

Supersymmetric quantum mechanics with non-Abelian self-dual fields

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Motivation

This talk is devoted to a new supersymmetric system with $\mathcal{N} = 4$ supersymmetry: a particle moving in a self-dual non-Abelian external gauge field in 4 space dimensions.

Topics:

- ▶ Hamiltonian description
- ▶ Superfield formalism
- ▶ A new way to express solutions of self-duality equation $\mathcal{F}_{\mu\nu} = \tilde{\mathcal{F}}_{\mu\nu}$ for SU(2) gauge group

$\mathcal{N} = 4$ supersymmetric quantum mechanics

A particle in external **self-dual** gauge field:

$$H = \frac{1}{2} (\hat{p}_\mu - \mathcal{A}_\mu)^2 + \frac{i}{4} \mathcal{F}_{\mu\nu} \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi}$$

Supercharges:

$$Q_\alpha = (\sigma_\mu \bar{\psi})_\alpha (\hat{p}_\mu - \mathcal{A}_\mu)$$

$$\bar{Q}^\alpha = (\psi \sigma_\mu^\dagger)^\alpha (\hat{p}_\mu - \mathcal{A}_\mu)$$

$$\{Q_\alpha, Q_\beta\} = 0, \quad \{Q_\alpha, \bar{Q}^\beta\} = 2\delta_\alpha^\beta H$$

Notations:

$$\sigma_\mu = \{i, \vec{\sigma}\}, \quad \sigma_\mu^\dagger = \{-i, \vec{\sigma}\}$$

$$\mu, \nu = 1, \dots, 4 \quad \alpha, \beta = 1, 2$$

$\mathcal{N} = 4$ supersymmetric quantum mechanics

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In spinorial notations

$$\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2), \quad \mathcal{F}_{\mu\nu} \rightarrow \left\{ \mathcal{F}_{\alpha\beta}, \mathcal{F}_{\dot{\alpha}\dot{\beta}} \right\}, \quad \mathcal{F}_{\alpha\beta} = 0$$

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E. Ivanov and O. Lechtenfeld, 2003

A. Kirchberg, J. D. Lange and A. Wipf, 2005

M. Konyushikhin and A. Smilga, 2009

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$\mathcal{F}_{\mu\nu}$ can have color indices (be a matrix)

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non-Abelian case?

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Add two complex **auxiliary fields** φ_α (SU(2) gauge group example):

$$\mathcal{A}_\mu \dot{x}_\mu \quad \longrightarrow \quad i\bar{\varphi}^\alpha \dot{\varphi}_\alpha + (\mathcal{A}_\mu)_\beta^\alpha \varphi_\alpha \bar{\varphi}^\beta \dot{x}_\mu$$

Quantization:

$$\left[\varphi_\alpha, \bar{\varphi}^\beta \right] = \delta_\alpha^\beta, \quad T_a = \frac{1}{2} \bar{\varphi} \sigma_a \varphi, \quad [T_a, T_b] = i\epsilon_{abc} T_c$$

Representation fixing:

$$\bar{\varphi}^\alpha \varphi_\alpha = k, \quad T_a T_a = \frac{k}{2} \left(\frac{k}{2} + 1 \right), \quad k - \text{integer}$$

A. P. Balachandran, Per Salomonson, Bo-Sture Skagerstam, Jan-Olof Winnberg, 1977

S. Fedoruk, E. Ivanov, O. Lechtenfeld, 2009

How to write the Lagrangian in non-Abelian case

Basic idea:

$$\mathcal{A}_\mu \dot{x}_\mu \quad \longrightarrow \quad i\bar{\varphi}^\alpha \dot{\varphi}_\alpha + (\mathcal{A}_\mu)_\beta^\alpha \varphi_\alpha \bar{\varphi}^\beta \dot{x}_\mu$$

A. P. Balachandran, Per Salomonson, Bo-Sture Skagerstam, Jan-Olof Winnberg, 1977

S. Fedoruk, E. Ivanov, O. Lechtenfeld, 2009

$\mathcal{N} = 4$ supersymmetric quantum mechanics (flat metric case):

$$L = \frac{1}{2} \dot{x}_\mu \dot{x}_\mu + i\bar{\psi}^{\dot{\alpha}} \dot{\psi}_{\dot{\alpha}} + i\bar{\varphi}^\alpha (\dot{\varphi}_\alpha + iB\varphi_\alpha) + kB \\ + (\mathcal{A}_\mu)_\beta^\alpha \varphi_\alpha \bar{\varphi}^\beta \dot{x}_\mu - \frac{i}{4} (\mathcal{F}_{\mu\nu})_\beta^\alpha \varphi_\alpha \bar{\varphi}^\beta \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi}$$

E. Ivanov, M. Konyushikhin, A. Smilga, 2009

Harmonic superspace

Idea: extend space coordinates with R -symmetry group:

$$\mathbb{R}^4 \rightarrow \mathbb{R}^4 \times \text{SU}(2)_R$$

Harmonics – coordinates on 3-sphere:

$$u^{+\alpha} u_{\alpha}^{-} = 1, \quad u_{\alpha}^{-} \equiv (u^{+\alpha})^*, \quad \alpha = 1, 2$$

Superspace:

$$\left\{ t, \theta_{\alpha}, \bar{\theta}^{\beta}, u_{\gamma}^{\pm} \right\} \quad \longleftrightarrow \quad \left\{ t_A, \theta^{\pm}, \bar{\theta}^{\pm}, u_{\alpha}^{\pm} \right\}$$

(standard basis) *(analytical basis)*

$$t_A = t + i(\theta^{+}\bar{\theta}^{-} + \theta^{-}\bar{\theta}^{+}), \quad \theta^{\pm} = u_{\alpha}^{\pm}\theta^{\alpha}, \quad \bar{\theta}^{\pm} = u_{\alpha}^{\pm}\bar{\theta}^{\alpha}$$

A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, 1984

A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Harmonic Superspace*, 2001

E. Ivanov and O. Lechtenfeld, 2003

Superfield description involves two prepotentials:

- ▶ First term produces kinetic term with conformally flat metric:

$$S_{\text{kin}} = \int dt d^4\theta du R(x, u) = \int dt \{ f^{-2}(x) \dot{x}_\mu \dot{x}_\mu + \dots \}$$

- ▶ The second term produces non-Abelian self-dual field:

$$\begin{aligned} S_{\text{int}} &= -\frac{1}{2} \int dt d^2\theta du K(x, u, v) \\ &= \int dt \left\{ i\bar{\varphi}^\alpha (\dot{\varphi}_\alpha + iB\varphi_\alpha) + (\mathcal{A}_\mu)_\beta^\alpha \varphi_\alpha \bar{\varphi}^\beta \dot{x}_\mu + \dots \right\} \end{aligned}$$

(v – auxiliary superfield for φ_α)

- ▶ Third term – Fayet–Iliopoulos term:

$$S_{\text{FI}} = -\frac{ik}{2} \int dt d^2\theta du V^{++} = k \int dt B(t)$$

There is a relation between
self-dual SU(2) gauge field $\mathcal{A}_\mu(x)$
and function $K(x, u, v)$

For example,

$$\int du d^2\theta K^{++} (x^{+\dot{\alpha}}, u_{\dot{\alpha}}^{\pm}, v^+ \widetilde{v}^+) = \int du d\bar{\theta}^+ d\theta^+ K (x^{+\dot{\alpha}}, u_{\dot{\alpha}}^{\pm}) v^+ \widetilde{v}^+$$

gives **t'Hooft ansatz** form:

$$\mathcal{A}_{\mu}^a(x) = -\bar{\eta}_{\mu\nu}^a \partial_{\nu} \ln h(x), \quad h(x) = \int du K (x^{+\dot{\alpha}}, u_{\dot{\alpha}}^{\pm})$$

$h(x)$ – arbitrary harmonic function, $\partial_{\mu}^2 h(x) = 0$.

Why is it interesting?

More on harmonics: solution of self-duality equation

$$\mathcal{F}_{\mu\nu}^a = \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} \mathcal{F}_{\rho\lambda}^a$$

The solutions $(\mathcal{A}_{\alpha\dot{\alpha}})^i_j(x)$ are expressed through arbitrary function $(K^{++})^i_j(x^{+\dot{\alpha}}, u)$.

$$K^{++} = -ie^{-i\lambda} \partial^{++} e^{i\lambda} \quad \longleftrightarrow \quad \mathcal{A}_{\dot{\alpha}}^\alpha u_\alpha^+ = -ie^{-i\lambda} \partial_{-\dot{\alpha}} e^{i\lambda}$$

A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, 1988

Bridge $\lambda(x^{\pm\dot{\alpha}}, u)$ is a matrix. The self-duality equation is equivalent to

$$\partial_{-\dot{\alpha}} K^{++} = 0$$

Notations:

$$\partial^{++} = u_\alpha^+ \frac{\partial}{\partial u_\alpha^-}, \quad \partial_{-\dot{\alpha}} = \frac{\partial}{\partial x^{-\dot{\alpha}}}$$

General n -instanton solution, $SU(2)$ gauge group

$$(K^{++})_j^i = i \sum_{A=1}^n \frac{\rho_A^2}{(x_A^{+-})^2} (M_A)_k^i u^{+k} u_\ell^+ (M_A^\dagger)_j^\ell$$

S. Kalitzin, E. Sokatchev, 1987

Instanton moduli are **explicit**.

Connection with ADHM or present construction is unknown

Summary

- ▶ A new $\mathcal{N} = 4$ SQM model is constructed.
- ▶ Superfield description is given for the particular case of t'Hooft ansatz. Generalization to general self-dual fields?
Generalization to $SU(N)$ gauge group?
- ▶ Non-Abelian **self-dual** gauge fields can be described in terms of **functions on extended space**.