The QCD equation of state at weak coupling

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Probing the Extremes of Matter with Heavy Ions
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Equilibrium thermodynamics

Conceptually simple goal: Evaluate the grand potential of QCD

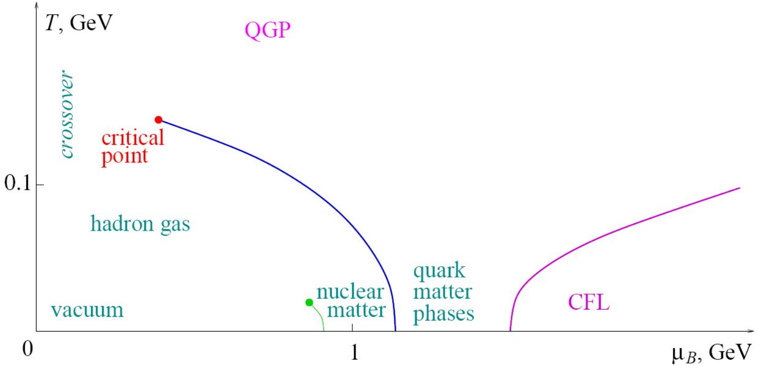
$$\begin{aligned}\Omega(T,\{\mu_f\},\{m_f\}) &= -T\log\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu e^{-\int_0^\beta d\tau\int d^3x\mathcal{L}_{\rm QCD}} \\ \mathcal{L}_{\rm QCD} &= \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\sum_f\bar{\psi}_f(\gamma_\mu D_\mu+m_f-\mu_f\gamma_0)\psi_f\end{aligned}$$

Bullk equilibrium thermodynamics from Ω:

$$\begin{aligned}pV &= -\Omega \\ sV &= -\partial_T\Omega \\ n_fV &= -\partial_{\mu_f}\Omega \\ \varepsilon &= -p+Ts+\mu_f n_f \\ \langle\bar{\psi}_f\psi_f\rangle V &= \partial_{m_f}\Omega\end{aligned}$$

Equilibrium thermodynamics

Unfortunately, QCD is a complicated theory, and no single method covers entire phase diagram ⇒ Need combination of (and interpolation between) several.



Corners of the phase diagram

Parametrize the phase diagram in terms of radial and angular variables: $r \equiv \sqrt{T^2 + \frac{\mu_B^2}{9\pi^2}}$, $\theta \equiv \arctan\frac{\mu_B}{T}$

- r measures, how strongly coupled the system is
  - $r \lesssim 100$ MeV: Confinement; hadron resonance gas model, chiral effective theories,...
  - $100\ \text{MeV} \lesssim r \lesssim 500$ MeV: Nonperturbative phase transition dynamics; lattice QCD, effective theories, models
  - $r \gtrsim 500$ MeV: Deconfinement, weakly interacting quasiparticles; weak coupling methods
- θ separates two distinct physically interesting regimes
  - $\theta \lesssim 1$: Quark-gluon plasma; heavy ion collisions, early universe
  - $\theta \approx \pi/2$: Cold nuclear and quark matter; neutron stars

Weak coupling methods

In this talk: Try to extend perturbatively determined EoS to as low energy densities as possible

- $\theta \lesssim 1$: Complement lattice simulations at very high T and/or $\mu \neq 0$
  - No sign problem: $\mu = 0$ results straightforwardly extendable to finite density
  - Guide for lattice simulations as $T \to \infty$
- $\theta \approx \pi/2$: Constrain nuclear matter EoSs above saturation density by providing information of the high r region
  - Challenges: Technically complicated, in particular in the presence of quark pairing effects

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Energy scales in a high temperature plasma

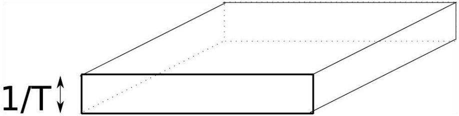
At $T \gg T_c$, hierarchy of three length scales in the QGP:

- $\lambda \sim 1/(\pi T)$: Wavelength of thermal fluctuations, inverse effective mass of non-static field modes ($p_0 \neq 0$)
  - $n_b(E)g^2(T) \sim g^2(T) \Rightarrow$ Contributes to the EoS perturbatively (naive loop expansion)
- $\lambda \sim 1/(gT)$: Screening length of static color electric fluctuations, inverse thermal ('Debye') mass of A_0
  - $n_b(E)g^2(T) \sim g(T) \Rightarrow$ Physics somewhat perturbative at high T
  - Requires resummation of EoS at three loop order $\Rightarrow g^3, g^4 \ln g$, ...
- $\lambda \sim 1/(g^2 T)$: Nonperturbative screening length of static color magnetic fluctuations, inverse 'magnetic mass'
  - $n_b(E)g^2(T) \sim g^0(T) \Rightarrow$ Physics non-perturbative at high T
  - Invalidates pert. expansion of EoS at four loops: 'Linde' problem

No further length scales due to confinement

Thermodynamics via dimensional reduction

- Scale hierarchy $\Rightarrow$ Natural to integrate out massive (non-static) modes (Appelquist, Pisarski)
  - Effective description accurate for $\Delta x \gtrsim 1/(gT)$



- Result: 3d eff. thy for static dof's (Kajantie et al; Braaten, Nieto):

$$\begin{aligned}\mathcal{L}_{\rm EQCD} &= g_{\rm E}^{-2}\left\{\frac{1}{2}\,\mathrm{Tr}\,F_{ij}^2 + \mathrm{Tr}\big[(D_iA_0)^2\big] + m_{\rm E}^2\,\mathrm{Tr}\,(A_0^2)\right. \\ &+ \left.\lambda_{\rm E}\,\mathrm{Tr}\,(A_0^4)\right\} + \delta\mathcal{L}_{\rm E}, \\ & g_{\rm E} \equiv \sqrt{T}g,\ m_{\rm E} \sim gT,\ \lambda_{\rm E} \sim g^2\end{aligned}$$

Thermodynamics via dimensional reduction

- EQCD valuable tool in reorganizing perturbation theory
  - No need for resummations in full theory, when dimensional reduction applicable
  - IR sector described by EQCD: Non-perturbative physics available from simulations in a 3d theory

$$p_{\rm QCD}(T,\mu) \;=\; p_{\rm E}(T,\mu) + \frac{T}{V}\ln\int \mathcal{D}A_i^a\,\mathcal{D}A_0^a\,\exp\left\{-S_{\rm E}\right\}$$

- Finite μ has only minor effects on structure of effective theory as long as $m_{\rm D} \lesssim T \Leftrightarrow g\mu \lesssim T$ (Ipp, Kajantie, Rebhan, AV)
  - $p_{\rm E}$ and other 3d parameters now functions of T, μ
  - One new operator generated in Lagrangian

The pressure at $\mu = 0$

$$
\begin{aligned}
p_{\rm QCD} &= T^4 \Big\{ p_0(\mu/T) + g^2\, p_2(\mu/T) + g^3\, p_3(\mu/T) + g^4 \ln g\, \tilde{p}_4(\mu/T) \\
&+ g^4\, p_4(\mu/T) + g^5\, p_5(\mu/T) + g^6 \ln g\, \tilde{p}_6(\mu/T) + g^6\, p_6(\mu/T) + \cdots \Big\}
\end{aligned}
$$

First non-perturbative contributions from scale $g^2 T$, requiring 3d lattice simulations and a complicated conversion of results to continuum regularization (di Renzo et al.)

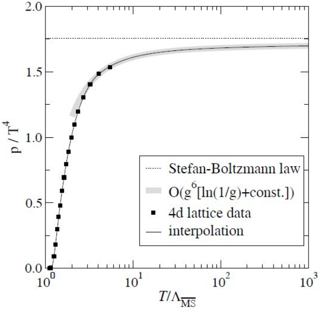
Contribution from scale gT computed perturbatively in EQCD (Kajantie, Laine, Rummukainen, Schröder)

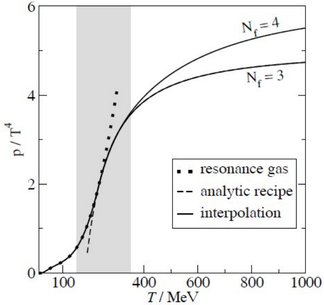
Contribution of scale πT remaining, available through strict loop expansion of full theory pressure up to 4 loops; only $\mathcal{O}(N_f^3)$ term known (Gynther, Kurkela, AV)

The pressure at $\mu = 0$

$$
\begin{aligned}
p_{\mathrm{QCD}} &= T^4 \Big\{ p_0(\mu/T) + g^2\, p_2(\mu/T) + g^3\, p_3(\mu/T) + g^4 \ln g\, \tilde{p}_4(\mu/T) \\
&+ g^4\, p_4(\mu/T) + g^5\, p_5(\mu/T) + g^6 \ln g\, \tilde{p}_6(\mu/T) + {\color{red} g^6\, p_6(\mu/T)} + \cdots \Big\}
\end{aligned}
$$

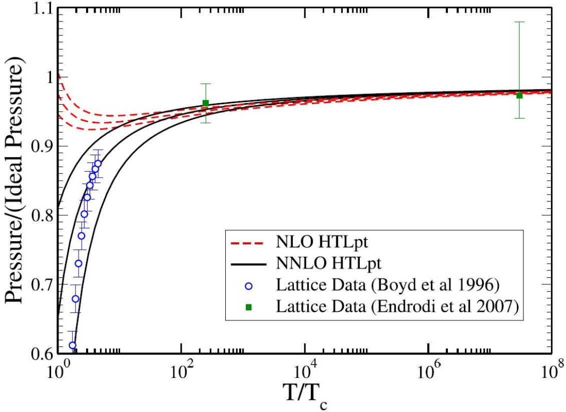
Fitting unknown $\mathcal{O}(g^6)$ term to lattice results and keeping EQCD parameters unexpanded gives an almost perfect match down to $T = 2T_c$ (Laine, Schröder)





Comparison with lattice

DR results in accordance with HTLpt (Andersen, Strickland, Su) and high-T lattice data (Endrodi et al.) — $\mu = 0$ story seems fairly complete on the perturbative side



Towards finite density: Susceptibilities

$\mathcal{O}(g^6 \ln g)$ EoS generalized to finite density (AV): μ-dependent part converges better than the $\mu = 0$ one due to absence of purely gluonic contributions

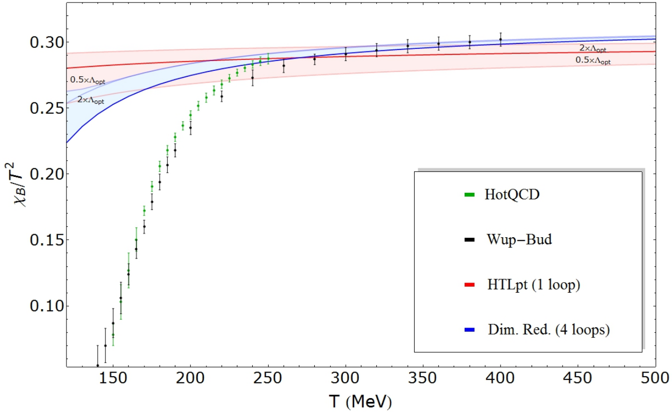
Natural observable allowing comparison with lattice data: Quark number susceptibilities

$$\chi_{ijk} \equiv -\frac{\partial^n \Omega(T, \{\mu_f\}, \{m_f\})}{\partial \mu_u^i \partial \mu_d^j \partial \mu_s^k}\bigg|_{\mu_f=0}, \quad n = i + j + k$$

Allows comparison of various resummation schemes (DR, HTLpt,...)

Towards finite density: Susceptibilities

Recent comparison of DR and HTLpt results for linear quark number susceptibility χ_{uu} (Andersen, Mogliacci, Su, AV) with lattice results shows agreement down to $\sim 2T_c$ (see talk by Sylvain Mogliacci)



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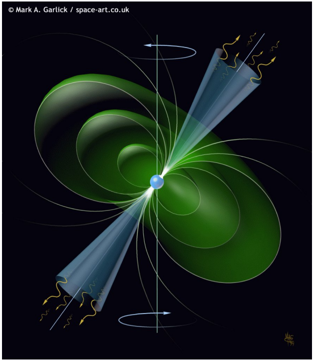
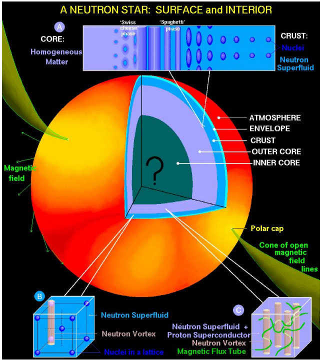
Nuclear matter EoSs

Most phenomenologically interesting applications of low-T strongly interacting matter at small or moderate r $\Rightarrow$ Role of weak coupling techniques mainly to constrain low energy EoSs





Nuclear matter EoSs

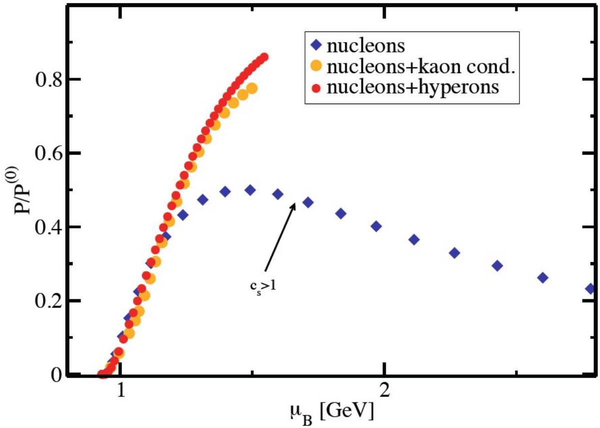
Most phenomenologically interesting applications of low-T strongly interacting matter at small or moderate r $\Rightarrow$ Role of weak coupling techniques mainly to constrain low energy EoSs

Low energy nuclear physics not derivable from first principles, but experimentally under excellent control $\Rightarrow$ Several model EoSs, which agree well at low densities

However: Many uncertainties with increasing density

- Composition of the matter: Hyperons, Kaon condensation,...
- Multi-nucleon interactions (often ignored)
- Form of variational ansatz
- Details of hyperon interactions, kaon condensation potential, etc.

Nuclear matter EoSs



Quark matter EoS

At high densities, gauge coupling small $\Rightarrow$ Use weak coupling techniques to evaluate EoS

Problem: Effects of quark pairing important to take into account ($\Delta \sim e^{-\#/g}$), but formulating weak coupling calculations with anomalous propagators and vertices difficult

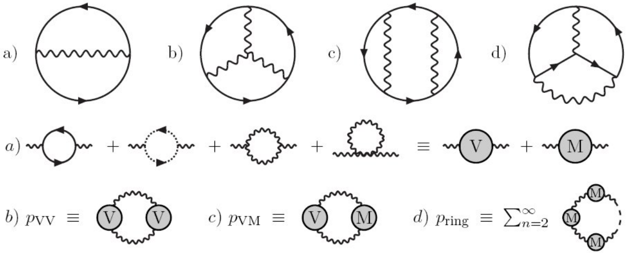
- At asymptopia, physical phase Color-Flavor-Locking (CFL)
- Two competing effects: Pairing increases pressure, but deformation of Fermi surfaces decreases it

Leading order solution: Add condensation energy term to the pressure of unpaired quark matter

$$p = p_{\rm pert} + \# \times \frac{\Delta^2 \mu_{\rm B}^2}{3\pi^2}$$

Quark matter EoS

Complication in evaluation of p_{pert}: For practical applications, must keep strange quark mass non-zero — state of the art three loops (Kurkela, Romatschke, AV)



Also need to enforce β-equilibrium and charge neutrality

Quark matter EoS

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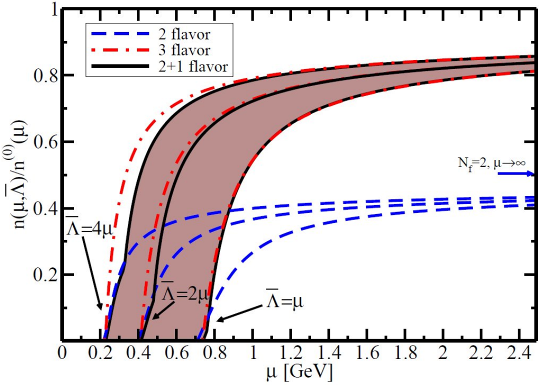


Interpolation to intermediate densities

For densities relevant for neutron star interiors, no quantitatively reliable method available

- Nuclear matter EoSs differ wildly
- Weak coupling expansions show poor convergence

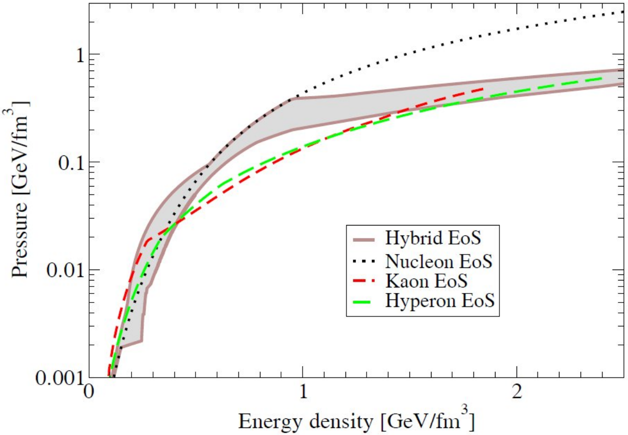
In particular, details of the phase transition remain unknown: μ_c, existence of mixed phase,...

Two possibilities:

1. Model calculations based on symmetries
    - Conceptually appealing, but validity hard to estimate quantitatively
2. Interpolation between trusted limits, requiring thermodynamically stable matching
    - Hope: Bulk thermo insensitive to details of phase structure

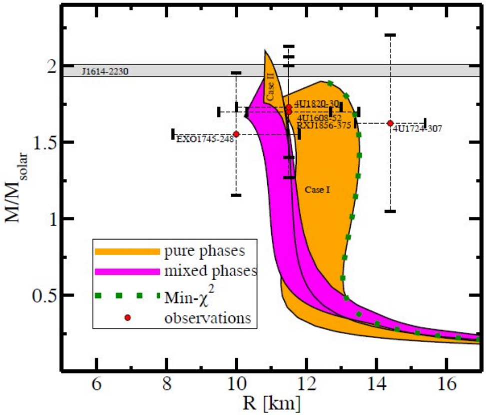
Interpolation to intermediate densities

Result of thermodynamic matching: EoS band for all densities; (Kurkela, Romatschke, AV, Wu)



Interpolation to intermediate densities

Ultimately, mass-radius measurements of neutron stars will determine the correct EoS



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Conclusions

(Resummed) perturbation theory important tool for description of equilibrium thermodynamics of deconfined QCD matter because

- Finite density no obstacle: Existing $\mu = 0$ results generalizable (and indeed, generalized) to $\mu \neq 0$
- Provides information on the approach of the system towards free theory limit as $T \to \infty$
- Only available tool for very high density quark matter — extremely useful constraint of nuclear matter EoSs

Status of perturbation theory well established at $\theta \lesssim 1$, much less so at low temperatures

- Main open challenge: Correct incorporation of quark pairing effects into a high order weak coupling calculation