Goldstone Bosons in Crystalline Chiral Phases

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International School of Nuclear Physics 2016
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Phase Diagram

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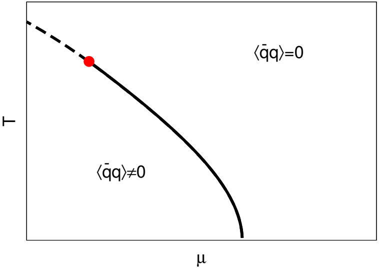
Focus on chiral symmetry

- Spontaneously broken in vacuum
- Order parameter:
  chiral condensate $\langle \bar{q} q \rangle$
- Believed to have first-order phase transition for low temperatures
- Critical endpoint



Most calculations: order parameter constant in space

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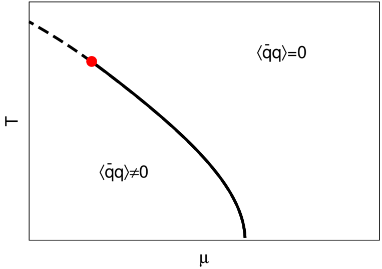

Most calculations: order parameter constant in space

What happens if we allow space dependence?

Inhomogeneous Phase in Nambu–Jona-Lasinio Model

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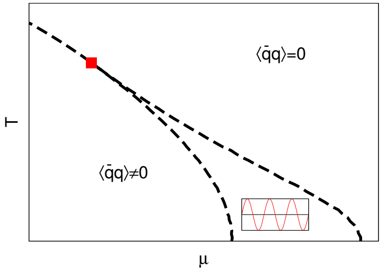
Nambu–Jona-Lasinio model

- Critical endpoint replaced by Lifschitz point
  [Nickel, PRD (2009)]
- First-order phase transition replaced by inhomogeneous region



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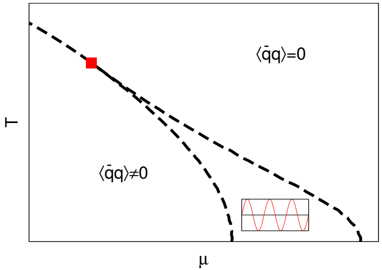

Here:

- Goldstone bosons from spontaneous broken symmetries (chiral and spatial)
- Important for transport properties (e.g. cooling in neutron stars)
- Could lead to instabilities (Landau-Peierls instability)

Nambu–Jona-Lasinio Model

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- NJL Lagrangian in chiral limit

$$\mathscr{L} = \overline{\psi}\, i\partial\!\!\!/\, \psi + G_S \left(\left(\overline{\psi}\psi\right)^2 + \left(\overline{\psi} i\gamma_5 \tau^a \psi\right)^2 \right)$$

- Derive thermodynamic properties from grand potential Ω
- Mean-field approximation

$$S(\vec{x}) = \langle \overline{\psi}\psi \rangle, \quad P(\vec{x}) = \langle \overline{\psi} i\gamma_5 \tau^3 \psi \rangle$$

  - keep space dependence, but neglect time dependence

$$\mathscr{L}_{MF} = \overline{\psi} S^{-1} \psi - V(\vec{x})$$

$$S^{-1} = i\partial\!\!\!/ - \underbrace{\left(-2G_S(S(\vec{x}) + i\gamma_5\tau^3 P(\vec{x}))\right)}_{=:\widehat{M}(\vec{x})}, \qquad V(\vec{x}) = G_S\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

Space Dependent Mass

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- Space dependent mass

$$M(\vec{x}) = -2G_S(S(\vec{x}) + iP(\vec{x}))$$

- Crystal with unit cell vectors $\vec{n}_i$, $i = 1, 2, 3$
- Periodicity in mass

$$M(\vec{x}) = M(\vec{x} + \vec{n}_i)$$

- Fourier transformation

$$M(\vec{x}) = \sum_{\vec{q}_k} M_{\vec{q}_k} e^{i\vec{q}_k \vec{x}}$$

- Wave vector $\vec{q}_k$ spans reciprocal lattice: $\vec{q}_k \vec{n}_i = 2\pi N_{ki}, \qquad N_{ki} \in \mathbb{Z}$

Propagator

- Write inverse Propagator

$$S^{-1} = \gamma^0(i\partial_0 - H(\vec{x}))$$

- Perform Fourier transformation on Hamiltonian

$$H(\vec{p}_m, \vec{p}_{m'}) = \begin{pmatrix} -\vec{\sigma}\vec{p}_m\delta_{\vec{p}_m,\vec{p}_{m'}} & -\sum_{\vec{q}_k} M_{\vec{q}_k}\delta_{\vec{q}_k,(\vec{p}_m-\vec{p}_{m'})} \\ -\sum_{\vec{q}_k} M_{\vec{q}_k}\delta_{\vec{q}_k,(\vec{p}_{m'}-\vec{p}_m)} & \vec{\sigma}\vec{p}_m\delta_{\vec{p}_m,\vec{p}_{m'}} \end{pmatrix}$$

$\rightarrow$ Matrix in momentum space

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$\rightarrow$ Matrix in momentum space

- Propagator has different incoming and outgoing momenta

$$S^{-1}(p_{out}, p_{in}) = \gamma^0 (p_0 \delta_{p_{out},p_{in}} - H(p_{out}, p_{in}))$$

Modulated Order Parameter

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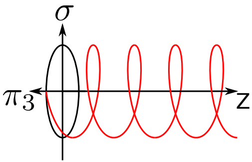
Chiral density wave (CDW)

$$\widehat{M}(\vec{x}) = M\, exp(i\gamma_5\tau^3\vec{q}\vec{x})$$



- Analytical diagonalization of Hamiltonian possible

$$H'(p,p') = U^\dagger H U = H'(p)\delta(p-p')$$

- Eigenvalues

$$E_\pm = \sqrt{\vec{p}^2 + M^2 + \vec{q}^2/4 \pm \sqrt{(\vec{p}\cdot\vec{q})^2 + \vec{q}^2 M^2}}$$

- *U*'s are rotation in chiral space [Dautry, Nyman, Nucl. Phys. A (1979)]

$$U = \exp(i\gamma_5\tau^3\vec{q}\vec{x}/2)$$

CDW Propagator

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CDW: analytic expression for propagator

- Inverse propagator

$$S^{-1}(p, p') = \gamma^0 \left(p_0 \delta(p - p') - H(p, p')\right)$$

- Apply chiral transformation

$$S = U S' U$$

- with

$$S'(k) = \frac{1}{N(k)} \left[A(k) + \gamma_5 \tau^3 B(k) + \gamma_\mu C^\mu(k) + \gamma_5 \tau^3 \gamma_\mu D^\mu(k) + \gamma_5 \tau^3 \gamma_\mu \gamma_\nu E^{\mu\nu}(k)\right]$$

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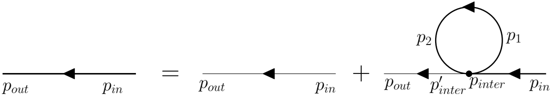
diagonal in momentum space

Gap Equation



- Gap equation

$$S^{-1}(p_{out}, p_{in}) = S_0^{-1}(p_{out}, p_{in}) - \Sigma(p_{out}, p_{in})$$

- Self energy

$$\Sigma(p_{out}, p_{in})$$
$$= 2G_S \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \left[\mathbb{1} \mathrm{Tr}\left(\mathbb{1} iS(p_1, p_2) \right) + i\gamma^5 \tau_3 \mathrm{Tr}\left(i\gamma_5 \tau^3 iS(p_1, p_2) \right) \right]$$
$$\delta(p_{out} + p_2 - p_{in} - p_1)$$

Gap Equation

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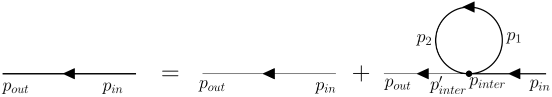


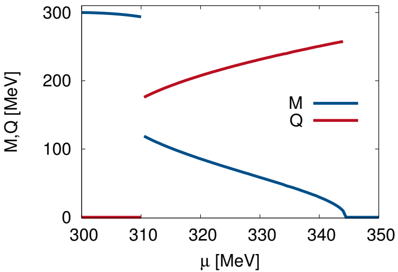
- Gap equation

$$S^{-1}(p_{out}, p_{in}) = S_0^{-1}(p_{out}, p_{in}) - \Sigma(p_{out}, p_{in})$$

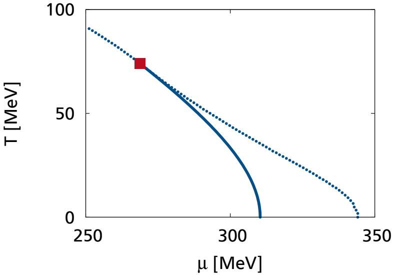
- Self energy

$$\begin{aligned}
&\Sigma(p_{out}, p_{in}) \\
&= 2G_S \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \left[\mathbb{1}\,\mathrm{Tr}\,(\mathbb{1}\, iS(p_1, p_2)) + i\gamma^5 \tau_3 \mathrm{Tr}\left(i\gamma_5 \tau^3 iS(p_1, p_2)\right)\right] \\
&\qquad \delta(p_{out} + p_2 - p_{in} - p_1) \\
&= iG_S\left[(\mathbb{1} - \gamma_5\tau_3)\delta(p_{out} - p_{in} - q) + (\mathbb{1} + \gamma_5\tau_3)\delta(p_{out} - p_{in} + q)\right] \int \frac{d^4 p_1}{(2\pi)^4} \,\mathrm{Tr} S'(p_1)
\end{aligned}$$

Results



300
200
100
0
M,Q [MeV]
300
310
320
330
340
350
μ [MeV]
M
Q



100
50
0
T [MeV]
250
300
350
μ [MeV]

Phonons and Goldstone Modes

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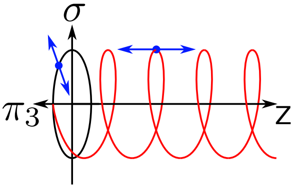
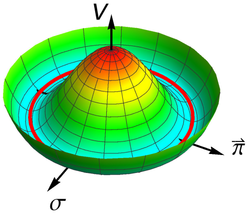
Inhomogeneous Phase



Rotational and translational symmetry broken

Homogeneous Phase

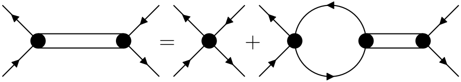


Chiral symmetry broken
Pions as Goldstone mode

Details in: [Lee, Nakano, Tsue, Tatsumi, Friman, PRD (2015)]

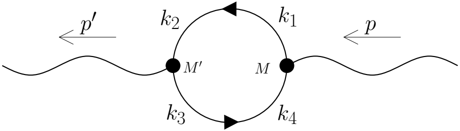
Mesons in NJL

- Start from Bethe-Salpeter equation



$$\hat{T} = \hat{K} + \hat{K}\hat{J}\hat{T}$$

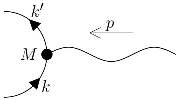
- Polarization loop $J_{M'M}(p', p)$



$$\propto \mathrm{Tr}\left[\Gamma_{M'} S \Gamma_M S\right]$$

Diagonalization in Momentum Space

$(\Gamma_M(p))_{k',k} =$



- Standard vertices from Lagrangian

$$\left(\Gamma_{\pi_3}(p)\right)_{k',k} = i\gamma_5\tau^3\delta(k - k' - p)$$

- Transformation on the vertex

$$\Gamma'_{\pi_3}(p) = \Gamma_M(k) W_{M\pi_3}(k, p)$$

- Transformed polarization loop

$$\underline{W}^\dagger(p', k')\underline{J}(k', k)\underline{W}(k, p) = \underline{J}'(p')\delta(p' - p)$$

Diagonalization in Momentum Space

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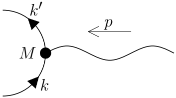

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- Transformed polarization loop

$$\underline{W}^\dagger(p',k')\underline{J}(k',k)\underline{W}(k,p) = \underline{J}'(p')\delta(p'-p) = \begin{pmatrix} J'_{\sigma\sigma}(p') & J'_{\sigma\pi_3}(p') \\ J'_{\pi_3\sigma}(p') & J'_{\pi_3\pi_3}(p') \end{pmatrix}\delta(p'-p)$$

Pions in the Inhomogeneous Phase

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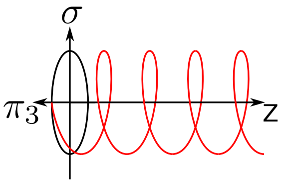
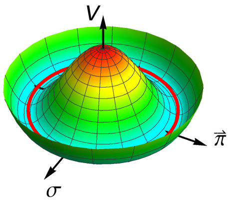




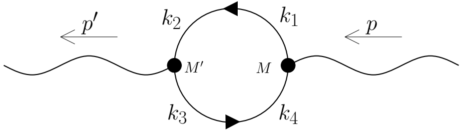
- For the CDW the standard π_3 vertex ($\Gamma_{\pi_3} = i\gamma_5\tau^3$) is no NG boson
- Instead

$$\Gamma_{\pi_1} = i\gamma_5\tau^1, \qquad \Gamma_{\pi_2} = i\gamma_5\tau^2, \qquad \Gamma'_{\pi_3}(z) = -\mathbb{1}\sin qz + i\gamma_5\tau^3\cos qz$$

- In momentum space

$$\Gamma'_{\pi_3}(p) = \frac{1}{2}\left[i\Gamma_\sigma(p-q) - i\Gamma_\sigma(p+q) + \Gamma_{\pi_3}(p-q) + \Gamma_{\pi_3}(p+q)\right]$$

Full Polarization Loop



$$J'_{\pi_3\pi_3}(p',p) = i\left(\prod_{j=1}^{4}\int\frac{d^4k_j}{(2\pi)^4}\right)\mathrm{Tr}\left[\left(\bar{\Gamma}'_{\pi_3}(p')\right)_{k_4,k_1} iS(k_1,k_2)\left(\Gamma'_{\pi_3}(p)\right)_{k_2,k_3} iS(k_3,k_4)\right]$$

- Inserting $S = US'U$
- Integrating over internal momenta

$$J'_{\pi_3\pi_3}(p',p) = i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[i\gamma_5\tau^3 iS'(k+p')i\gamma_5\tau^3 iS'(k)\right]\delta(p'-p)$$

Goldstone Mode

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- Poles in meson propagator (on-shell)

$$D_M'^{-1}(p_0', \vec{p}') = 0$$
$$\propto \det\left(\mathbb{1} - 2G_S \underline{J}'(p')\right)$$

- Dispersion relation $\quad p_0' \equiv p_0'(\vec{p}')$
- Goldstone mode $\quad p_0' = 0$
- For vanishing $\vec{p}'$

$$\lim_{p' \to 0} J'_{\sigma\pi_3}(p') = \lim_{p' \to 0} J'_{\pi_3\sigma}(p') = 0$$
$$\lim_{p = p' \to 0} (1 - 2G_S J'_{\pi_3\pi_3}(p')) = 0$$

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Goldstone boson

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Goldstone boson

Transformation needed

$$\Gamma'_{\pi_3}(p=0) = \frac{1}{2}\left[i\Gamma_\sigma(-q) - i\Gamma_\sigma(+q) + \Gamma_{\pi_3}(-q) + \Gamma_{\pi_3}(+q)\right]$$

Summary and Outlook

Mean-Field Calculations:

- Crystalline phase replaces first order phase transition and critical endpoint in phase diagram

Bosonic excitations

- Explicit construction of Goldstone modes in CDW
- Difficult due to non-diagonal structure of propagator
- Simplified by chiral transformations
- Goldstone mode identified

Outlook

- Calculate dispersion relations of Goldstone Bosons
- Derive transport properties
- Applications for beyond mean-field calculations

Thank you

Full CDW propagator

$$S'(k) = \frac{1}{N}\left[A + \gamma^5\tau^3 B + \gamma_\mu C^\mu + \gamma^5\tau^3\gamma_\mu D^\mu + \gamma^5\tau^3\gamma_\mu\gamma_\nu E^{\mu\nu}\right]$$

$$A(k) = M\left(k^2 - M^2 - \frac{1}{4}q^2\right), \quad B(k) = -Mq\cdot k$$

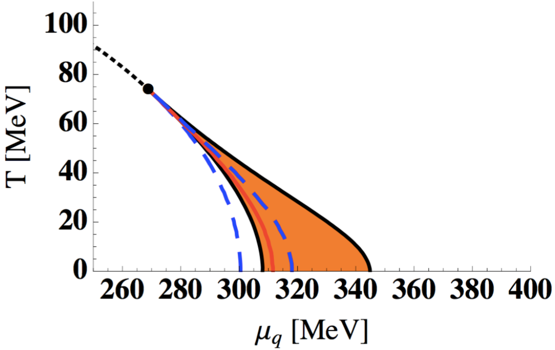
$$C_\mu(k) = k_\mu\left(k^2 - M^2 + \frac{1}{4}q^2\right) - \frac{1}{2}q_\mu\; q\cdot k$$

$$D_\mu(k) = -k_\mu\; q\cdot k + \frac{1}{2}q_\mu\left(k^2 + M^2 + \frac{1}{4}q^2\right)$$

$$E_{\mu\nu}(k) = q_\mu k_\nu M$$

$$N(k) = \left(k^2 - M^2 - \frac{1}{4}q^2\right)^2 + q^2k^2 - (q\cdot k)^2$$

Spinodials



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