RMS-radii from electron scattering

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Interest

R integral quantity describing size
reference for isotope shifts from many sources
comparison to radii from electronic+muonic atoms

Scope of talk

light nuclei, but emphasize proton
lessons how (*not*) to determine R
motivation: discrepancy (e,e) + H-atom $\Leftrightarrow$ μX 0.88 $\Leftrightarrow$ 0.84fm

state right from outset: no solution to problem for p
but: much more solid R's from (e,e)

Determination of R a priori looks simple:

fit low-q data with some parameterization for $G_e(q)$, $G_m(q)$
$q = 0$ slope of $G_e(q)$ $\rightarrow$ rms-radius R

Problem: many not-understood results + discrepancies

- **Power-expansion of $G(q)$**

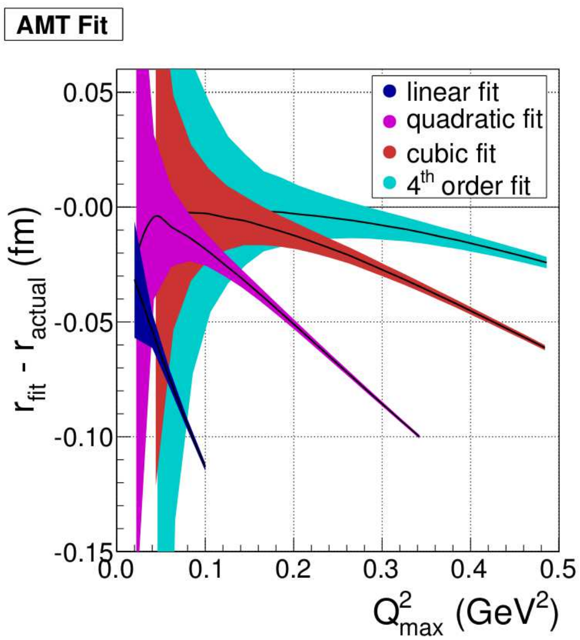
  always gives low R, depends on q_{max}

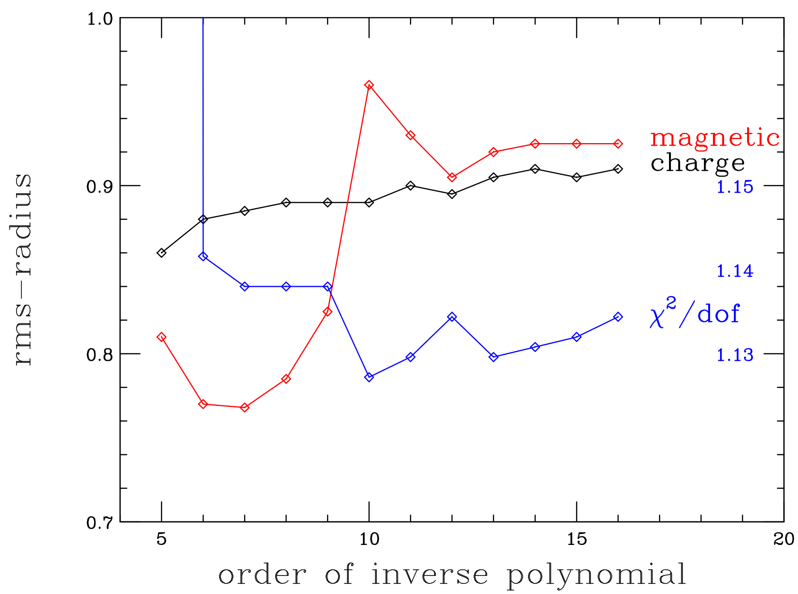
  Kraus et al.

- **Inverse-polynomial fit:**

  shows jump of R

  Bernauer et al.

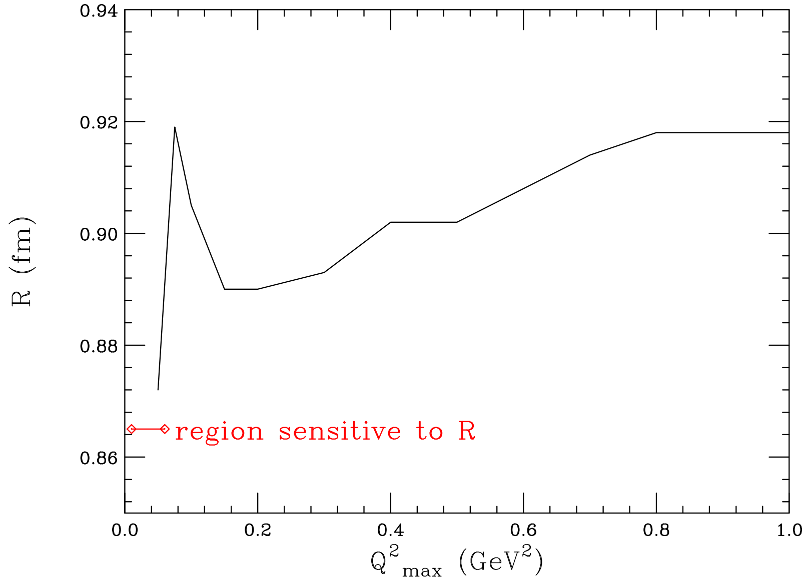




- R from conformal mapping:

  unexplained q_{max} dependence, gives even larger R: 0.92fm

  Lee et al.



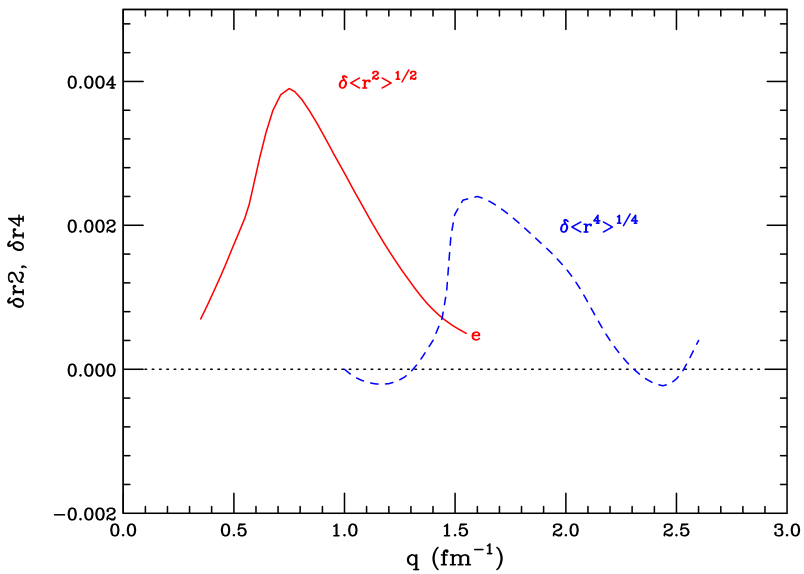
- various VDM-fits: give R around 0.84fm
- Own Pade-fit ($q_{max} = 2\text{fm}^{-1}$): yields 1.48fm
- Bayesian interference: 0.899±0.003fm

Worrisome!

.... but explained by insights discussed below

Important consideration (most often ignored!)
which q-region is sensitive to rms-radii?

Sensitivity to R: explored via notch-test



Region of sensitivity

$$0.5 < q < 1.3 \; fm^{-1}$$
$$0.01 < Q^2 < 0.06 \; GeV^2/c^2$$

lower q: $G(q) \sim 1 - q^2R^2/6 + ...$ measures only the "1"
higher q: $G(q)$ measures higher moments

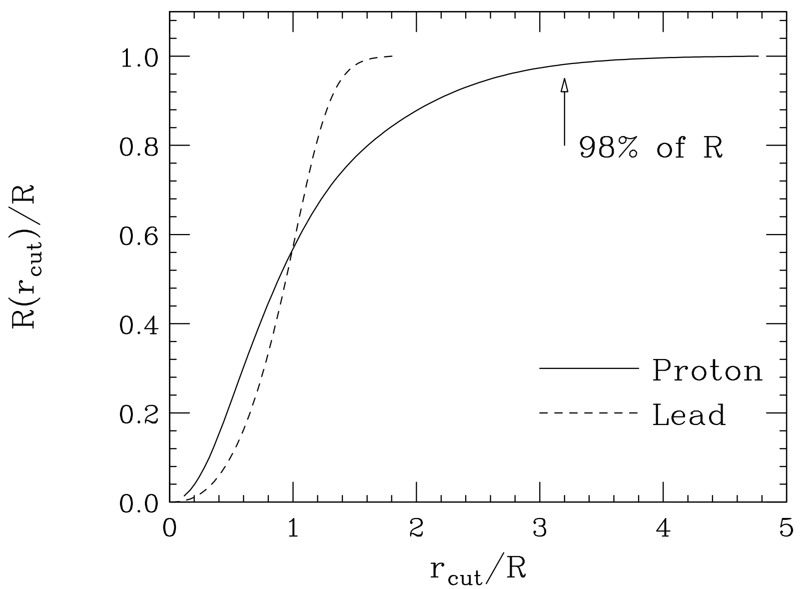
Recent data, all at $Q^2 > 0.06 GeV^2/c^2$ not relevant for R!

Second important point: need extrapolation to $q = 0$
particularly difficult for proton

form factor $\sim$ dipole $\sim 1/(1+q^2c^2)^2$
for *qualitative* discussion ignore rel. corr., 2γ, ...
$\Rightarrow \rho(r)$=Fourier-transform of $G(q) \Rightarrow \rho \sim$ exponential $\sim e^{-r/c}$

Long tail of exp. density causes serious problems

Illustration: study $[\int_0^{r_{cut}} \rho(r)\ r^4 dr / \int_0^{\infty} \rho(r)\ r^4 dr]^{1/2}$ as function of cutoff r_{cut}

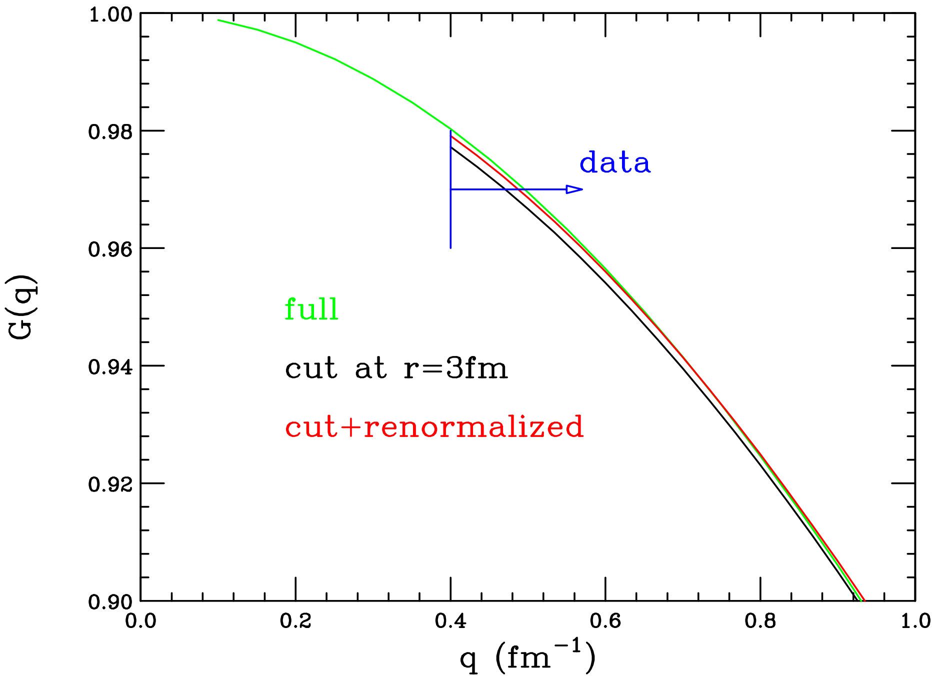


to get 98% of rms-radius R must integrate out to $r_{cut} \sim 3.2 \cdot R \sim 3fm$

$\Longrightarrow$ R sensitive to very large r where $\rho(r)$ poorly determined

$\Longrightarrow$ large r affect $G(q)$ at very low q, below q_{min} $\Longrightarrow$ affects extrapolation to $q = 0$
$\Longrightarrow$ model dependence of extrapolation

Large-r contribution not measurable in practice



$\rightarrow$ uncertainties of order 2% not reachable

Even worse for deuteron

for 98% of R must know density for $r > 7fm$!!

(r = distance to CM)

Problems with R of deuteron of past should have been a warning

Problem enhanced by peculiarity:

Traditional for proton, deuteron

parameterize $G(q)$, fit to data, get slope

Standard for A>2

parameterize $\rho(r)$, calculate σ, fit data, get slope of $G(q=0)$

Not equivalent!

For A>2 use implicit constraint on ρ for large r

standard parameterizations have $\rho(r > R_{max}) = 0$

the more physical parameterizations use $\pm$ exponential fall-off

For parameterized $G(q)$ this physics is *not* enforced

chosen $G(q)$ *can* imply large ρ at large r

large r strongly affect R

can imply unphysical curvature of $G(q < q_{min})$

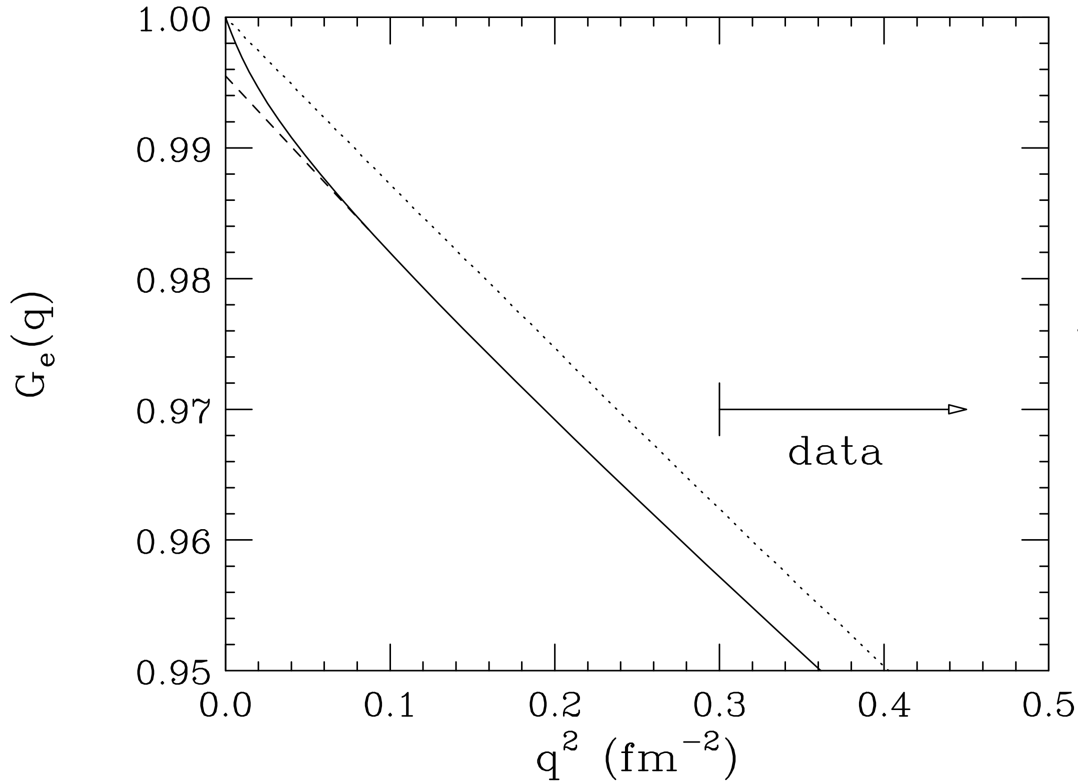
Most extreme demonstration case

own 4-parameter Pade-fit of Bernauer data, $q < 2fm^{-1}$

covers full region sensitive to R

$G(q) = (1 + a_1 q^2)/(1 + b_1 q^2 + b_2 q^4 + b_3 q^6)$

Fit has excellent $\chi^2 \sim 1.06/dof$ (as good as Spline fit)
no pole (a disease discussed below)



—— Pade-fit
.... Fit with 'normal' rms-radius
Note: data are floating

***rms*-radius $= 1.49 fm$!!**

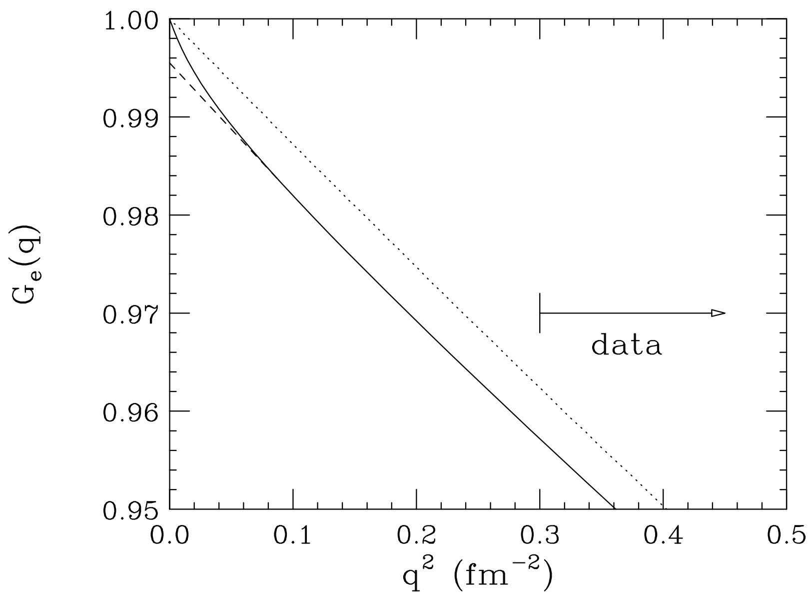
Pathology (?) visible by naked eye in $G(q)$ at *very* low q

Understanding

split fit into two contributions $G_1 + G_2$:

G_1 = Pade for $q^2 > 0.06$ plus dashed line for $q^2 < 0.06$

G_2 = Pade – G_1



G_1 has 'normal' q=0 slope, norm of 0.995

$G_2 \sim e^{-q^2/(0.02 fm^2)}$

corresponds to $\rho \sim e^{-r^2/(200 fm^2)}$

G_2 leads to large rms-radius despite small norm $\sim$ 0.005

Note: data are floating, solid and dotted curve give both excellent χ^2

(illustrates that *absolute* cross sections are *much* more valuable)

Solution: 3 ingredients

1. Use parameterization accessible in *both* q- and r-space

   SOG, Laguerre, MD (as in VDM fits)
   then can control behavior at large r
   can fit data + large-r constraint simultaneously

2. Constrain large-r behavior via physics knowledge

   for multi-constituent systems $\rho(r >>)$ given by wave function
   of least-bound constituent
   shape (not absolute norm) given by removal energy
   can be imposed at asymptotic r
   where $\rho(r) < 1\%$ of ρ(center)

3. Fit data to largest q possible

   full $G(q)$ helps also to fix tail of $\rho(r)$ (to some degree)

See PRC 89 (14) 012201

Return below to above points

1. Importance of considering r-space

some q-space parameterizations have serious problems

example: inverse polynomial (Bernauer): has pole at $q > q_{max}$

leads to (oscillatory) ρ to extremely large r

affects R

example: VDM fit of Lorenz et al.

$G(q)$ has pronounced maximum at $q > q_{max}$

leads to structure of $\rho(r)$ at large r

affects shape of $G(q < q_{min})$, affects extrapolation to $q = 0$

in addition: χ^2 factor 1.4 too large, does not really *fit* data

example: fit after conformal mapping (Lee et al.)

$G(q)$ for $q \to \infty \sim 3!$

does not even correspond to density

same problem as for popular power-series expansion in q^2

example: own Pade fit (discussed above)

disease only visible in r-space

2. Large-r constraint

consider Fock-component with smallest removal energy

for case of proton: $\pi^+ + n$ configuration, dominates large r

Shape of wave function given by Whittaker function $W_{-\eta,3/2}(2\kappa r)/r$

κ given by removal energy

Potential complication: relativistic effects

$G(q)$ not simply Fourier transform of $\rho(r)$

Relativistic corrections:

1. Determine $\rho(r)$ in Breit-frame, to account for Lorentz contraction

use as momentum transfer $\tilde{q}^2 = q^2/(1+\tau), \quad \tau = q^2/4M^2$

2. For composite systems boost operator depends on structure

various theoretical results (Licht, Mitra, Ji, Holzwarth,...), all of form

$G_e(q) \rightarrow G_e(q)(1+\tau)^\lambda$, λ=1 or 2

numerical test: λ=1 or 2 makes little difference for ρ at large r

is as expected, because large $r \sim$ low k, where rel. effects small

aside: correction fixes unphysical behavior at $r \sim 0$!

Allows for reliable calculation of *shape* of ρ at very large r

"Refinements" of model (not needed, nice consistency check)

allow also for $\Delta + \pi$ contribution

coefficients of various terms from Dziembowski e al.

include all states: $\pi^+ n$, $\pi^- p$, $\pi^- \Delta^{++}$, $\pi^+ \Delta^0$, $\pi^- \Delta^+$, $\pi^+ \Delta^-$

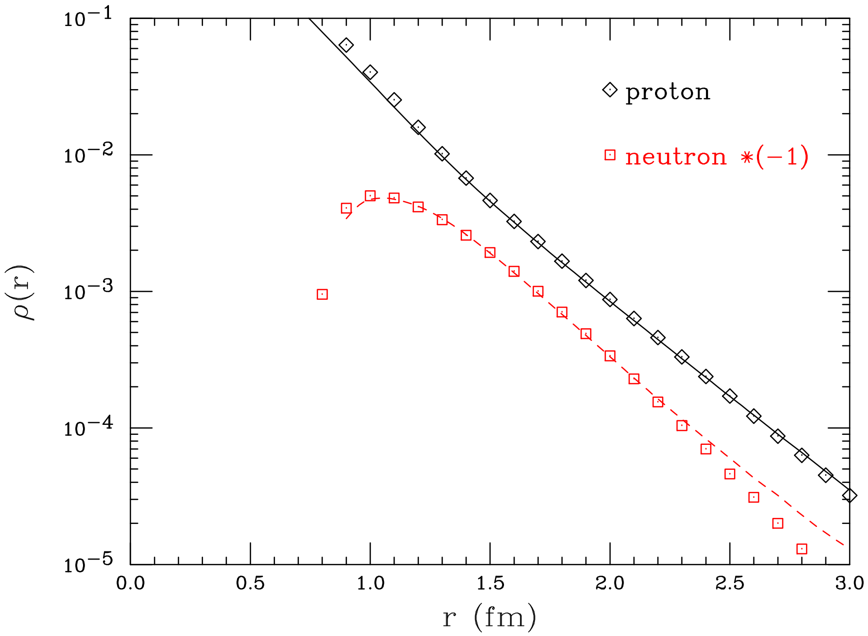
effect on p-tail: small, tail even a bit closer to ρ_{exp} at small r

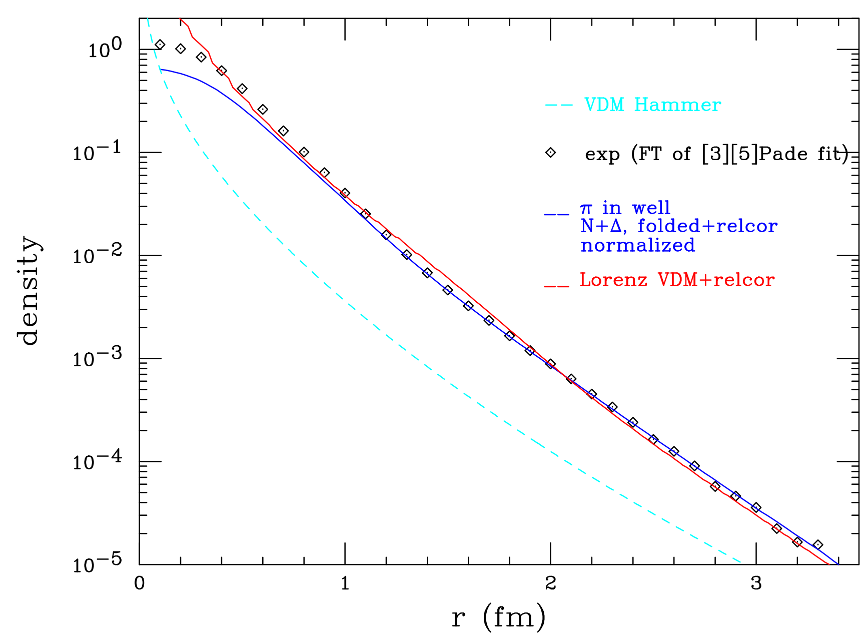
effect on n-tail: larger, gets close to ρ_{exp} with *same* parameters

nice consistency check

will ignore n since components $\neq \pi^- p$ too important

◇ $\rho_p(r)$, ◇ $-\rho_n(r)$, ——— shape tail, compares nicely to (new) VDM

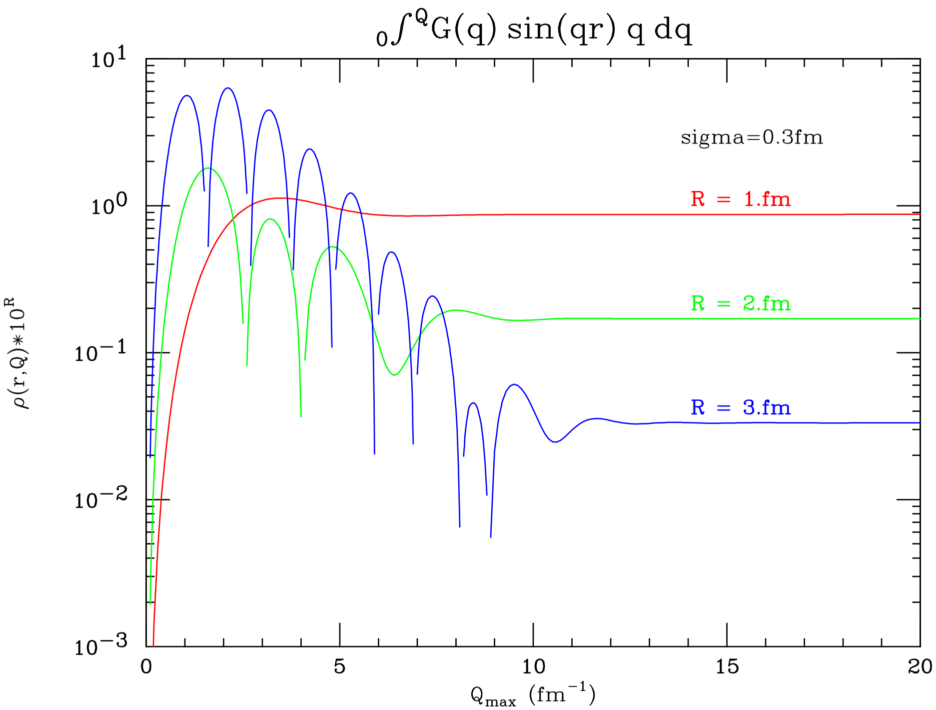




3. Fit data to maximum q

data up to $q_{max} = \infty$ would fix shape of entire ρ
for finite q_{max}: ρ at large r better constrained when using larger q_{max}

Pedagogical example: integral $\rho(r) = \int^{Q_{max}} G(q)....dq$ as function of Q_{max}



explains peculiar influence of large q on R (e.g. Lee et al., Kraus et al.)
for low q_{max} parameterized $G(q)$ implies strange ρ at medium r
data at larger q enforce more realistic behavior

BUT: large q_{max} only helpful, does not replace large-r constraint!

Analysis respecting above rules

Data used

- *world* (e,e) data up to 12 fm^{-1}
  - both cross sections and polarization data, 605 data points
- in general w/o Bernauer (problems with background subtraction)
- two-photon exchange corrections
  - needed to make G_{ep} from σ and P agree
  - includes both soft+hard photons, Melnitchouk+Tjon
- (relative) tail density for $r > 1.3 fm$

Parameterization for G_e and G_m

- sum of Laguerre polynomials (in r-space), most efficient
  - natural parameterization with quasi-exponential large-r fall-off
- equivalent results with SOG, VDM-type $G(q)$

Results

- average over various combinations of data sets
- floating or not of normalization

$R^{ch} = .886 \pm 0.008\ fm$ $\qquad$ $R^{m} = .858 \pm .024\ fm$

Conclusion: disagreement with μ-H confirmed

Deuteron

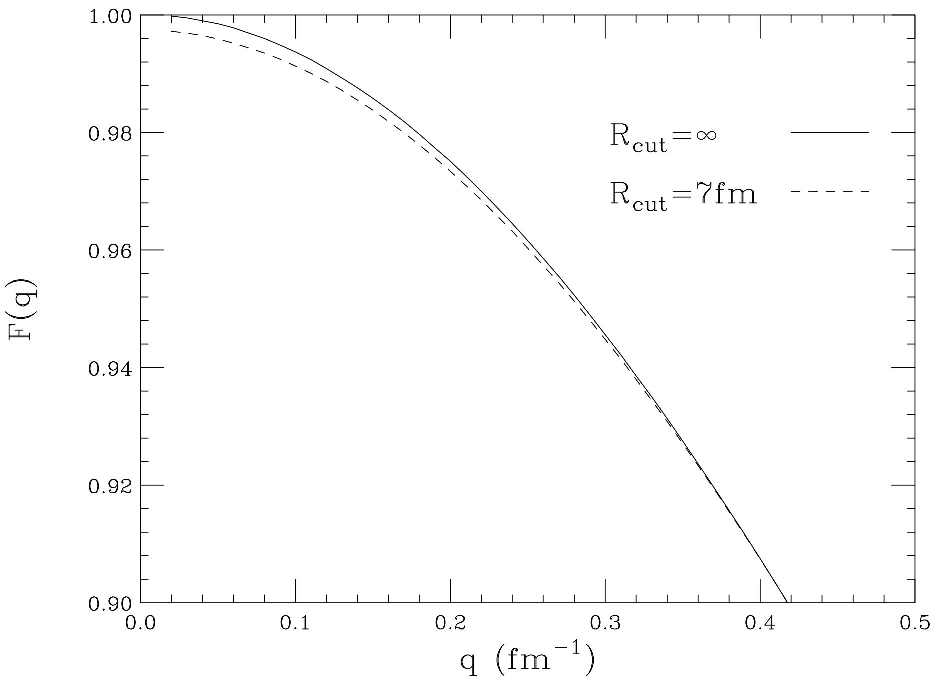
interest: comparison e-μ, exact calculations starting from V_{NN}

Problematic: large scatter of results in past

main problem: large-r tail (see above)

last 2% of R come from r >7 fm!

corresponding contribution in $G(q)$ not measurable



Added complication: 3 form factors, need T_{20} data

separation of C0 enhances uncertainty

Determination of R

use *world* data
use tail constraint

(e,e)	$2.130 \pm 0.010\, fm$
μH	$2.1289 \pm 0.0012 fm$ (prelim.)
a_{n-p}	$2.131 fm$

Find perfect agreement with μX data from Pohl et al.

agreement within .01fm significant given .04fm discrepancy for proton

Helium 4

Interest: comparison e-μ, exact calculations starting from V_{NN}

Great: ^{4}He data most precise of all light nuclei

Simple-most case: only *one* form factor

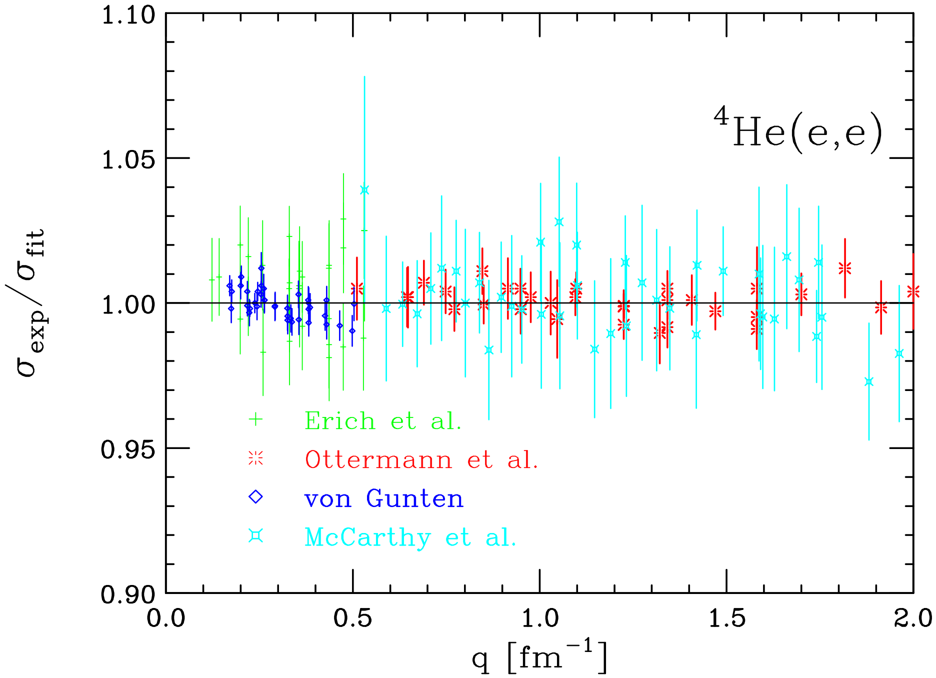
no error-enhancing separation needed

Most helpful: FDR analysis of *world* data on p -^{4}He scattering

determines residuum of closest singularity
corresponding to exchange scattering at 0°
yields *absolute* normalization of tail to $\pm$10%

Consequence: get most precise rms-radius of all nuclei

R=1.681±0.004 fm
μX value (Antognini et al.) well within error bar



Highly significant: difference (e,e)-μX < 0.25%
as compared to 4% for proton

Conclude: problem is not (e,e) vs. μX, problem is with proton

Conclusions

Determination of R from (e,e)

more difficult than appreciated

results more ambiguous than desirable

Source of problem

large-radius tail of density

affects shape of $G(q)$ below sensitivity range, affects extrapolation to $q = 0$

Non-solutions

- different mathematical/formal description
  data *are* compatible with large range of R
  shown by too many examples
- more data at lower q
  with realistic experimental errors sensitivity to R too small

Real solution

add physics *i.e.* knowledge on large-r shape

constrain shape of $G(q)$ at low q

$\longrightarrow$ reliable extrapolation to $q = 0$ where R extracted