Euclidean Relativistic Quantum Mechanics

W. N. Polyzou
Philip Kopp

The University of Iowa
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Thanks!
Amand Faessler
Jochen Wambach
U.S. D.O.E. Office of Science

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Motivation

Study

Few-hadron systems at the few GeV scale

using

well-defined mathematical models

motivated by quantum field theories

This talk: Scattering using Euclidean Green functions

Theoretical framework

Relativistic Quantum Mechanics

- **Unitary representation of the Poincaré group (invariance of quantum probabilities with respect to change of inertial coordinate system).**
- **Spacelike cluster properties (required for tests of relativity on isolated subsystems).**
- **Spectral condition (required for stability of theory).**

Models motivated by field theory (missing from "potential" models that satisfy the "•" conditions).

Strategy

Minkowski		Euclidean		Minkowski
Green functions	$\Leftrightarrow$	Schwinger functions	$\Leftrightarrow$	Wightman functions
$\Downarrow$		$\Downarrow$		$\Downarrow$
QFT phenomenology		Quantum theory		Quantum theory

Construction of relativistic quantum theory from Euclidean Green functions does not require locality[1] or analytic continuation!
([1] Osterwalder and Schrader - C.M.P. 31(1973)83.)

Schwinger functions constrained by DSE, lattice, $\cdots$

Model Assumption

Dynamics given by set of Euclidean-invariant reflection-positive Green functions or their generating functional.

Relation to formal Lagrangian field theory

$$Z[f] := \int D_e[\phi] e^{-A[\phi]+i\phi(f)}$$

$$= \sum_n \frac{(i)^n}{n!} G_n \underbrace{(f, \cdots, f)}_{\text{n times}} = \prod_n \exp\left(\frac{i^n}{n!} C_n(f, \cdots, f)\right)$$

Assumed properties of $Z[f]$

$$\mathrm{x} := (\tau, \mathbf{x}) \qquad f(\mathrm{x}) := f(\tau, \mathbf{x}) \qquad f(\mathrm{x}) \to f_{O,\mathrm{a}}(\mathrm{x}) := f(O(\mathrm{x} - \mathrm{a}))$$

$$Z[f] = Z[f_{O,\mathrm{a}}]$$ **Euclidean invariance**

$$\mathcal{S}_+ := \{f(\tau, \mathbf{x}) \in \mathcal{S} | f(\tau, \mathbf{x}) = 0, \quad \tau < 0\} \qquad \Theta f(\tau, \mathbf{x}) := f(-\tau, \mathbf{x})$$

$$\{f_i\}_{i=1}^N \in \mathcal{S}^+ \qquad M_{ij} = Z[f_i - \Theta f_j] \geq 0$$ **reflection positivity**

$$g_{\mathbf{a}}(\tau, \mathbf{x}) := g(\tau, \mathbf{x} - \mathbf{a}) \qquad f, g \in \mathcal{S}^+$$

$$\lim_{|\mathbf{a}| \to \infty} (Z[f + g_{\mathbf{a}}] - Z[f]Z[g]) \to 0$$ **cluster properties**

Hilbert Space

Vectors (wave functionals)

$$B[\phi] = \sum_{j=1}^{N_b} b_j e^{i\phi(f_j)} \qquad C[\phi] = \sum_{k=1}^{N_c} c_k e^{i\phi(g_k)}$$

$$b_j, c_k \in \mathbb{C} \qquad f_j, g_k \in \mathcal{S}_+ \qquad N_b, N_c < \infty$$

Hilbert space inner product

$$\langle B|C\rangle := \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \Theta f_j] =$$

$$\int D_e[\phi] e^{-A[\phi]} B^*[\phi \circ \theta] C[\phi]$$

Fröhlich - Helv. Phys. Acta. 47(1974)265 .

Observations

$$B[\phi] \qquad = \qquad \text{“wave functionals”}$$

$$\langle B|B\rangle \geq 0 \qquad \text{reflection positivity}$$

- **The inner product is the physical (Minkowski) inner product!**
- **The generating functional and test functions are Euclidean!**
- **All integrals are over Euclidean space-time variables!**
- **Analytic continuation not required to calculate the Minkowski scalar product!**

Poincaré and Euclidean invariance

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \qquad \mathsf{X} = \begin{pmatrix} i\tau+z & x-iy \\ x+iy & i\tau-z \end{pmatrix}$$

$$\det(X) = t^2 - \mathbf{x}^2 \qquad \det(\mathsf{X}) = -(\tau^2 + \mathbf{x}^2)$$

Complex Poincaré group = Complex Euclidean group

$$\mathsf{X}' = A\mathsf{X}B \qquad X = AXB \qquad \det(A) = \det(B) = 1$$

$B = A^\dagger$ **real Lorentz group,** A, B **unitary real orthogonal group**

Real Euclidean group = complex subgroup of P.G. in $\langle B|C\rangle$

One-parameter groups and semigroups

Euclidean time translations - contractive Hermitian semigroup

$$e^{-\beta H} \qquad \beta > 0$$

Rotations and space translations - unitary one-parameter groups

$$e^{i\mathbf{a}\cdot\mathbf{P}} \qquad e^{i\mathbf{J}\cdot\hat{\mathbf{n}}\psi}$$

Rotations in space-time planes - local symmetric semigroups

$$e^{\mathbf{K}\cdot\hat{\mathbf{n}}\psi}$$

Generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ are self-adjoint on $\mathcal{H}$ and satify Poincaré commutation relations

One-particle states

(mass)^2 eigenfunctional (point spectrum)

$$B_\lambda[\phi] = \sum_n b_n e^{i\phi(f_n)}$$

mass-momentum eigenfunctional

$$B_\lambda(\mathbf{p})[\phi] = \int \frac{d^3a}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} B_{\lambda,I,\mathbf{a}}[\phi]$$

mass-momentum-spin eigenfunctional

$$B_{\lambda,j}(\mathbf{p},\mu)[\phi] := \int_{SU(2)} dR \sum_{\nu=-j}^{j} B_{\lambda,R,0}(R^{-1}\mathbf{p})[\phi] D^{j*}_{\mu\nu}(R)$$

Two Hilbert space Haag-Ruelle scattering

$$|\Psi_\pm(f_1,\cdots f_n)\rangle := \lim_{t\to\infty} e^{iHt}\Phi e^{-iH_0 t}|\mathbf{f}\rangle = \Omega_\pm|\mathbf{f}\rangle$$

$$\Phi|\mathbf{f}\rangle[\phi] =$$

$$\int\sum\prod_k \left(\omega_{\lambda_k}(\mathbf{p}_k)B_{\lambda_k,j_k}(\mathbf{p_k},\mu_k)[\phi] - [H, B_{\lambda_k,j_k}(\mathbf{p}_k,\mu_k)[\phi]]\right)\tilde{f}_k(\mathbf{p}_k,\mu_k)d\mathbf{p}_k$$

Cook condition - existence of $\Omega_\pm[\phi]$

$$\int_0^{\pm\infty}\|(H\Phi-\Phi H_0)e^{-iH_0 t}|\mathbf{f}\rangle\|dt < \infty.$$

$$G_4 = C_2 C_2 + C_4$$

regularity condition on connected parts of Euclidean Green functions

Scattering calculations in Euclidean QM

Approximation 1: Use sharply peaked (in momentum) normalizable states to approximate plane-wave on-shell transition matrix elements.

$$\langle \mathbf{f}_f | S | \mathbf{f}_i \rangle = \langle \mathbf{f}_f | \mathbf{f}_i \rangle - 2\pi i \langle \mathbf{f}_f | \delta(E_+ - E_-) T | \mathbf{f}_i \rangle$$

$$\langle \mathbf{p}_1', \mu_1', \cdots, \mathbf{p}_n', \mu_n' | T | \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle \approx \frac{\langle \mathbf{f}_f | S | \mathbf{f}_i \rangle - \delta_{ab} \langle \mathbf{f}_f | \mathbf{f}_i \rangle}{2\pi i \langle \mathbf{f}_f | \delta(E_+ - E_-) | \mathbf{f}_i \rangle}$$

Scattering in Euclidean QM

$$H \to g(H) \qquad g(H) = -e^{-\beta H} \qquad \beta > 0$$

Kato-Birman invariance principle

$$\Omega_\pm := \lim_{t\to\pm\infty} e^{-iHt}\Phi e^{iH_0 t} = \lim_{n\to\pm\infty} e^{ine^{-\beta H}}\Phi e^{-ine^{-\beta H_0}}$$

Approximation 2: Replace $\lim_{n\to\infty}$ by large fixed n.

$$\langle \mathbf{f}_f | S | \mathbf{f}_i \rangle = \langle \mathbf{f}_f | \Omega_+^\dagger \Omega_- | \mathbf{f}_i \rangle$$

$$\approx \langle \mathbf{f}_f | e^{-ine^{-\beta H_f}} \Phi^\dagger e^{2ine^{-\beta H}} \Phi e^{-ine^{-\beta H_f}} | \mathbf{f}_i \rangle$$

Approximation 3: Uniform polynomial approximation

$$e^{2ine^{-\beta H}} \approx \sum c_m(n)(e^{-\beta mH})$$

note $\sigma(e^{-\beta H}) \in [0,1]$ **(compact)**

$$e^{2inx} \approx \sum c_m(n)x^m \qquad x \to e^{-\beta H}$$

$$|e^{2inx} - \sum c_m(n)x^m| < \epsilon(n) \qquad \forall x \in [0,1]$$

$$\Downarrow$$

$$\|[e^{2ine^{-\beta H}} - \sum c_m(n)(e^{-\beta mH})]|\psi\rangle\| < \epsilon(n)\||\psi\rangle\|$$

Approximation 3: Use Chebyshev polynomials

$$f(x) \approx \frac{1}{2} c_0 T_0(x) + \sum_{k=1}^{N} c_k T_k(x)$$

$$c_j = \frac{2}{N+1} \sum_{k=1}^{N} f\Big(\cos\big(\frac{2k-1}{N+1}\frac{\pi}{2}\big)\Big) \cos\big(j \frac{2k-1}{N+1}\frac{\pi}{2}\big)$$

$$f(e^{-\beta H}) \approx \frac{1}{2} c_0 T_0(e^{-\beta H}) + \sum_{k=1}^{N} c_k T_k(e^{-\beta H})$$

$$f(x) = e^{2inx}$$

$$|e^{2inx} - P_N(x)| < 2\frac{n^{N+1}}{(N+1)!}$$

Combine all three approximations

$$\langle \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_n, \mu'_n | T | \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle \approx \frac{\langle \mathbf{f}_f | S | \mathbf{f}_i \rangle - \delta_{ab} \langle \mathbf{f}_f | \mathbf{f}_i \rangle}{2\pi i \langle \mathbf{f}_f | \delta(E_+ - E_-) | \mathbf{f}_i \rangle}$$

$$\langle \mathbf{f}_f | S | \mathbf{f}_i \rangle \approx$$

$$= \sum c_m(n) \langle \mathbf{f}_f | e^{-ine^{-\beta H_f}} \Phi^\dagger (e^{-\beta m H}) \Phi e^{-ine^{-\beta H_f}} | \mathbf{f}_i \rangle$$

Each approximation converges - the order of the approximations is important $(1) \rightarrow (2) \rightarrow (3)$**.**

Practical considerations

Is it possible to use these methods to do GeV-scale scattering calculations?

First test of method: relativistic separable potential (solvable so all approximations can be tested)

$$M^2 = 4(\mathbf{k}^2 + m^2) - |g\rangle\lambda\langle g|$$

$$\langle \mathbf{k}|g\rangle = \frac{1}{m_\pi^2 + \mathbf{k}^2}$$

Calculate $\langle \mathbf{k}'|T(k^+)|\mathbf{k}\rangle$ using matrix elements of $e^{-\beta H}$ in normalizable states.

Convergence with respect to wave packet width

Table 1

k_0 [GeV]	α [GeV^{-2}]	k_w [GeV]	% error	k_w/k_0
0.1	105000	0.00308607	0.1	0.030
0.3	10500	0.009759	0.1	0.032
0.5	3000	0.0182574	0.1	0.036
0.7	1350	0.0272166	0.1	0.038
0.9	750	0.0365148	0.1	0.040
1.1	475	0.0458831	0.1	0.041
1.3	330	0.0550482	0.1	0.042
1.5	250	0.0632456	0.1	0.042
1.7	190	0.0725476	0.1	0.042
1.9	150	0.0816497	0.1	0.042

Convergence with respect time "*n*"

Table 2: $k_0 = 2.0$**[GeV]**, $\alpha = 135$**[GeV^{-2}]**

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-2.60094316473225e-6	1.94120750171791e-3
100	-2.82916859895010e-6	2.35553585404449e-3
150	-2.83171624670953e-6	2.37471383801820e-3
200	-2.83165946257657e-6	2.37492460997990e-3
250	-2.83165905312632e-6	2.37492527186858e-3
300	-2.83165905257121e-6	2.37492527262432e-3
350	-2.83165905190508e-6	2.37492527262493e-3
400	-2.83165905234917e-6	2.37492527262540e-3
ex	-2.83165905227843e-6	2.37492527259701e-3

Table 3: Parameter choices

k_0 [GeV]	β[GeV^{-1}]	$k_0 \times \beta$	n
0.1	40.0	4.0	450
0.3	5.0	1.5	330
0.5	3.0	1.5	205
0.7	1.6	1.2	200
0.9	1.05	.945	190
1.1	0.95	1.045	200
1.3	0.85	1.105	200
1.5	0.63	0.945	200
1.7	0.5	0.85	200
1.9	0.42	0.798	200

Table 4: Convergence with respect to Polynomial degree e^{inx}

x	n	deg	poly error %
0.1	200	200	3.276e+00
0.1	200	250	1.925e-11
0.1	200	300	4.903e-13
0.1	630	630	2.069e+00
0.1	630	680	5.015e-08
0.1	630	700	7.456e-11
0.5	200	200	1.627e-13
0.5	200	250	3.266e-13
0.5	630	580	1.430e-14
0.5	630	680	9.330e-13
0.9	200	200	3.276e+00
0.9	200	250	1.950e-11
0.9	200	300	9.828e-13
0.9	630	630	2.069e+00
0.9	630	680	5.015e-08
0.9	630	700	7.230e-11

Table 15: Final calculation

k_0	Real T	Im T	% error
0.1	-2.30337e-1	-4.09325e-1	0.0956
0.3	-3.46973e-2	-6.97209e-3	0.0966
0.5	-6.44255e-3	-3.86459e-4	0.0986
0.7	-1.88847e-3	-4.63489e-5	0.0977
0.9	-7.28609e-4	-8.86653e-6	0.0982
1.1	-3.35731e-4	-2.30067e-6	0.0987
1.3	-1.74947e-4	-7.38285e-7	0.0985
1.5	-9.97346e-5	-2.76849e-7	0.0956
1.7	-6.08794e-5	-1.16909e-7	0.0964
1.9	-3.92110e-5	-5.42037e-8	0.0967

Summary

- Phenomenology based on model reflection-positive Euclidean Green functions can be used to formulate a relativistic quantum theory.
- Analytic continuation is not necessary.
- Explicit expressions for sharp momentum GeV-scale transition matrix elements. Existence can be established using Cook's method.
- Finite Poincaré transformations on single particle and scattering states can be performed.
- Models can be constrained by field-theory based phenomenology.
- A test using an exactly solvable model suggests that GeV scale scattering calculations are possible in this framework.

Future studies

- **Thirring model - solvable, 2d, with reflection positivity and non-trivial S-matrix.**
- **NN scattering -** $G_4 \to C_{20}C_{20} + C_{20}C_{20}VC_{20}C_{20}$, **unitary S!**
- **NN scattering -** $G_4 \to C_2C_2 + C_2C_2VC_2C_2$, **non-trivial one-body problem.**
- **Nakanishi representation and reflection positivity.**
- **Gauge theories - reflection positivity only for color singlets.**
- **More information: nucl-th/1106.4086.**