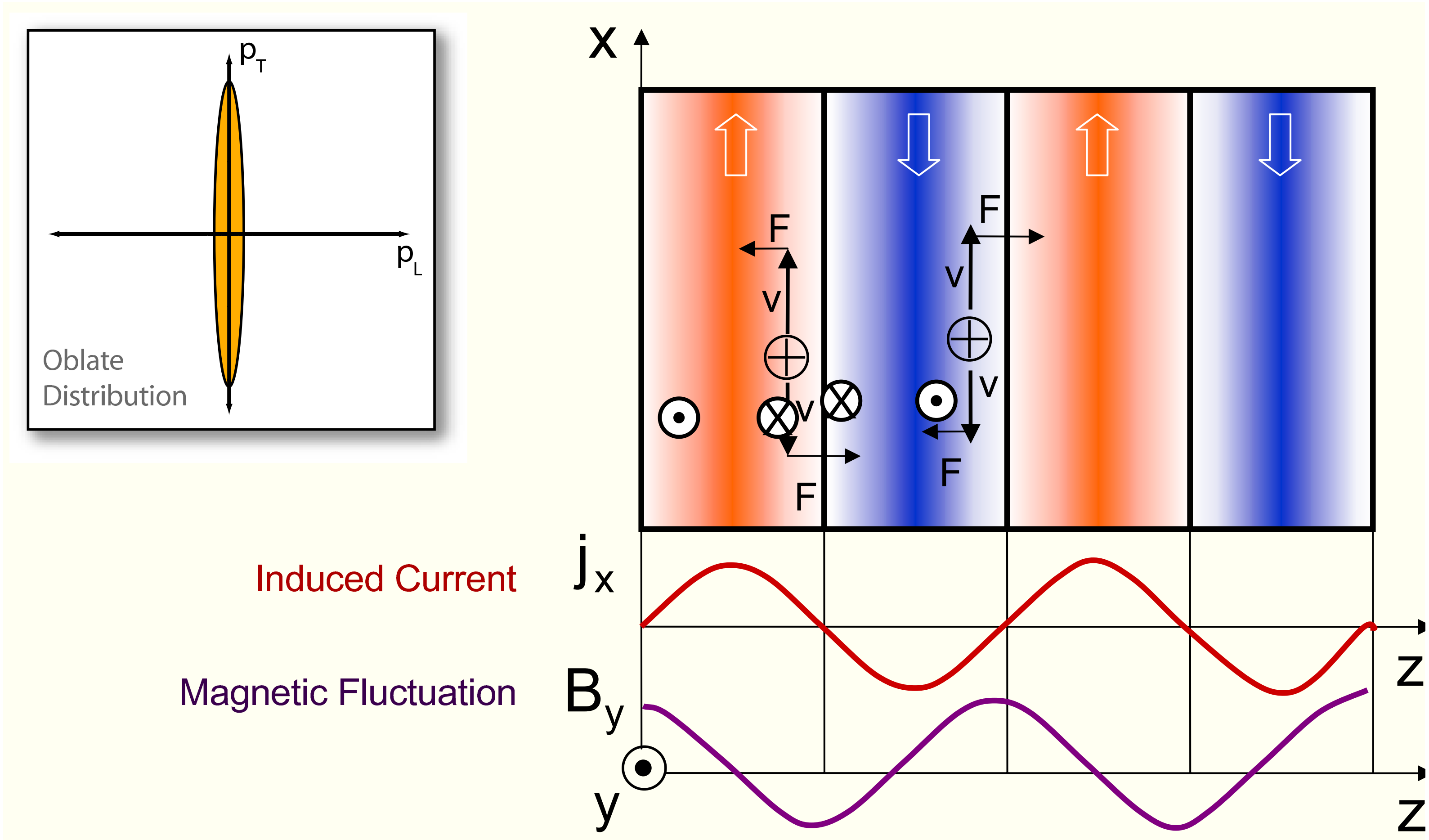
- $\xi > 0$ (oblate):

  "tachyonic" magnetic mass — **instability!**



Filamentation (Weibel) instabilities

E.g.: ensemble of counterstreaming currents unstable against filamentation



Abelian plasma: exponential growth of currents and magnetic fields
until magnetic fields strong enough to bend trajectories $\longrightarrow$ fast isotropization

Full anisotropic polarization tensor for $\mathbf{k}||\mathbf{n}$

For full dispersion laws (for $\mathbf{k}||\mathbf{n}$ which contains the most unstable modes) need complete frequency dependences ($\eta \equiv \omega/k$) [Romatschke & Strickland 2004]

$$\begin{aligned}
\alpha &= \frac{m^2}{4\sqrt{\xi}(1+\xi\eta^2)^2}\Big[\left(1+\eta^2+\xi(-1+(6+\xi)\eta^2-(1-\xi)\eta^4)\right)\arctan\sqrt{\xi} \\
&\qquad +\sqrt{\xi}\,(\eta^2-1)\left(1+\xi\eta^2-(1+\xi)\eta\ln\frac{\eta+1+i\epsilon}{\eta-1+i\epsilon}\right)\Big], \\
\beta &= -\frac{\eta^2 m^2}{2\sqrt{\xi}(1+\xi\eta^2)^2}\Big[(1+\xi)(1-\xi\eta^2)\arctan\sqrt{\xi} \\
&\qquad +\sqrt{\xi}\left((1+\xi\eta^2)-(1+\xi)\eta\ln\frac{\eta+1+i\epsilon}{\eta-1+i\epsilon}\right)\Big]
\end{aligned}$$

more complicated: $\mathbf{k} \not\parallel \mathbf{n}$

- second branch of poles in $\Delta_{\mathcal{L}}$ which can contain *electric* (Buneman) instability

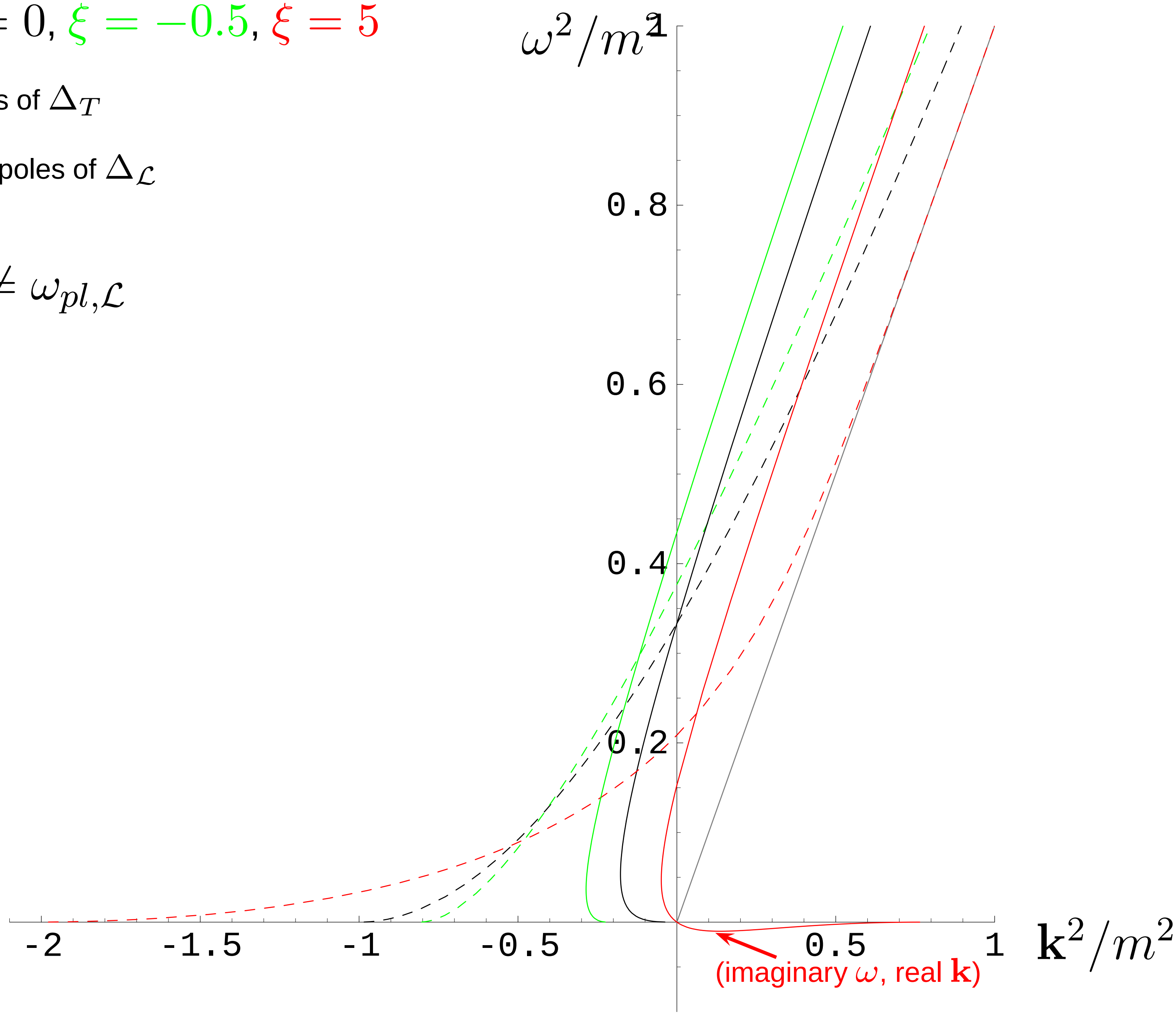
Dispersion laws for $\mathbf{k}||\mathbf{n}$

Comparing $\xi = 0$, $\xi = -0.5$, $\xi = 5$

full lines: poles of Δ_T

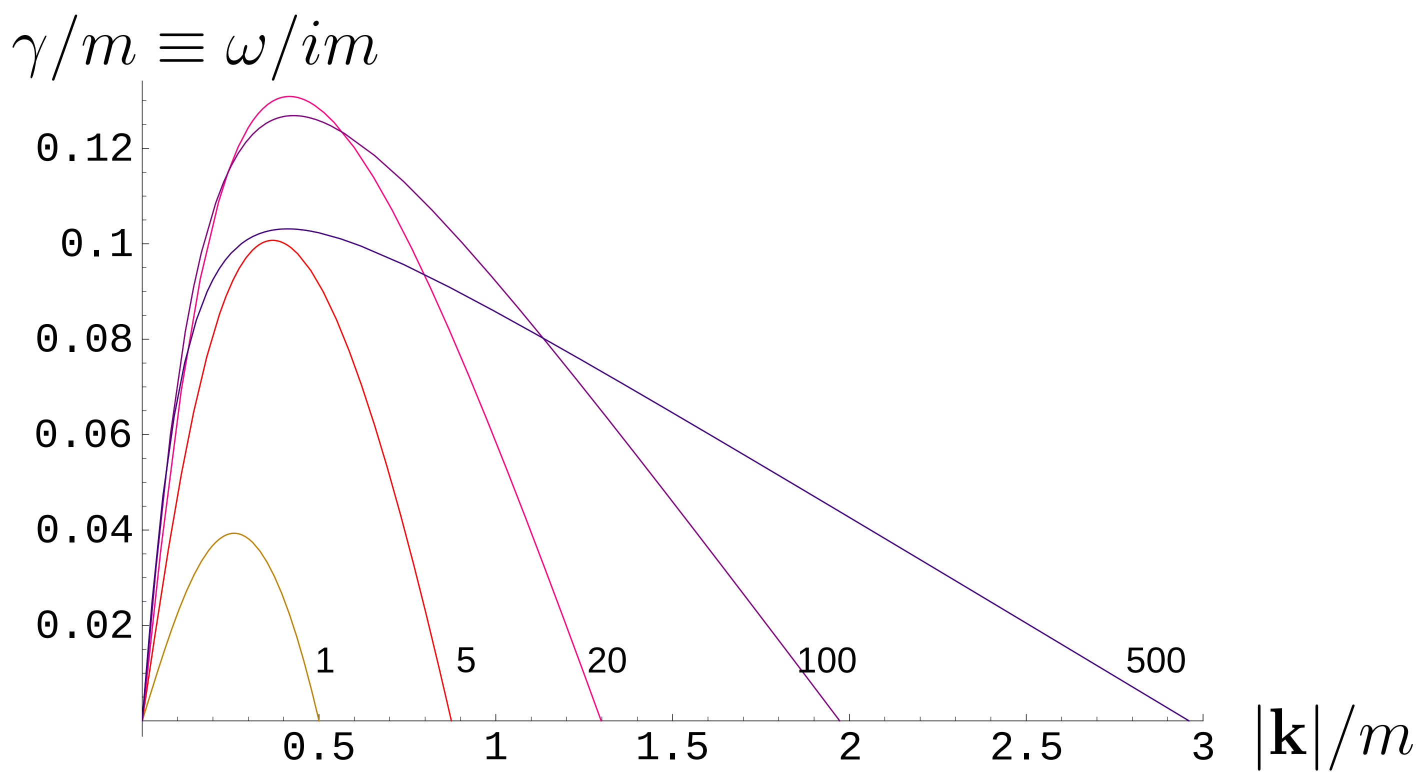
dashed lines: poles of $\Delta_{\mathcal{L}}$

$\xi \neq 0$: $\omega_{pl,T} \neq \omega_{pl,\mathcal{L}}$



Growth rates for Weibel instabilities (k||n)

Anisotropy parameter $\xi = 1,\ 5,\ 20,\ 100,\ 500$ (increasing oblateness)



large ξ behavior: $k_{\max}/m \sim \xi^{1/4}$, $\quad k/m|_{\gamma=\gamma_{\max}} \sim 1$

compared to asymptotic gluon mass m_∞: $k_{\max}/m_\infty \sim \sqrt{\xi}$

$\gamma_{\max}/m_\infty \to 1/\sqrt{2}$

Hard Anisotropic Loops from Boltzmann-Vlasov

With color-neutral background distribution $v \cdot \partial\, f_0(\mathbf{p}, \mathbf{x}, t) = 0$, $v^\mu = p^\mu / p^0$

gauge covariant Boltzmann-Vlasov:

$$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = g v_\mu F_a^{\mu\nu} \partial_\nu^{(p)} f_0(\mathbf{p}, \mathbf{x}, t) = -g(\mathbf{E}_a + \mathbf{v} \times \mathbf{B}_a) \cdot \nabla_{\mathbf{p}} f_0,$$

$$D_\mu F_a^{\mu\nu} = j_a^\nu = g \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{2p^0} \delta f_a(\mathbf{p}, \mathbf{x}, t).$$

Linear response: **Hard loop gauge boson self energy**

$$j^\mu(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \partial_{(p)}^\beta f(\mathbf{p}) \left(g_{\gamma\beta} - \frac{v_\gamma k_\beta}{k \cdot v + i\epsilon} \right) A^\gamma(k) = \Pi^{\mu\nu}(k) A_\nu(k)$$

Beyond linear response (unavoidable for QCD with exponentially growing instabilities):

Full hard-loop effective theory (infinitely many vertex functions)!

Discretized Hard Loop Effective Theory

Useful:

auxiliary field formulation: [Nair; Blaizot & Iancu 1994; Mrówczyński, AR & Strickland 2004]

$$\delta f^a(x;p) = -gW_\mu^a(t,\mathbf{x};\mathbf{v})\partial^\mu_{(p)} f_0(\mathbf{p})$$

$$\boxed{[v\cdot D(A)]W_\mu(x;\mathbf{v}) = F_{\mu\gamma}(A)v^\gamma}$$

$$v^\mu \equiv p^\mu/|\mathbf{p}| = (1,\mathbf{v})$$

$$\boxed{D_\rho(A)F^{\rho\mu} = j^\mu(x) = -g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{2|\mathbf{p}|}\,p^\mu\,\frac{\partial f(\mathbf{p})}{\partial p^\nu}W^\nu(x;\mathbf{v})}$$

Hard Loop effective theory: (hard) scale $|\mathbf{p}|$ can be integrated out

Auxiliary field version: local in terms of field living also on velocity space S_2

Nonlinear response $\longrightarrow$ real-time lattice simulation

$\longrightarrow$ discretize also velocity space

$$D_\rho(A)F^{\rho\mu} = j^\mu(x) = \frac{1}{\mathcal{N}}\sum_{\mathbf{v}} v^\mu \mathcal{W}_{\mathbf{v}}(x)$$

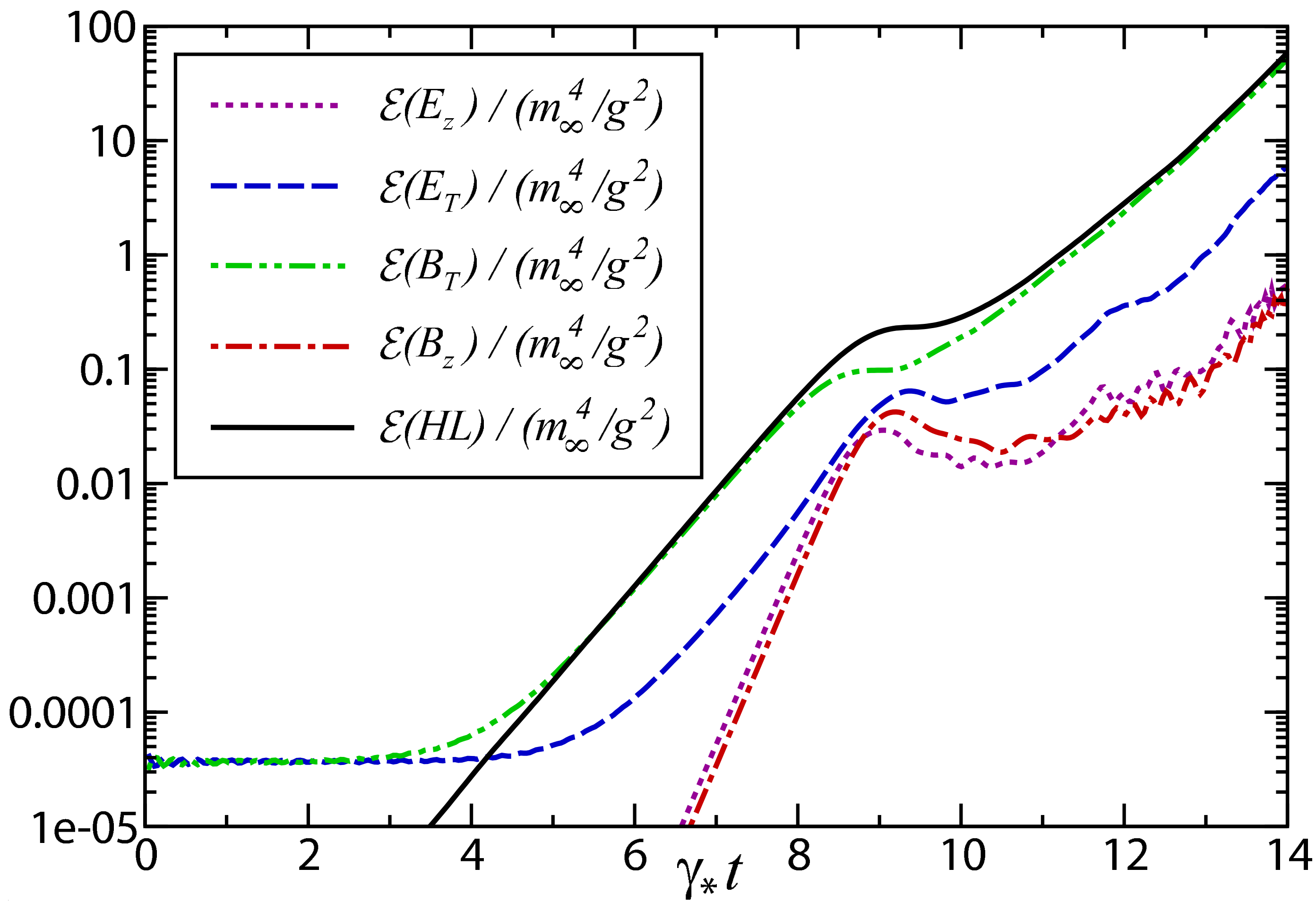
"disco balls"

Transversely constant modes: 1D+3V

Most unstable modes in linear response: $\mathbf{k} \parallel \mathbf{n}$

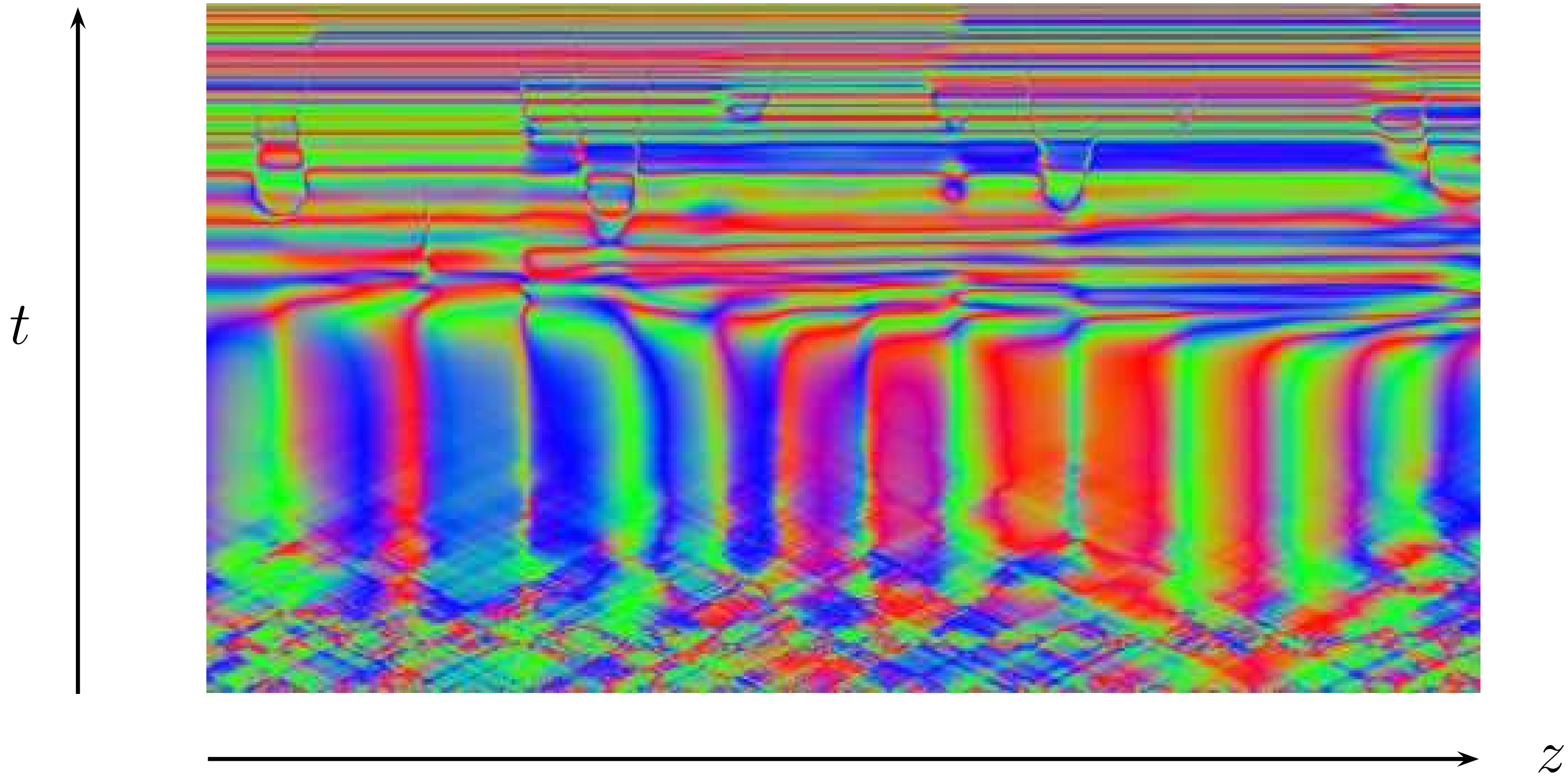
$\Longrightarrow$ no dependence on transverse coordinates;
dimensional reduction to 1 spatial dimension

[AR, Romatschke & Strickland, PRL 94 ('05) 102303]



Transversely constant modes: 1D+3V

Evolution of color degrees of freedom:
(parallel-transported color from fixed spatial point)



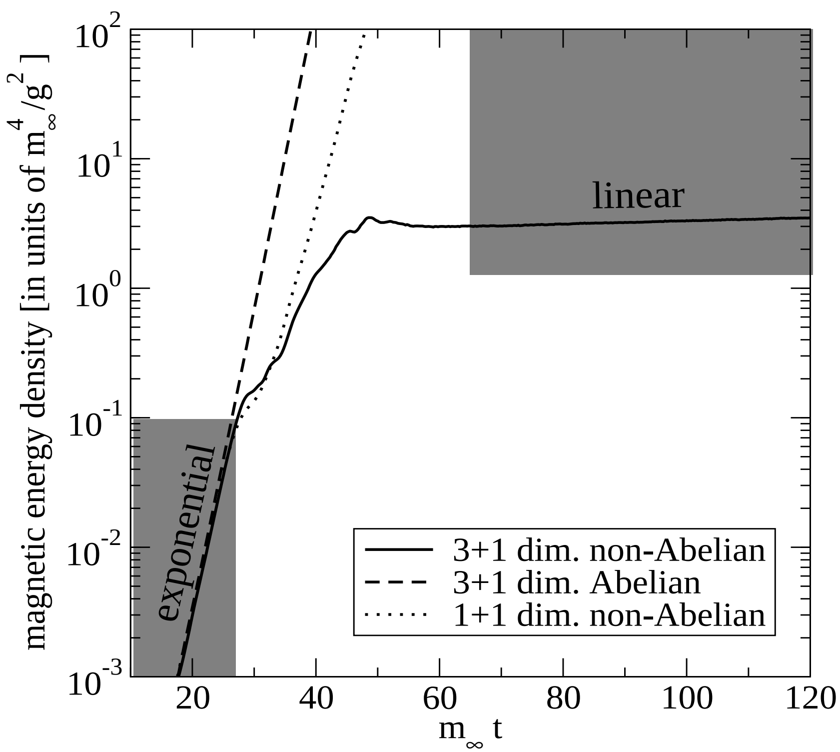
Late-time (non-linear) regime: Abelianization over extended spatial domains
– responsible for continued Abelian-like growth in non-linear regime

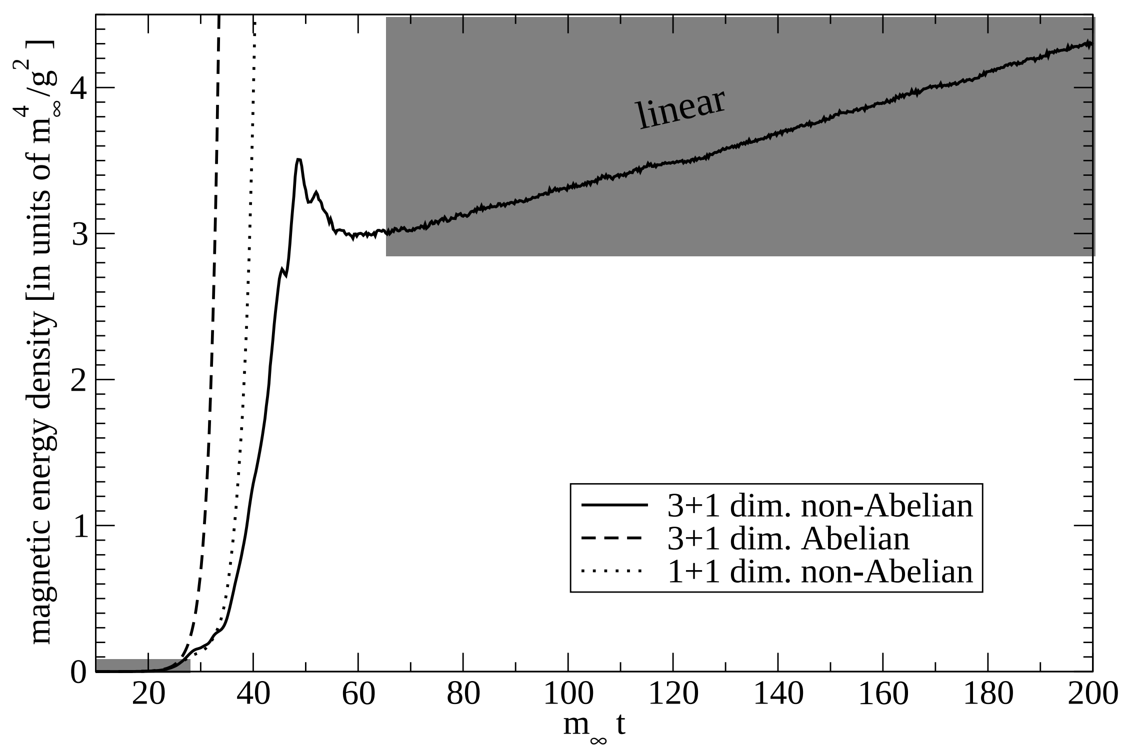
3D+3V

However: local Abelianization can be destroyed by interactions with not perfectly transversely constant modes

$\longrightarrow$ attenuation of exponential growth to only linear one:

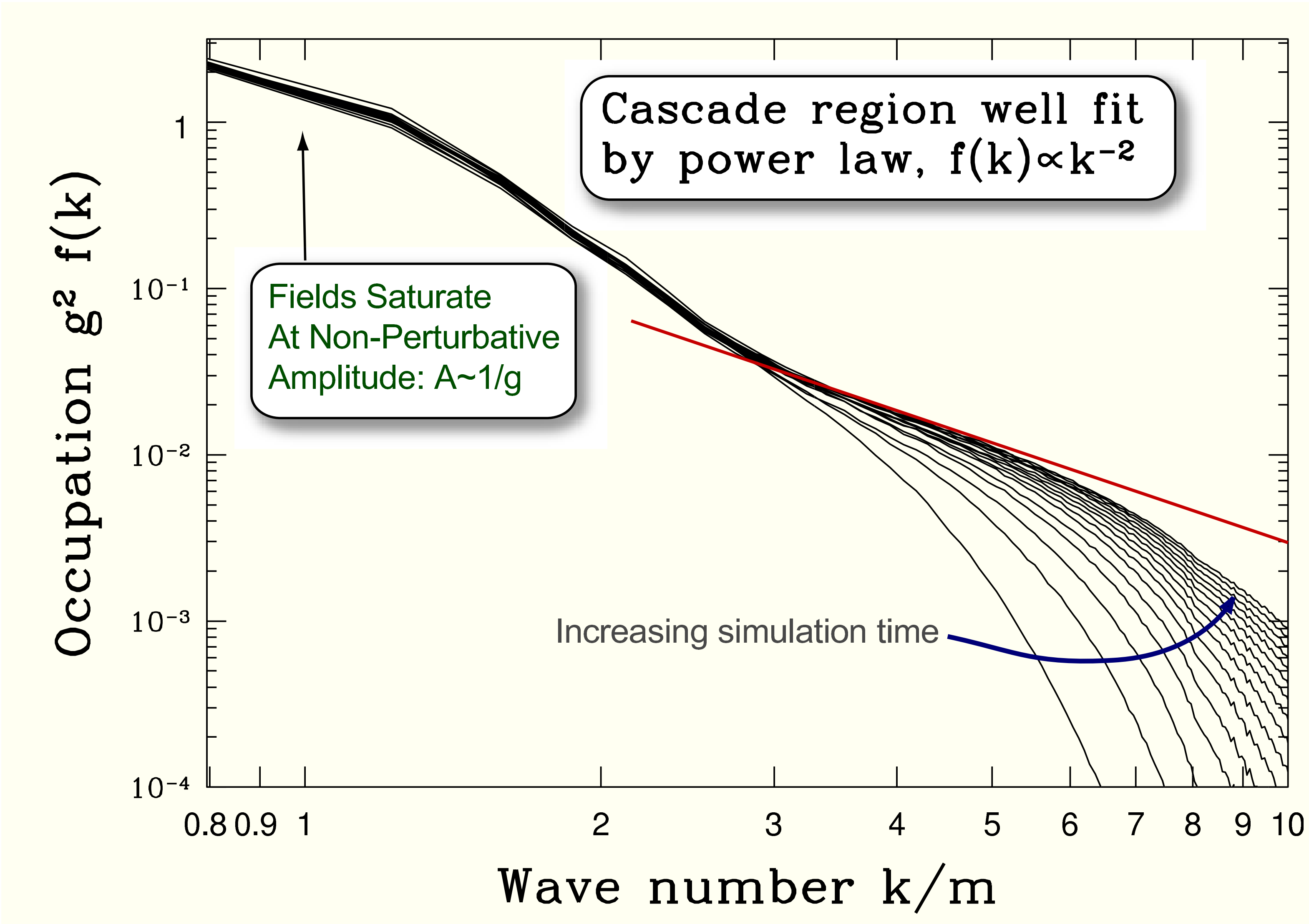
[Arnold, Moore & Yaffe, PRD72 ('05) 054003]





(btw different discretization method: finite number of spherical harmonics W_{lm})

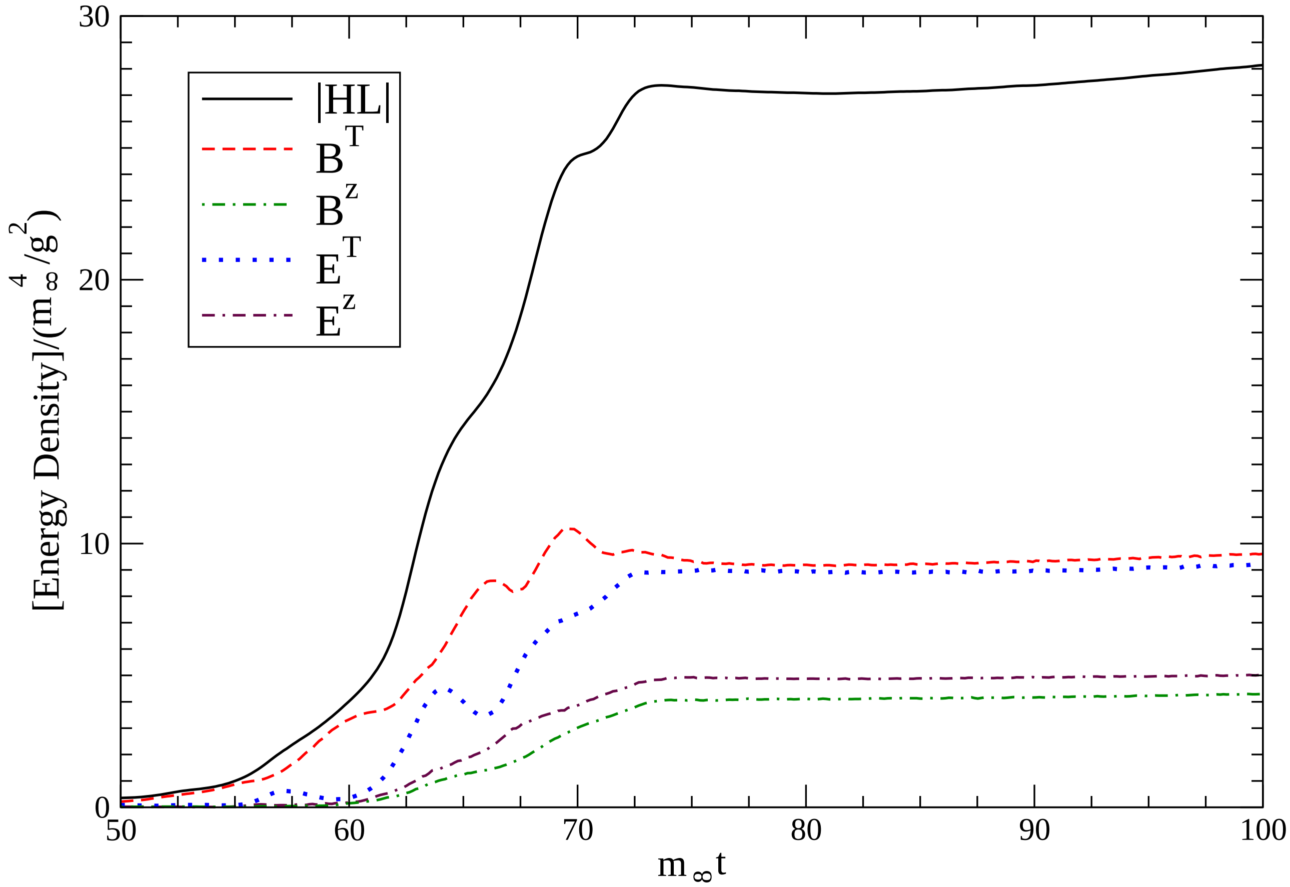
Cascade


Cascade region well fit by power law, $f(k) \propto k^{-2}$
Fields Saturate At Non-Perturbative Amplitude: A~1/g
Increasing simulation time
Occupation $g^2 f(k)$
Wave number k/m
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0.8 0.9 1 2 3 4 5 6 7 8 9 10

3D+3V

Similar results with discoball discretization (using somewhat larger anisotropy)

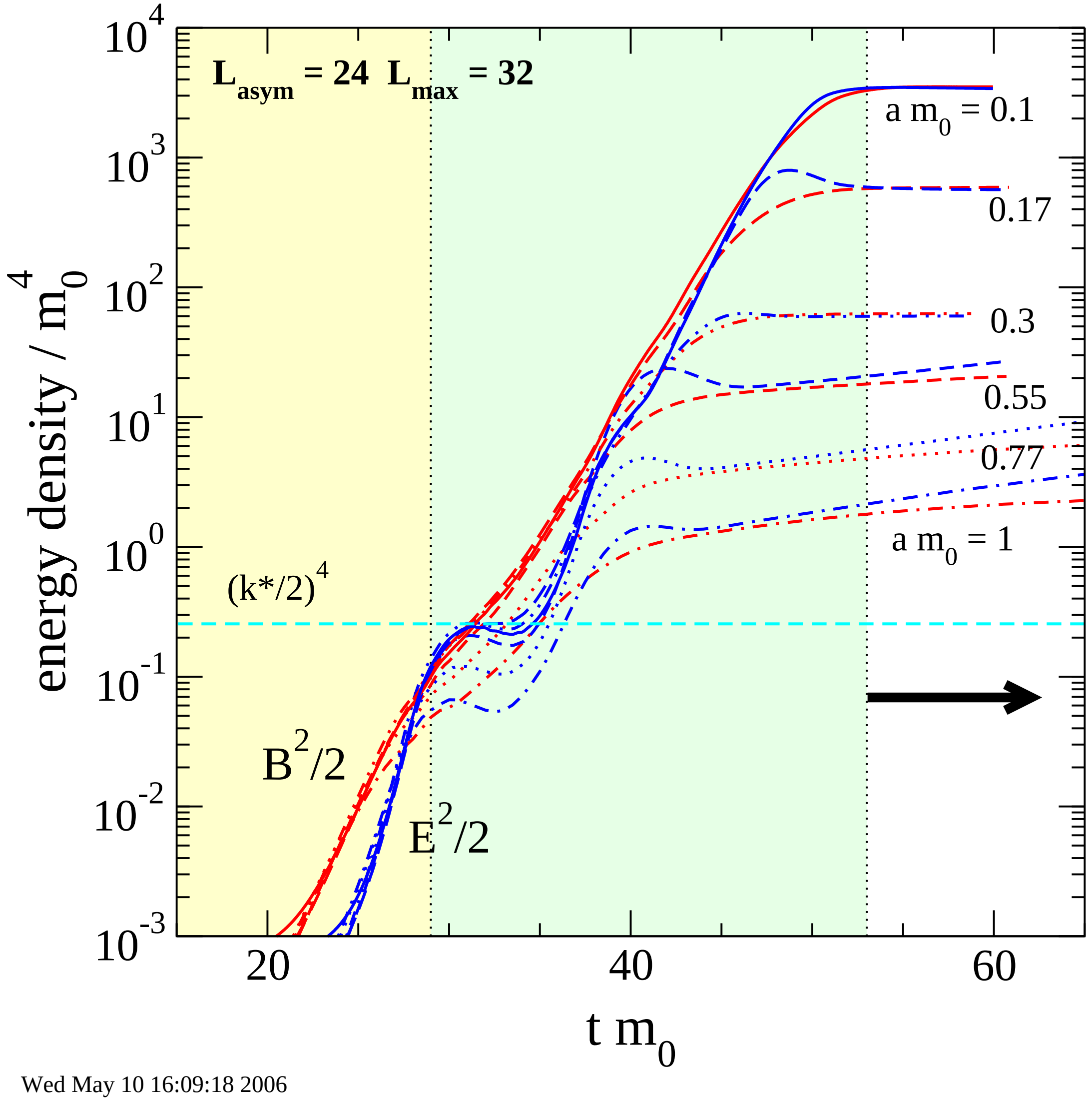
[AR. Romatschke & Strickland. JHEP 09 (2005) 041]



3D+3V - Strong anisotropy

More recently – Strong anisotropy: no saturation before lattice saturation

[Bödeker & Rummukainen, JHEP 07 (2007) 022]



Plasma instabilities in Bjorken expansion

Longitudinal (Bjorken) expansion: Competition between

- increasing anisotropy (more and more modes become unstable)
- and decreasing density ($\leftrightarrow$ growth rate)

[Romatschke & AR, PRL 97 (2006)]

Notation: proper time $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\eta = \mathrm{atanh}\frac{z}{t}$

$$x^\mu \to x^\alpha = (\tau, x^i, \eta) \text{ with } g_{\alpha\beta} = (1, -1, -1, -\tau^2)$$

momentum rapidity $y = \mathrm{atanh}\frac{p^0}{p^z}$:

$$p^\mu \to p^\alpha = |\mathbf{p}_\perp|(\cosh(y-\eta), \cos\phi, \sin\phi, \tau^{-1} \underbrace{\sinh(y-\eta)}_{p'^z/|\mathbf{p}_\perp|})$$

Boost invariant and transversely isotropic $f_0(\mathbf{p}, x) = f_0(p_\perp, p'^z, \tau)$

$$\boxed{p^\mu \partial_\mu f_0(x,p) = p^\alpha \partial_\alpha f_0 \Big|_{fixed\ p^\mu} = 0}$$

solved by $f_0(\mathbf{p}, \mathbf{x}, t) = f_0(\mathbf{p}_\perp, p_\eta(x))$ because $(p^\alpha \partial_\alpha) p_\eta(x)|_{fixed\ p^\mu} = 0$

Boost-invariant free-streaming background

Will use:

$$f_0(\mathbf{p},x) = f_{\rm iso}\left(\sqrt{p_\perp^2 + p_\eta^2/\tau_{\rm iso}^2}\right) = f_{iso}\left(\sqrt{p_\perp^2 + (p'^z\tau/\tau_{iso})^2}\right)$$

space-time dependent anisotropy parameter $\xi(\tau) = (\tau/\tau_{iso})^2 - 1$

increasingly oblate momentum space anisotropy at $\tau > \tau_{iso}$

(but prolate anisotropy for $\tau < \tau_{iso}$)

Will start at finite τ_0 (mostly $\gg \tau_{iso}$)

as motivated by CGC initial conditions at $\tau_0 \sim Q_s^{-1}$

$n \propto \alpha_s^{-1}$ — particle interpretation/kinetic theory actually only appropriate for $\tau \gg \tau_0$

strong initial anisotropy which gets even stronger, $\xi \sim \tau^2$

(bottom-up scenario: $\xi \sim \tau^{(<2/3)}$

Bödeker: $\tau^{1/2}$; Arnold & Moore: $\tau^{1/4}$)

Hard-Expanding-Loop formalism

Since $p^\beta \partial_\beta \left[\partial^\alpha_{(p)} f_0(\mathbf{p}_\perp, p_\eta)\right]\big|_{p^\mu = const.} = 0$ (with index α upstairs!) can solve

$$p \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t)\big|_{p^\mu = const.} = g p^\beta F^a_{\beta\alpha} \partial^\alpha_{(p)} f_0(\mathbf{p}, \mathbf{x}, t),$$

by introducing auxiliary fields

$$\delta f^a(x; p) = -g W^a_\alpha(\tau, x^i, \eta; \phi, y) \partial^\alpha_{(p)} f_0(p_\perp, p_\eta)$$

that obey

$$\boxed{v \cdot D\, W_\alpha(\tau, x^i, \eta; \phi, y)\big|_{\phi, y} = v^\beta F_{\alpha\beta},}$$

where $v^\alpha \equiv \frac{p^\alpha}{|\mathbf{p}_\perp|} = \left(\cosh(y - \eta), \cos\phi, \sin\phi, \frac{\sinh(y-\eta)}{\tau}\right).$

Discretized HEL

For $f_0(\mathbf{p}, x) = f_{\rm iso}\left(\sqrt{p_\perp^2 + p_\eta^2/\tau_{\rm iso}^2}\right)$

$$j^\alpha(\tau, x^i, \eta) \;=\; -\frac{m_D^2(\tau{=}\tau_{iso})}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_{-\infty}^{\infty} dy\, v^\alpha\left(1+\frac{\tau^2}{\tau_{iso}^2}\sinh^2(y-\eta)\right)^{-}$$

$$\times\underbrace{\left\{\cos\phi\, W_1 + \sin\phi\, W_2 - \frac{\tau}{\tau_{iso}^2}\sinh(y-\eta)\, W_\eta\right\}}_{\mathcal{W}(\tau, x^i, \eta; \phi, y)}$$

instead of discoballs [$\mathcal{W}(t, \mathbf{x}; \phi_n, \theta_m)$ with equally spaced ϕ_n, $\cos\theta_m$]

now *disco cylinders*: $\mathcal{W}(\tau, x^i, \eta; \phi, y)$ with equally spaced ϕ_n, y_m

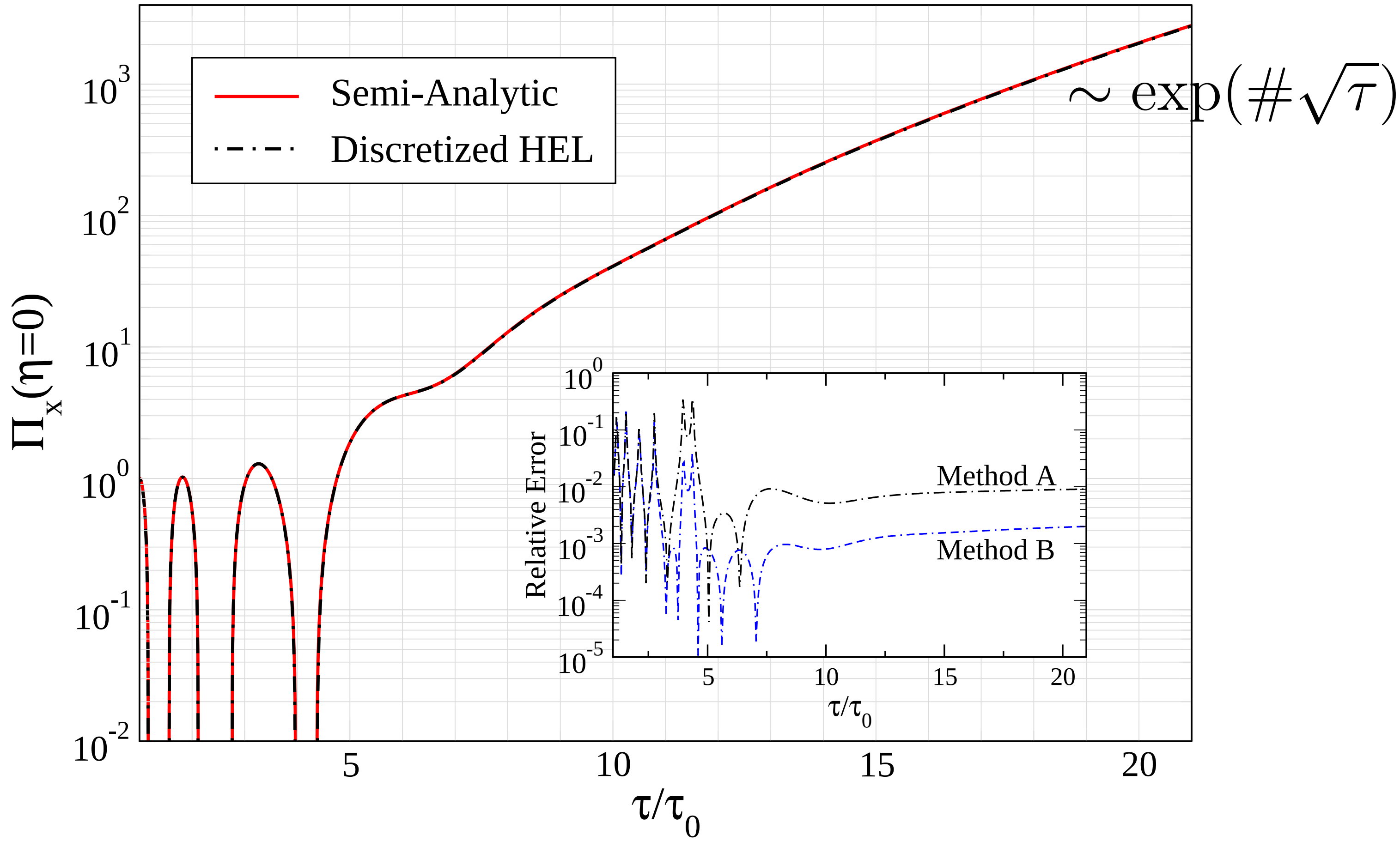
finite rapidity interval for $y - \eta$ because of exponential suppression

$\longrightarrow$ numerical simulation on space-time & $\phi, y - \eta$ grid

AR, M. Strickland, M. Attems: PRD78 (2008)

Discretized HEL - Abelian checks

Abelian: can solve e.o.m. for $\mathcal{W}$ to give 1D integro-differential equation ("semi-analytic")

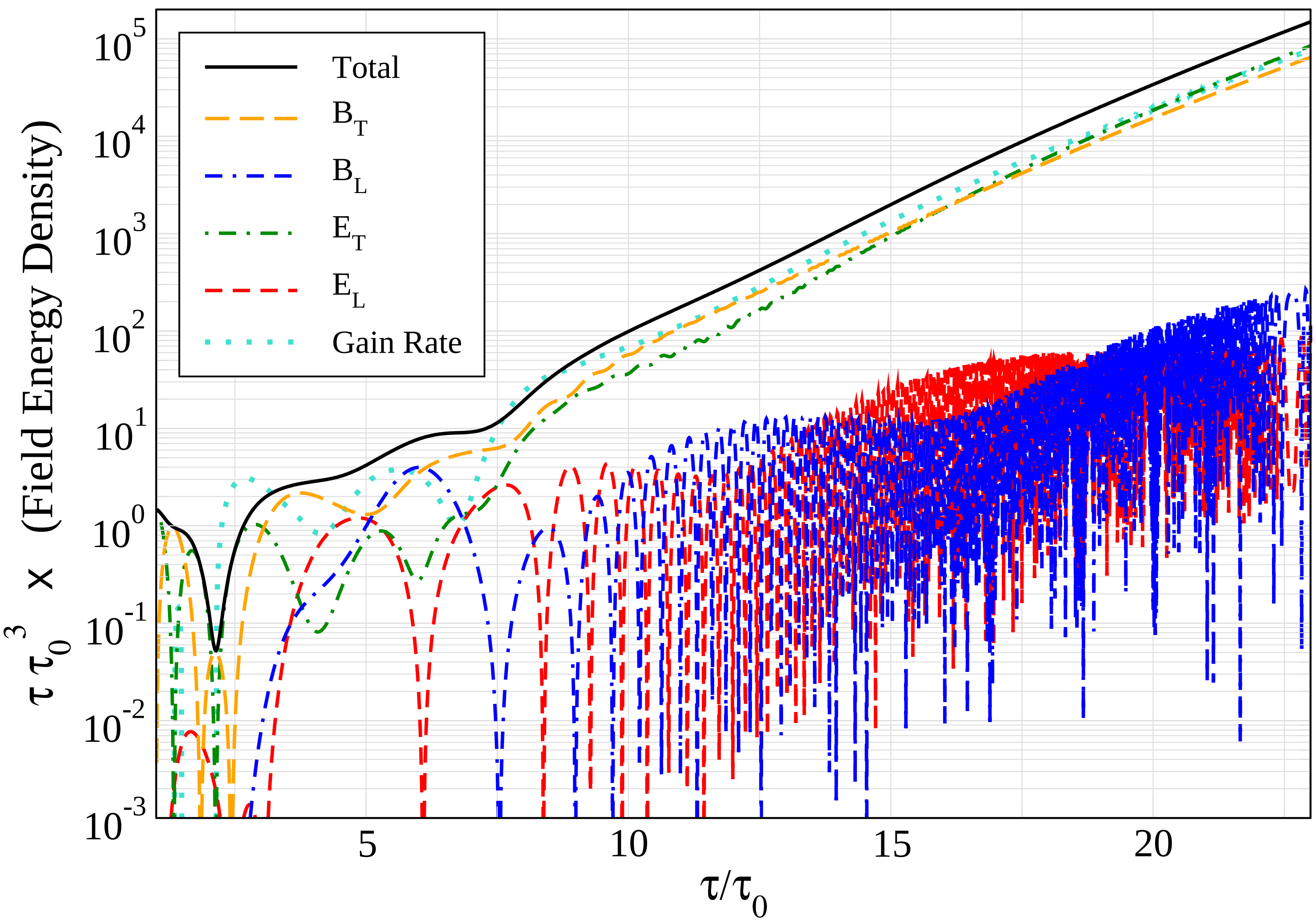


$\tau_{\rm iso} = 0.1$, $\tau_0 = 1.0$, $m_D = 10$, $a = 0.0025$, $\epsilon = 0.001$, $N_\eta = 250$, $N_\phi = 8$,

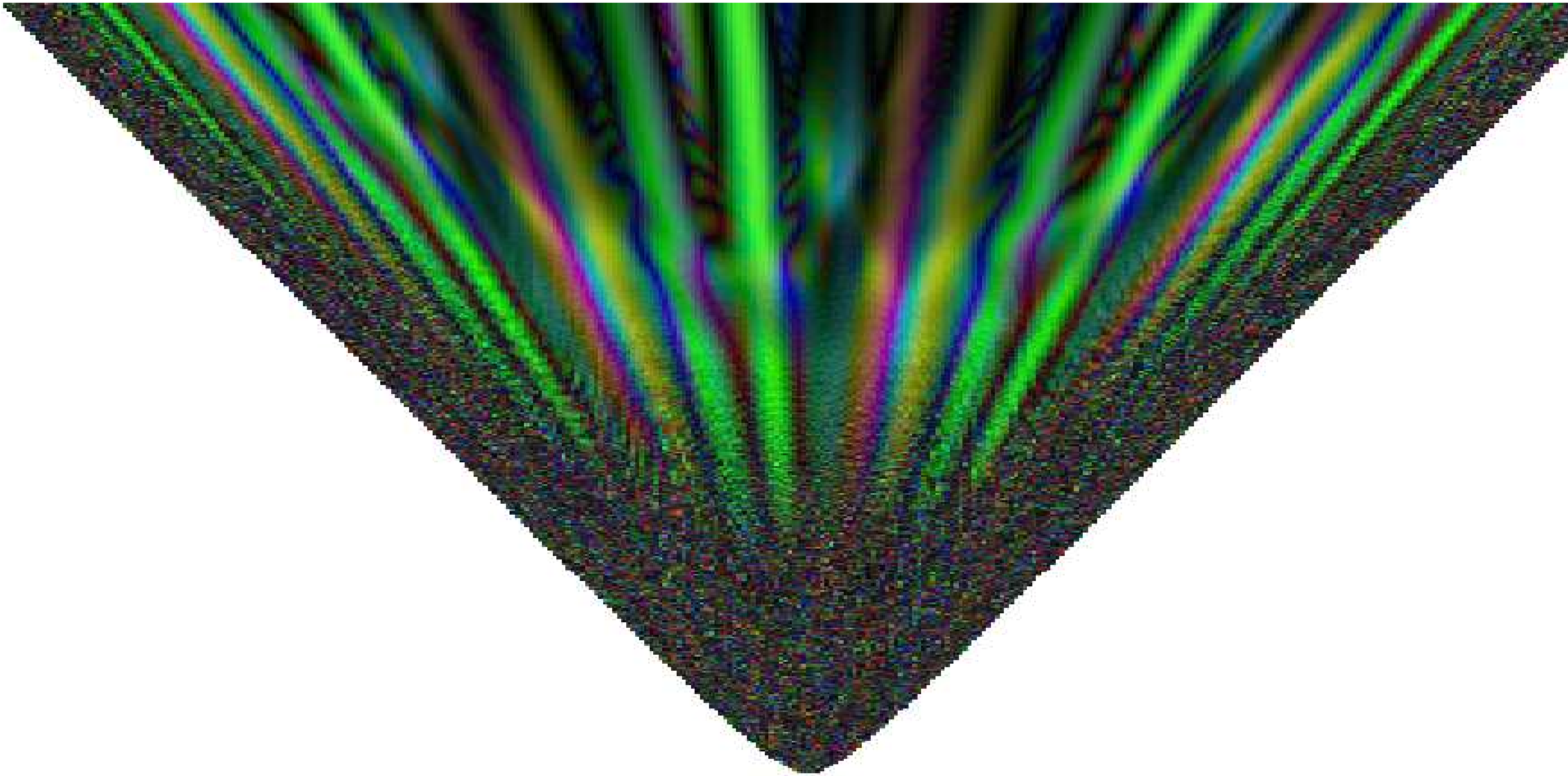
and (Method A) $N_y = 1000$, (Method B: $x = \tanh(y - \eta)$) $N_x = 1000 \ldots$ $N_{\mathcal{W}} = 8000$

Non-Abelian Discretized HEL

Single nonabelian mode: (unrealistically dense plasma)



Non-Abelian Discretized HEL — Visualization in Lab Frame



Non-Abelian Discretized HEL

Hard gluon number density and initial fluctuation spectrum from **CGC** $\longrightarrow$

Parameters from saturation scenario $\tau_0 \simeq Q_s^{-1}$:

$n(\tau_0) = c \frac{(N_c^2-1)Q_s^3}{4\pi^2 N_c \alpha_s (Q_s \tau_0)}$

with gluon liberation factor $c = \begin{cases} 0.5 & \text{Krasnitz '99 et al.} \longrightarrow \text{1.1 Lappi '07 (numerical)} \\ \underline{2\ln 2} \approx 1.39 & \text{Kovchegov (analytical estimate)} \end{cases}$

$f_{\rm iso} = \mathcal{N} f_{\rm thermal}$ with (transverse) temperature $T = 0.47 Q_s$ [Krasnitz et al.]

$$\text{pure glue} \quad \longrightarrow \quad \mathcal{N} = \frac{1}{\alpha_s} \frac{c}{8N_c (0.47)^3 \zeta(3)} \frac{\tau_0}{\tau_{iso}} \frac{1}{Q_s \tau_0}$$

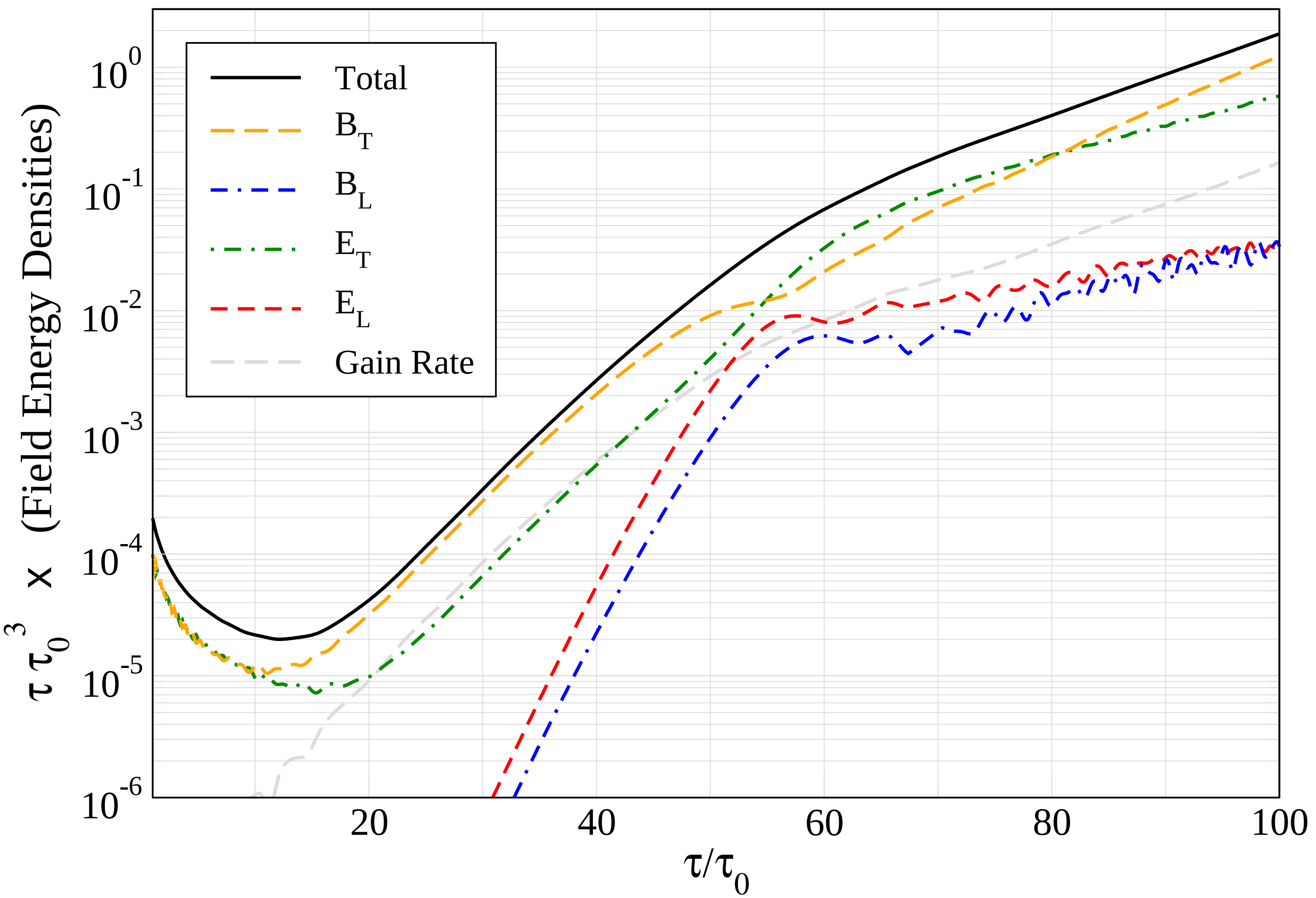
$$\longrightarrow \quad \frac{\mu}{Q_s} = \frac{1}{8} m_D^2 \pi \tau_{\rm iso} Q_s^{-1} = \frac{\pi^2}{48 \cdot 0.47 \cdot \zeta(3)} c \approx \begin{cases} 0.182 & (c = 0.5) \\ \underline{0.505} & (c = 2\ln 2) \end{cases}$$

$Q_s \simeq 1$ GeV (RHIC) . . . 3 GeV (LHC) ?

+ form of initial fluctuation spectrum from Fukushima, McLerran & Gelis 2007

Non-Abelian Discretized HEL – Chromofield energy densities

Hard gluon number density and initial fluctuation spectrum from **CGC** $\longrightarrow$

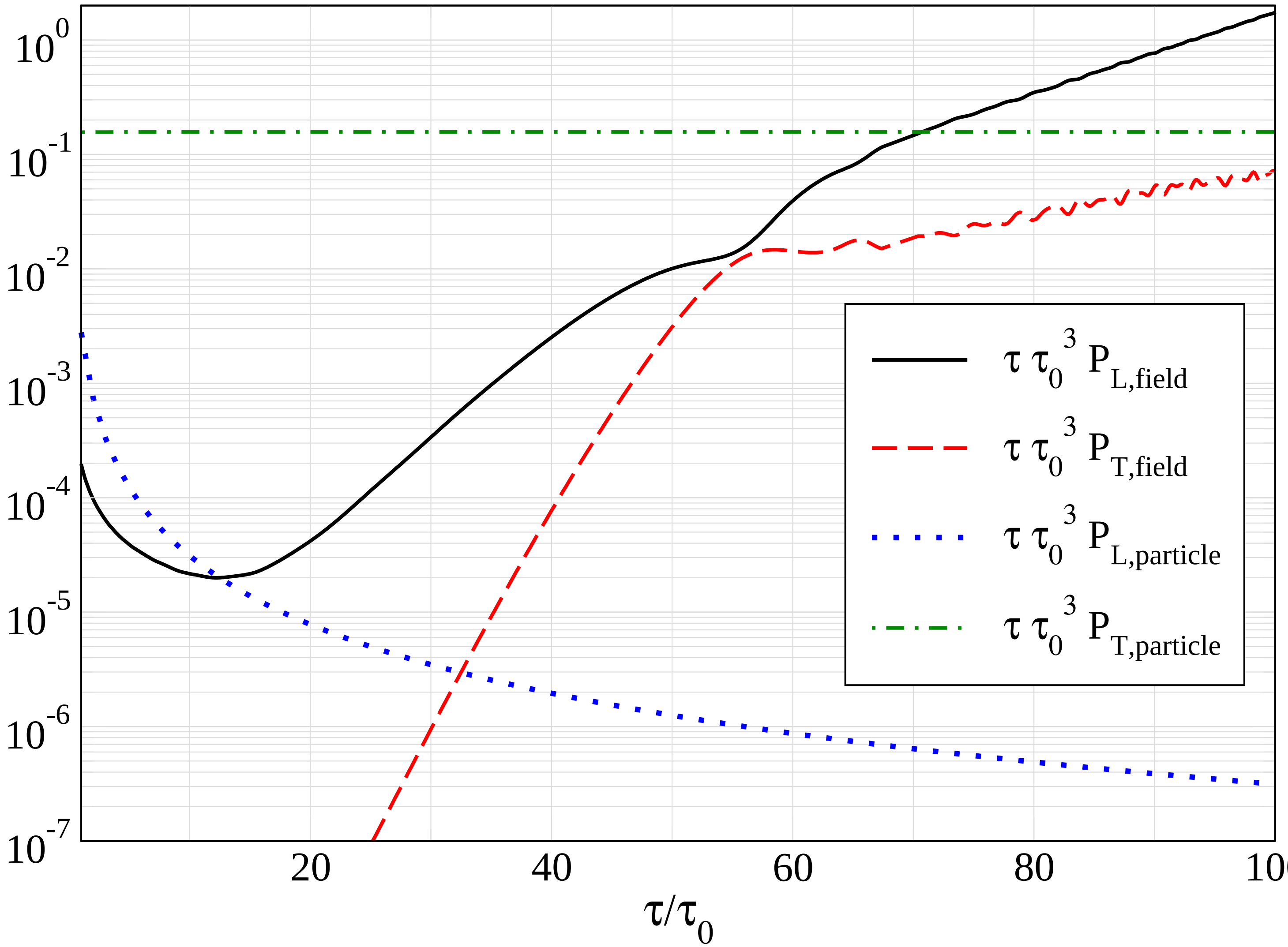


$= \begin{cases} \text{20 fm/c (RHIC)} \\ \text{7 fm/c (LHC)} \end{cases}$

Non-Abelian Discretized HEL — Pressure components

Effective growth rate of P_L up to $(0.7\,\mathrm{fm}/c)^{-1}$ (RHIC), $(0.3\,\mathrm{fm}/c)^{-1}$ (LHC)

BUT: uncomfortable delay of onset of exponential (in $\sqrt{\tau}$) growth



$= \begin{cases} 20\ \mathrm{fm/c\ (RHIC)} \\ 7\ \mathrm{fm/c\ (LHC)} \end{cases}$

Conclusions and Outlook

Conclusions:

- Plasma instabilities are parametrically dominant phenomenon in anisotropic plasmas with interesting characteristic time scales
- Fate of nonabelian Weibel instabilities depends strongly on degree of anisotropy
- Uncomfortably long delay of onset of plasma instabilities in Bjorken expansion to explain early isotropization at RHIC
- More important role for plasma instabilites at LHC?

Forthcoming:

- Full 3D+3V
- needed for analysis of generic large initial fields

Open challenge:

- Complete perturbative bottom-up thermalization scenario

Supplement: Transversely constant modes in linear (Abelian) regime

Most unstable modes for $\tau > \tau_{\rm iso}$ have $\boxed{\partial_i A^\alpha \equiv 0}$

Linearize ($A^\tau = 0$): $\boxed{\left[\frac{1}{\tau}\partial_\tau \tau \partial_\tau - \frac{1}{\tau^2}\partial_\eta^2\right] A^i(\tau,\eta) = j^i,}$ $\boxed{\partial_\tau \frac{1}{\tau}\partial_\tau A_\eta = \frac{j_\eta}{\tau},}$

Solving $v \cdot \partial W = v^\beta F_{\alpha\beta}$:

$$W_\alpha(\tau,\eta;\phi,y) = \int_{\tau_0}^{\tau} d\tau' \frac{v^\beta F_{\alpha\beta}\big|_{\tau',\eta(\tau')}}{\cosh(y-\eta(\tau'))}\,, \qquad y - \eta(\tau') = \mathrm{asinh}\left(\frac{\tau}{\tau'}\sinh(y-\eta)\right),$$

$$\longrightarrow \quad j^i[W] \;=\; -\frac{m_D^2}{4}\int_{-\infty}^{\infty} dy \left(1 + \frac{v_\eta^2}{\tau_{\rm iso}^2}\right)^{-2} \int_{\tau_0}^{\tau} d\tau'$$

$$\times \left[\left(\partial'_\tau - \frac{\tanh\bar\eta'}{\tau'}\partial_{\eta'}\right) A^i(\tau',\eta') + \frac{v_\eta}{\tau_{\rm iso}^2}\frac{\partial_{\eta'}A^i(\tau',\eta')}{\cosh\bar\eta'}\right],$$

$$j^\eta[W] \;=\; -\frac{m_D^2}{2\tau_{\rm iso}^2}\int \frac{dy\, v^\eta v_\eta}{\left(1 + \frac{v_\eta^2}{\tau_{\rm iso}^2}\right)^2}\int_{\tau_0}^{\tau} d\tau' \partial_{\tau'} A_\eta(\tau',\eta'),$$

where $\eta' = \eta(\tau')$ and $\bar\eta' = \eta(\tau') - y$.

Transversely constant modes in linear (Abelian) regime

Fourier transform in space-time rapidity $(\nu \sim k_z \tau$ at $\eta \sim 0)$

$$A^i(\tau,\eta) = \int \frac{d\nu}{2\pi} \exp(i\nu\eta) \widetilde{A}^i(\tau,\nu),$$

$\Rightarrow$

$$\widetilde{j}^i(\tau,\nu) = -\frac{m_D^2}{4} \int \frac{dy}{\left(1 + \frac{\tau^2 \sinh^2 y}{\tau_{\text{iso}}^2}\right)^2} \left\{ \widetilde{A}^i(\tau,\nu) - \int_{\tau_0}^{\tau} d\tau' \frac{\widetilde{A}^i(\tau',\nu)\tau'^2}{\tau_{\text{iso}}^2} \partial_{\tau'} e^{i\nu\left[y - \text{asinh}\left(\frac{\tau}{\tau'} \sinh y\right)\right]} \right.$$

(similar equation for $\widetilde{j}^\eta(\tau,\nu)$)

Integro-differential equations, solved by numerical leap-frog algorithm

$$\tau \partial_\tau \widetilde{A}^i(\tau,\nu) = \widetilde{\Pi}^i(\tau,\nu) \qquad \text{and}$$

$$\partial_\tau \widetilde{\Pi}^i(\tau,\nu) = -\nu^2 \tau^{-1} \widetilde{A}^i(\tau,\nu) + \tau \widetilde{j}^i(\tau,\nu)$$

Transversely constant modes in linear regime: *Analytical results*

Late-time behavior: approximate 4th order ODE

$$\tau \gg \tau_0 \gtrsim \tau_{\rm iso}: \left[\partial_\tau^2 \tau \partial_\tau \tau \partial_\tau + \nu^2 \partial_\tau^2 + \mu \partial_\tau^2 \tau - \mu\nu^2 \tfrac{1}{\tau}\right] \widetilde{A}^i(\tau,\nu) \approx 0,$$

$$\left[\partial_\tau \tfrac{1}{\tau} \partial_\tau + \mu \tfrac{2}{\tau^2}\right] \widetilde{A}_\eta(\tau,\nu) \approx 0, \text{ where } \boxed{\mu = \tfrac{1}{8} m_D^2 \pi \tau_{\rm iso}}.$$

Stable plasma oscillations for $\nu \ll 1$:

$$\widetilde{A}^i(\tau,\nu) = c_1 J_0\left(2\sqrt{\mu\tau}\right) + c_2 Y_0\left(2\sqrt{\mu\tau}\right),$$

$$\tau^{-1}\widetilde{A}_\eta(\tau,\nu) = c_1 J_2\left(2\sqrt{2\mu\tau}\right) + c_2 Y_2\left(2\sqrt{2\mu\tau}\right)$$, indeed: $\lim_{\xi\to\infty} \omega_{\rm pl}^\ell / \omega_{\rm pl}^t = \sqrt{2}$

[Romatschke & Strickland, PRD68]

Unstable transverse modes for $\nu \gtrsim 1$:

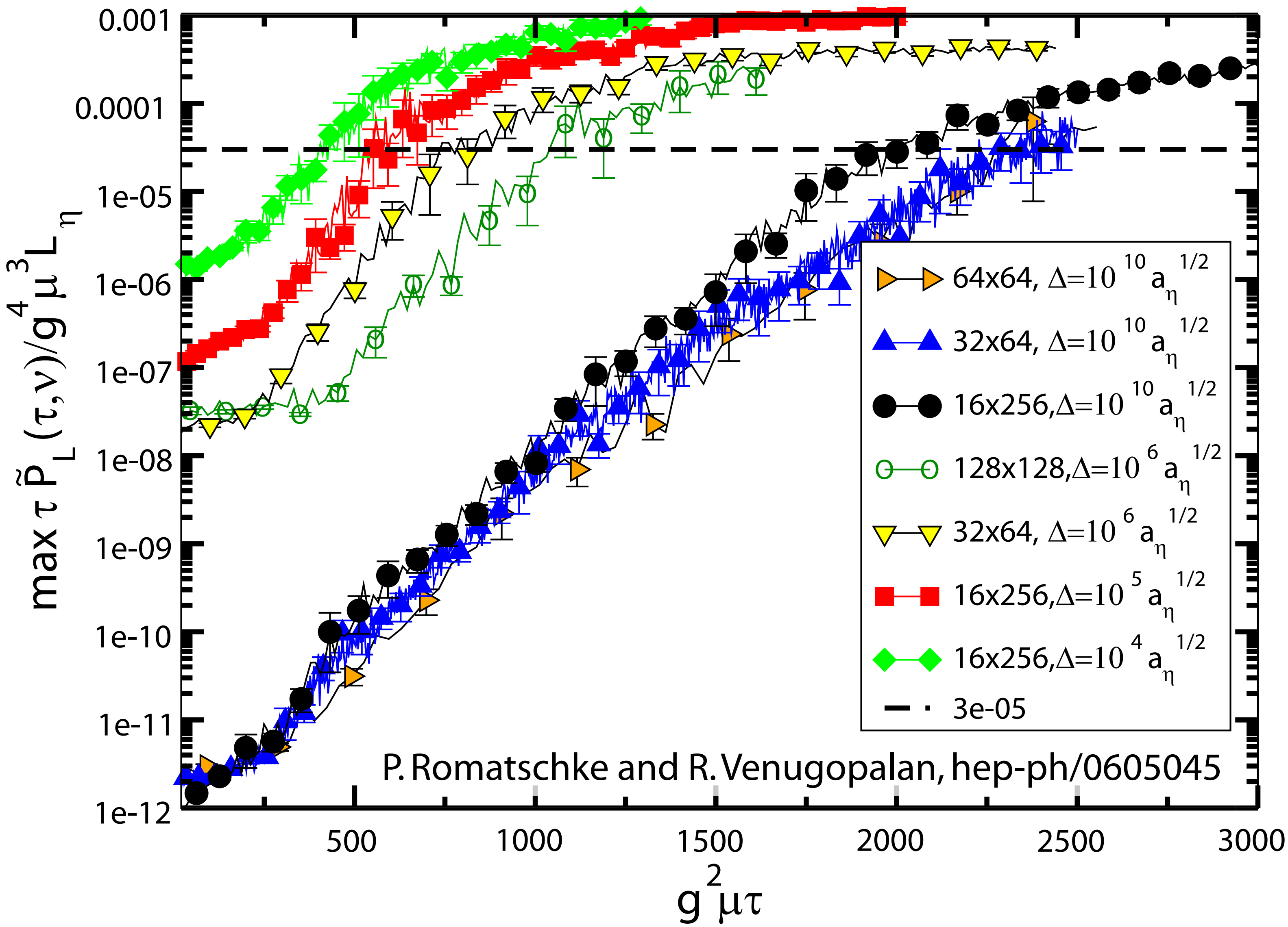
$$\widetilde{A}^i(\tau,\nu) \sim \tau\ {}_2F_3\left(\tfrac{3-\sqrt{1+4\nu^2}}{2}, \tfrac{3+\sqrt{1+4\nu^2}}{2}; 2, 2-i\nu, 2+i\nu; -\mu\tau\right)$$

$$\to \tau^{1/4} \exp\left(2\sqrt{\mu\tau}\right) \quad \text{for } \nu \gg 1$$

qualitative agreement with unstable melting color glass-condensate
of [P. Romatsche & R. Venugopalan, PRL96(2006)062302; hep-ph/0605045]

Unstable glasma

P. Romatschke and R. Venugopalan, PRL 96, PRD 74 (2006)



Transversely constant modes in linear regime: *Numerical results*

Numerical result *vs.* asymptotic ${}_2F_3$ behavior (thin bright lines) $(c = 0.5)$