Numerical Stochastic Perturbation Theory

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in collaboration with

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Outline

- Stochastic Quantization
- NSPT
  - Fermion contribution
- Applications:
  - Application: computation of critical mass with improved clover action.

Stochastic Quantization

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in Euclidean space-time

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To get the equilibrium we evolve the fields (configurations) according to *Langevin equation*

$$\frac{\partial\phi(x,t)}{\partial t} = -\frac{\delta S[\phi]}{\delta\phi(x,t)} + \eta(x,t)$$

where $\eta(x,t)$ is a random noise s.t.

$$\langle\eta(x,t)\rangle = 0 \qquad \langle\eta(x,t)\eta(x',t')\rangle = 2\delta(x-x')\delta(t-t')$$

Solved the Langevin equation at some initial condition, the observables of interest are obtained by means correlation functions

$$\langle \phi(x_1, t_1) \dots \phi(x_n, t_n) \rangle_\eta = \int \mathcal{D}[\eta] \, e^{-\frac{1}{4} \int dx' \, dt' \, \eta^2(x', t')} \phi(x_1, t_1) \dots \phi(x_n, t_n)$$

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The main assertion of stochastic quantization is

$$\lim_{t\to\infty}\langle\phi(x_1,t)\ldots\phi(x_n,t)\rangle_\eta = \langle\phi(x_1)\ldots\phi(x_n)\rangle$$

F. Di Renzo, L. Scorzato, *"Numerical Stochastic Perturbation Theory for full QCD"* (2004).

Consider Lattice Gauge Theory and try to apply Stochastic Quantization to the gauge fields

$$U_{x\mu} \longrightarrow U_{x\mu}(t;\eta)$$

put them into Langevin equation with i.c. $U_{x\mu}(0) = 1$

$$\frac{\partial U_{x\mu}(t,\eta)}{\partial t} = \left[i\nabla_{x\mu}S[U] - i\eta_{x\mu}(t)\right] U_{x\mu}(t,\eta)$$

the stochastic time can be discretized $t = n\epsilon$. The solution can be found as

$$U_{x\mu}(n+1,\eta) = e^{F_{x\mu}[U,\eta]} U_{x\mu}(n,\eta)$$

By expanding the link in a power series

$$U_{x\mu} \to 1 + \sum_{k>0} g^k U_{x\mu}^{(k)}$$

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Then we convert differential equations into integral ones and perform the integration numerically in a perturbative MonteCarlo simulation.

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Numerical Stochastic Perturbation Theory: Quark contributions

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$$\nabla_{x\mu} S \longrightarrow \nabla_{x\mu} S_G + \nabla_{x\mu} S_{\mathit{eff}}$$

The Langevin equation receives a contribution

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$$\text{Tr}\left[\nabla_{x\mu}(M)M^{-1}\right]$$

The trace can be stochastically evaluated inserting a gaussian source ξ s.t. $\langle \xi_i \xi_j \rangle = \delta_{ij}$:

$$\text{Tr}\left[\nabla_{x\mu}(M)M^{-1}\right] = Re\langle \xi^\dagger \nabla_{x\mu}(M) M^{-1} \xi \rangle_\xi$$

The Lie derivative can be analytically computed for almost all the actions while the inversion of M usually have to be evaluated by means numerical inversion.

Numerical Stochastic Perturbation Theory: Quark contributions

In NSPT the procedure for including fermions is the following

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The inversion of M^{-1} are iteratively constructed

$$\begin{aligned}
{M^{-1}}^{(0)} &= {M^{(0)}}^{-1} \\
{M^{-1}}^{(1)} &= {M^{(0)}}^{-1} M^{(1)} {M^{(0)}}^{-1} \\
&\dots \\
{M^{-1}}^{(n)} &= {M^{(0)}}^{-1} \left(\sum_{m=0}^{n-1} M^{(n-m)} {M^{(m)}}^{-1} \right)
\end{aligned}$$

The algorithm is then :

- compute $\psi^{(0)} = {M^{(0)}}^{-1}\xi$
- $\psi^{(n)} = {M^{(0)}}^{-1}\left[\sum_{m=0}^{n-1} M^{(n-m)}\psi^{(m)}\right]$
- find the solution via Euler integration

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One has to face only with the inversion on $M^{(0)}$ which is analytically known in some cases.

Even if there is no analytical solution the inversion of three level usually requires only few steps.

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In collaboration with Di Renzo, Brambilla and Guagnelli we want to calculate the clover coefficient beyond 1-loop, but as a preliminary check one has to control the efficiency of the program for the clover action.
Let's see a simple application of NSPT for computation of the renormalized critical mass of quark for 1-loop and 2-loop.

Fermion action

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S_W must be improved!

Symanzik's improvement - application

$$S_{eff} = S_0 + aS_1 + a^2 S_2 + \dots$$
$$\tilde{S}_{eff} = S_{eff} - aS_1 = S_0 + O(a^2) + \dots$$

A possible choice for improving S_W is to add

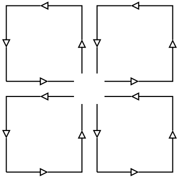
$$O = a^5 \sum_x c_{SW} \bar{\psi}(x) \frac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu}(x) \psi(x)$$

and obtain the *Sheikholeslami-Wolhert* action

$$S = S_W + a^5 c_{SW} \sum_x \bar{\psi}(x) \frac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu}(x) \psi(x)$$

In PT the coefficient c_{SW} can Taylor-expanded in power of the bare coupling

$$c_{SW} = c_{SW}^{(0)} + c_{SW}^{(1)} g_0^2 + c_{SW}^{(2)} g_0^4 + \dots$$



NSPT could be the good choice to compute clover-improved observables.

First check: computation of critical mass

The quark critical mass can be extracted from the propagator

$$S(q^2, m, g_0)^{-1} = i\not{p} + m - \Sigma(q^2, m, g_0)$$
$$m_{cr} = \Sigma(0, m, g_0) \simeq g_0^2 \Sigma^{(1)} + g_0^4 \Sigma^{(2)} + \ldots$$

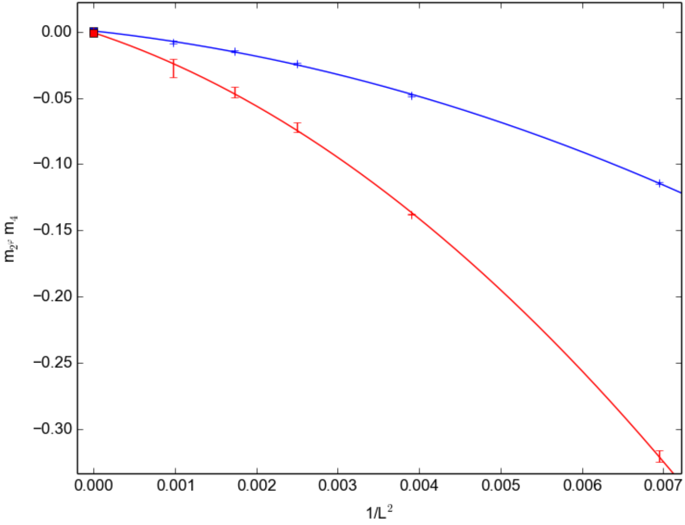
$$\Sigma^{(1)} \to c_{SW}^{(0)}$$
$$\Sigma^{(2)} \to c_{SW}^{(1)}$$

$$\delta\Sigma_c^{(k-loop)} = \Sigma_c^{(k-loop)} - dm^{(k-loop)}$$

In summary, simulations consisted in:

- Generation of gauge configurations with NSPT in quenched approximation considering two massless quark;
- Computation of fermionic propagators;
- Extrapolation of the critical mass (three extrapolation needed).

Critical mass: linear extrapolation

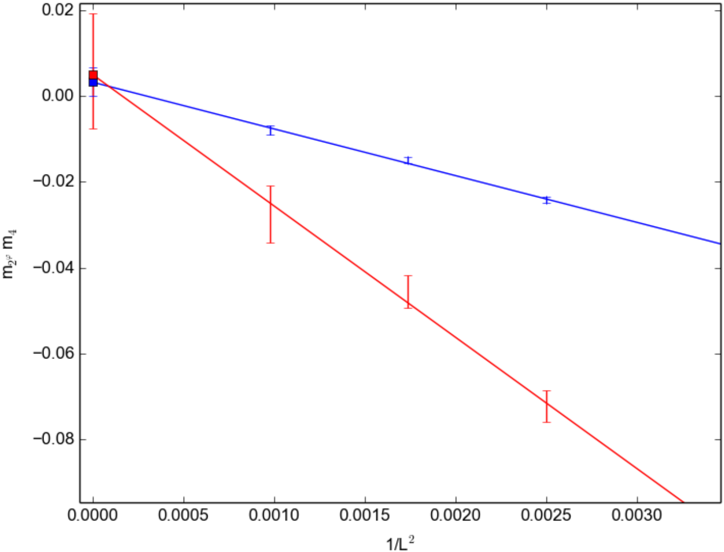


Linear lattice size $L = 12, 16, 20, 24, 32$

$$\delta\Sigma_c^{(1-loop)} = \Sigma_c^{(1-loop)} - dm^{(1-loop)} = 0.0007(21)$$

$$\delta\Sigma_c^{(2-loop)} = \Sigma_c^{(2-loop)} - dm^{(2-loop)} = 0.0004(76)$$

Critical mass: linear extrapolation



Linear lattice size $L = 20, 24, 32$

$$\delta\Sigma_c^{(1-loop)} = \Sigma_c^{(1-loop)} - dm^{(1-loop)} = 0.0033(34)$$

$$\delta\Sigma_c^{(2-loop)} = \Sigma_c^{(2-loop)} - dm^{(2-loop)} = 0.0058(133)$$

Conclusion

In this talk I present basics of NSPT and show a simple application.

- NSPT is an effective tool to perform lattice perturbative computations;
- NSPT can be applied to other topics, such that
  - matching between perturbative results obtained in a continuum regolarization with non perturbative one;
  - computing perturbative renormalization factors of bare parameters and operators;
  - Determination the so-called improvement coefficients for lattice actions and operators;

Actually the improved clover fermions are (partially) available in NSPT:

- the clover term is implemented;
- 2-loop computations are feasible since 1-loop c_{SW} is known;
- unquenched dynamic is almost ready.

Final goal of the project (in collaboration with F. Di Renzo, M. Brambilla and M. Guagnelli) is the computation of c_{SW} to higher orders.