Flavor states for oscillating neutrinos

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Summary

Motivations

- CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;

- Theoretical interest: origin of mixing in the Standard Model;

- Bargmann superselection rule*: coherent superposition of states with different masses is not allowed in non-relativistic QM;

- Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles†; oscillation formulas‡;

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*V.Bargmann, Ann. Math. (1954); D.M.Greenberger, Phys. Rev. Lett. (2001).
†C.W.Kim and A.Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood, 1993). C.Giunti, J. Phys. G (2007).
‡M.Beuthe, Phys. Rep. (2003).

Flavor states vs mass states

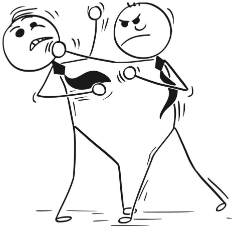
Flavor states

M. Blasone, G. Vitiello (1995)

C. Ji et al. (2002)

C. Lee (2017)

⋮



Mass states

R. E. Shrock (1980)

C. Giunti (2005)

C. Kim et al. (2007)

⋮

Flavor states and Poincaré group

- Particles states in relativistic quantum field theory (QFT) are usually assumed to belong to unitary irreducible representations of Poincaré group[*]
- Flavor states, not having a definite mass, do not fit in such a scheme
- A possibility is to consider flavor states as belonging to irreducible representations of an extended Poincaré group [†]
- Another possibility is that Poincaré symmetry is spontaneously broken when field mixing is dynamically generated [‡].

[*] V. Bargmann, E.P. Wigner, Proc. Natl. Acad. Sci. U.S.A. (1948).

[†] A.E Lobanov, Ann. Phys. **403**, 82 (2019).

[‡] M. B., P. Jizba, N.E Mavromatos and L.Smaldone. Phys. Rev. D (2019)

QFT of fermion mixing

Neutrino oscillations in QM *

Pontecorvo mixing relations

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle \,+\, \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle \,+\, \cos\theta\,|\nu_2\rangle$$

– Time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1 t}\,|\nu_1\rangle \,+\, \sin\theta\, e^{-iE_2 t}\,|\nu_2\rangle$$

– Flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 \,=\, 1 - \sin^2 2\theta\, \sin^2\left(\frac{\Delta E}{2}t\right) \,=\, 1 - P_{\nu_e\to\nu_\mu}(t)$$

– Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 \,+\, |\langle\nu_\mu|\nu_e(t)\rangle|^2 \,=\, 1$$

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*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

Mixing of neutrino fields

– Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

ν_1, ν_2 are fields with definite masses.

– Mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1\,(i\,\not{\partial} - m_1)\,\nu_1 + \bar{\nu}_2\,(i\,\not{\partial} - m_2)\,\nu_2$$

and

$$\mathcal{L} = \bar{\nu}_e\,(i\,\not{\partial} - m_e)\,\nu_e + \bar{\nu}_\mu\,(i\,\not{\partial} - m_\mu)\,\nu_\mu - m_{e\mu}\,(\bar{\nu}_e\,\nu_\mu + \bar{\nu}_\mu\,\nu_e)$$

with

$$m_e = m_1\cos^2\theta + m_2\sin^2\theta,\; m_\mu = m_1\sin^2\theta + m_2\cos^2\theta,\; m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta.$$

– ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \Big[u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \;+\; v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \Big], \quad i = 1, 2.$$

– Anticommutation relations:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\,; \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}\delta_{ij}$$

– Orthonormality and completeness relations:

$$u^r_{\mathbf{k},i}(t) \;=\; e^{-i\omega_{k,i} t}\, u^r_{\mathbf{k},i} \quad ; \quad v^r_{\mathbf{k},i}(t) \;=\; e^{i\omega_{k,i} t}\, v^r_{\mathbf{k},i} \quad ; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2}$$

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha *}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha *}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta}\ .$$

– Fock space for ν_1, ν_2:

$$\mathcal{H}_{1,2} = \Big\{ \alpha^\dagger_{1,2}\ ,\ \beta^\dagger_{1,2}\ ,\ |0\rangle_{1,2} \Big\}.$$

– Vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$.

Rotation

– Pontecorvo mixing can be seen as arising by the application to the vacuum state $|0\rangle_{1,2}$ of the rotated operators:

$$
\begin{aligned}
R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},1}R(\theta) &= \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1} + \sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2},\\
R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},2}R(\theta) &= \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2} - \sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1},
\end{aligned}
$$

and similar ones for $\beta^{r\dagger}_{\mathbf{k},i}$.

– The generator $R(\theta)$ is:

$$
R(\theta) \equiv \exp\left\{\theta\sum_{\mathbf{k},r}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right) - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)\right]\right\},
$$

The above unitary operator leaves the vacuum invariant:
$R(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}$

Consider the action of the rotation on the field ν_1 for example:

$$R^{-1}(\theta)\nu_1(x)R(\theta) = \cos\theta\,\nu_1(x)$$

$$+\sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\left(\alpha^r_{\mathbf{k},2}\,u^r_{\mathbf{k},1}(t)+\beta^{r\dagger}_{\mathbf{k},2}\,v^r_{-\mathbf{k},1}(t)\right),$$

• Problem in the last term in the r.h.s. which appears as the expansion of the field in the “wrong” basis.

Bogoliubov transformation

– We can recover the wanted expression by means of a Bogoliubov transformation:

$$\tilde{\alpha}^{r\dagger}_{\mathbf{k},i} = \cos\Theta_{\mathbf{k},i}\, \alpha^{r\dagger}_{\mathbf{k},i} - \epsilon^r\, \sin\Theta_{\mathbf{k},i}\, \beta^{r}_{-\mathbf{k},i},$$

$$\tilde{\beta}^{r\dagger}_{-\mathbf{k},i} = \cos\Theta_{\mathbf{k},i}\, \beta^{r\dagger}_{-\mathbf{k},i} + \epsilon^r\, \sin\Theta_{\mathbf{k},i}\, \alpha^{r}_{\mathbf{k},i}, \qquad i = 1, 2,$$

with $\tilde{\alpha}^{r\dagger}_{\mathbf{k},i} \equiv B^{-1}_i(\Theta_i)\, \alpha^{r\dagger}_{\mathbf{k},i}\, B_i(\Theta_i)$, etc..

– Generator

$$B_i(\Theta_i) = \exp\left\{ \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \Theta_{\mathbf{k},i}\; \epsilon^r \left[\alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i} \right] \right\}.$$

Let us see this for the field ν_1.

$$B_2^{-1}(\Theta_2)\, R^{-1}(\theta)\, \nu_1(x)\, R(\theta)\, B_2(\Theta_2) =$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\tilde{\alpha}^r_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t) + \tilde{\beta}^{r\dagger}_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t) \right)$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\alpha^r_{\mathbf{k},2}\, \tilde{u}^r_{\mathbf{k},1}(t) + \beta^{r\dagger}_{\mathbf{k},2}\, \tilde{v}^r_{-\mathbf{k},1}(t) \right) \,,$$

where

$$\tilde{u}^r_{\mathbf{k},1}(t) = \cos\Theta_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t) + \epsilon^r \sin\Theta_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t) \,,$$

$$\tilde{v}^r_{-\mathbf{k},1}(t) = \cos\Theta_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t) - \epsilon^r \sin\Theta_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t) \,.$$

For

$$\tilde{\Theta}_{\mathbf{k},2} = \cos^{-1}\left(u^{r\dagger}_{\mathbf{k},2}(t) u^r_{\mathbf{k},1}(t)\right)$$

the above Bogoliubov transformation implements the mass shift

$$\Delta m = m_2 - m_1$$

such that $\tilde{u}^r_{\mathbf{k},1}(t) = u^r_{\mathbf{k},2}(t)$ and $\tilde{v}^r_{-\mathbf{k},1}(t) = v^r_{-\mathbf{k},2}(t)$.

• The action of $B_2^{-1}(\tilde{\Theta}_2)\, R^{-1}(\theta)$ produces the desired transformation (rotation) of the field ν_1.

– Similar reasoning for ν_2, using $B_1^{-1}(\tilde{\Theta}_1)\, R^{-1}(\theta)$.

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

can be written as[†]

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\, \nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\Big|_{\theta=0} = \nu_2^\alpha$.

[†] M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\;{}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty} e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\;\ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \;\;\equiv\;\; \sum_{r,s} |\; v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\;|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite $\sharp$ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite $\sharp$ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

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*F.Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (W. Sc., 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_\mathbf{k}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_\mathbf{k}||U_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_\mathbf{k}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; _{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_\mathbf{k}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

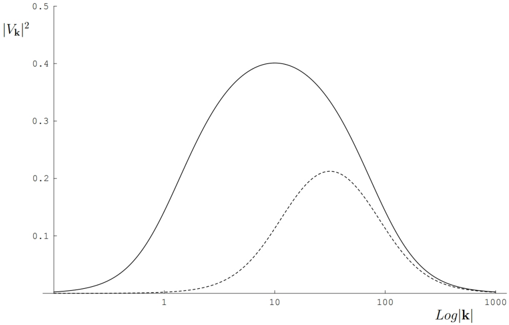
– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensates: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \;=\; u^{r\dagger}_{\mathbf{k},2}u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\; u^{r\dagger}_{\mathbf{k},1}v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

Mixing generator function of m_1, m_2, and θ. Try to disentangle the mass dependence from the one by the mixing angle.

Let us define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\ \epsilon^r \left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1,2$$

Since $[B_1, B_2] = 0$ we put

$$B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$$

*M.B., M.V.Gargiulo and G.Vitiello, Phys. Lett. B (2017)

• We find:

$$G_\theta \;=\; B(\Theta_1, \Theta_2) \;\; R(\theta) \;\; B^{-1}(\Theta_1, \Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k} \;\cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}} \;\sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})$$

The $B_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are Bogoliubov transformations implementing a mass shift, and $R(\theta)$ is a rotation.

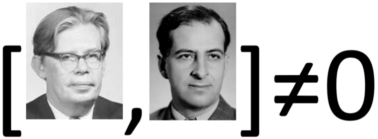
– Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} \;\;=\;\; \prod_{\mathbf{k},r,i} \Big[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\Big]|0\rangle_{1,2}$$

$$R^{-1}(\theta)|0\rangle_{1,2} \;\;=\;\; |0\rangle_{1,2}\;.$$

Bogoliubov vs Pontecorvo

Bogoliubov and Pontecorvo do not commute!



As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \;=\; |0\rangle_{1,2} \;+\; \Big[B(m_1, m_2)\,,\; R^{-1}(\theta)\Big]\;|\widetilde{0}\rangle_{1,2}$$

- Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\;\tilde{V}_{\mathbf{k}} \sum_r \epsilon^r \Big(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$.

Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\nu}_m \left(i \not\partial - M_d \right) \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge Q= total charge.

  – Consider now the $SU(2)$ transformation:

$$\nu_m' = e^{i\alpha_j \tau_j} \nu_m \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

*M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} = i\alpha_j\,\bar{\nu}_m\,[\tau_j, M_d]\,\nu_m = -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} = \bar{\nu}_m\,\gamma^\mu\,\tau_j\,\nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] = i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$$

– Casimir operator proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$Q_1 = \frac{1}{2}Q + Q_{m,3} = \int d^3\mathbf{x}\,\nu_1^\dagger(x)\,\nu_1(x)$$

$$Q_2 = \frac{1}{2}Q - Q_{m,3} = \int d^3\mathbf{x}\,\nu_2^\dagger(x)\,\nu_2(x).$$

These are the flavor charges in the absence of mixing.

The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \left(i \not\partial - M \right) \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu_f' = e^{i\alpha_j \tau_j} \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x}\, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i\, \epsilon_{jkl}\, Q_{f,l}(t).$$

– Casimir operator proportional to the total charge $C_f = C_m = \frac{1}{2} Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$Q_e(t) \;\equiv\; \frac{1}{2}Q \,+\, Q_{f,3}(t) \,=\, \int d^3\mathbf{x}\; \nu_e^\dagger(x)\,\nu_e(x)$$

$$Q_\mu(t) \;\equiv\; \frac{1}{2}Q \,-\, Q_{f,3}(t) \,=\, \int d^3\mathbf{x}\; \nu_\mu^\dagger(x)\,\nu_\mu(x)$$

where $Q_e(t) \,+\, Q_\mu(t) \,=\, Q$.

– We have:

$$Q_e(t) \;=\; \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) \;=\; \sin^2\theta\; Q_1 + \cos^2\theta\; Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\equiv\; \int d^3\mathbf{x} \; :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^{r}_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^{r}_{-\mathbf{k},\sigma}(t) \right) , \quad \sigma = e, \mu.$$

Here $:: \ldots ::$ denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv A - {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\,|0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle = |\nu^r_{\mathbf{k},\sigma}\rangle$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$

$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}\, t\right) - |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}\, t\right)$$

$$\mathcal{Q}_{\mathbf{k},\mu}(t) = |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}\, t\right) + |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0$.

*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P = \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle + \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P = -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle + \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}| :Q_e(0): |\nu^r_{\mathbf{k},e}\rangle_P = \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\,\sin^2\theta\cos^2\theta < 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†]M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Weak process states

A more elaborate choice are the *weak process states*[‡]:

$$|\nu_\sigma^r\rangle_{WP} \;\equiv\; \sum_j \mathcal{A}_{\sigma j}\, |\nu^r_{\mathbf{k}_j, j}\rangle$$

with

$$\mathcal{A}_{\sigma j} \;=\; \langle \nu_j \, l_\sigma^+ \, P_F | \hat{S} | P_I \rangle$$

- Once more, these are not flavor eigenstates

[‡]C. Giunti and C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford Univ. Press, 2007)

Other results

- Rigorous mathematical treatment for any number of flavors [*]
- Three flavor fermion mixing: CP violation[†];
- QFT spacetime dependent neutrino oscillation formula[‡];
- Boson mixing[§];
- Majorana neutrinos[¶];

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[*] K. C. Hannabuss and D. C. Latimer, J. Phys. A (2000); J. Phys. A (2003);
[†] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)
[‡] M.B., P. Pires Pachêco and H. Wan Chan Tseung, Phys. Rev. **D**, (2003).
[§] M.B., A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. **D**(2001); M.Binger and C.R.Ji. Phys. Rev. **D**(1999); C.R.Ji and Y.Mishchenko, Phys. Rev. **D**(2001); Phys. Rev. **D**(2002).
[¶] M.B. and J.Palmer, Phys. Rev. **D** (2004)

- Flavor vacuum and cosmological constant[*]
- Flavor vacuum induced by condensation of D-particles.[†]
- Geometric phase for mixed particles[‡].

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[*] M.B., A.Capolupo, S.Capozziello, S.Carloni and G.Vitiello Phys. Lett. A (2004);
[†] N.E.Mavromatos and S.Sarkar, New J. Phys. (2008); N.E.Mavromatos, S.Sarkar and W.Tarantino, Phys. Rev. D (2008); Phys. Rev. D (2011).
[‡] M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999)

Dynamical generation of flavor mixing

• The non trivial nature of flavor vacuum should result from a SSB process;

• We consider dynamical symmetry breaking in a model with chiral symmetry*

• The symmetry breaking pattern associated to the flavor vacuum condensate is $SU(2)_L \times SU(2)_R \times U(1)_V \longrightarrow U(1)_V$.

*M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2019)

Neutrino mixing in accelerated proton decay

Motivations

- Testing the consistency of QFT in curved background by comparing the decay rate of accelerated protons (*inverse β decay*) in the inertial and comoving frames: a 'theoretical check" of the Unruh effect*

- Clarifying some conceptual issues concerning the inverse β decay in the context of neutrino mixing†

- Investigating the dichotomy between mass and flavor neutrinos as fundamental "objects" in QFT.

*G. E. A. Matsas and D. A. T. Vanzella, Phys. Rev. D (1999)

†D. V. Ahluwalia, L. Labun and G. Torrieri, Eur. Phys. J. A (2016)

The Unruh effect[‡]

...the behavior of particle detectors under acceleration a is investigated where it is shown that an accelerated detector even in flat spacetime will detect particles in the vacuum...

...This result is exactly what one would expect of a detector immersed in a thermal bath of temperature

$$T_{\rm U} = a/2\pi$$

[‡] W.G.Unruh, Phys. Rev. D (1976)

The Unruh effect

- Rindler coordinates

$$x^0 = \xi \sinh\eta, \quad x = \xi\cosh\eta$$

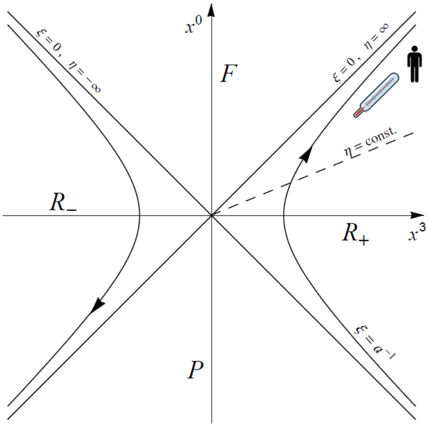
- Rindler vs Minkowski

$$ds^2_{\mathrm{M}} = \left(dx^0\right)^2 - (dx)^2$$

$$\Longrightarrow \; ds^2_{\mathrm{R}} = \xi^2 d\eta^2 - d\xi^2$$

- Rindler worldline

$$\eta = a\tau, \quad \xi = \text{const} \equiv a^{-1}$$



- Minkowski vacuum is a thermal bath for the Rindler observer

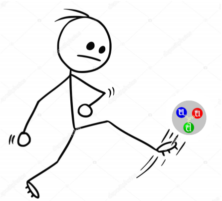
$$\langle 0_{\mathcal{M}}|\widehat{N}(\omega)|0_{\mathcal{M}}\rangle = \frac{1}{e^{a\,\omega/T_{\mathrm{U}}} + 1}$$

Decay of accelerated particles

Decay properties are not universal[§]

$$\tau_{proton} \gg \tau_{universe} \sim 10^{10}\,\text{yr}$$

However, if we "kick" the proton...



$$p \rightarrow n + e^+ + \nu_e$$

...the proton decay is kinematically allowed!

acceleration	lifetime
a_{LHC}	$\tau_p \sim 10^{3\times10^8}\,\text{yr}$
a_{pulsar}	$\tau_p \sim 10^{-1}\,\text{s}$

[§] R. Muller, Phys. Rev. D (1997)

Accelerated proton decay and existence of Unruh effect*

Basic assumptions:

- Massless neutrino
- $|\mathbf{k}_e| \sim |\mathbf{k}_{\nu_e}| \ll M_{p,n}$
- Current-current Fermi theory

$$\widehat{S}_I = \int d^2x \sqrt{-g}\, \widehat{j}_\mu \left(\widehat{\overline{\Psi}}_\nu \gamma^\mu \widehat{\Psi}_e + \widehat{\overline{\Psi}}_e \gamma^\mu \widehat{\Psi}_\nu \right)$$

$$\widehat{j}^\mu = \widehat{q}(\tau)\, u^\mu\, \delta\big(u - a^{-1}\big)\,, \quad \widehat{q}(\tau) = e^{i\widehat{H}\tau}\, \widehat{q}_0\, e^{-i\widehat{H}\tau}$$

$$\widehat{H}\,|n\rangle = m_n\,|n\rangle\,, \quad \widehat{H}\,|p\rangle = m_p\,|p\rangle\,, \quad G_F = |\langle p|\, \hat{q}_0\,|n\rangle|$$

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*G.E.A. Matsas and D.A.T. Vanzella, Phys. Rev. D (1999); D.A.T. Vanzella and G.E.A. Matsas, Phys. Rev. D (2000); Phys. Rev. Lett. (2001).

• Laboratory frame

$$p \to n + e^{+} + \nu_e$$

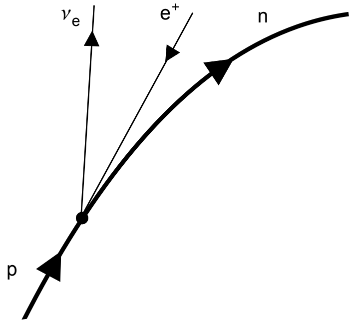

Figure 1: The decay occurs since the acceleration supplies the p-n rest mass difference

- Comoving frame

$p + e \rightarrow n + \nu_e$ $\qquad$ $p + \bar{\nu}_e \rightarrow n + e^+$ $\qquad$ $p + e + \bar{\nu}_e \rightarrow n$

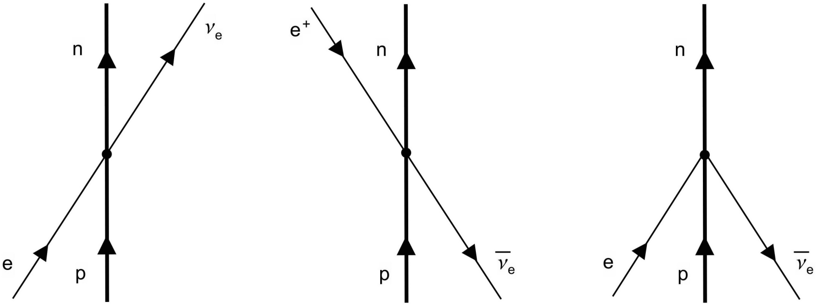

Figure 2: The decay occurs since p interacts with the Unruh thermal bath of e^- and ν_e

- Tree-level transition amplitude

$$\mathcal{A}^{p\to n} = \langle n| \otimes \langle e^{+}_{k_e\,\sigma_e}, \nu_{k_\nu\,\sigma_\nu}| \widehat{S}_I\, |0\rangle \otimes |p\rangle$$

- Differential transition rate

$$\frac{d^2\mathcal{P}^{p\to n}_{in}}{dk_e dk_\nu} = \frac{1}{2}\sum_{\sigma_e=\pm}\sum_{\sigma_\nu=\pm} |\mathcal{A}^{p\to n}|^2$$

- Scalar decay rate (inertial frame)

$$\Gamma^{p\to n}_{in} \equiv \frac{\mathcal{P}^{p\to n}_{in}}{T} = \frac{4G_F^2 a}{\pi^2 e^{\pi\Delta m/a}} \int_0^\infty d\tilde{k}_e \int_0^\infty d\tilde{k}_\nu K_{2i\Delta m/a}\left[2\left(\tilde{\omega}_e + \tilde{\omega}_\nu\right)\right]$$

- Scalar decay rate (comoving frame)

$$\Gamma_{com}^{p\to n} = \Gamma_{(i)}^{p\to n} + \Gamma_{(ii)}^{p\to n} + \Gamma_{(iii)}^{p\to n}$$

$$= \frac{G_F^2\, m_e}{a\, \pi^2\, e^{\pi \Delta m/a}} \int_{-\infty}^{+\infty} d\omega\, \frac{K_{i\omega/a+1/2}(m_e/a) K_{i\omega/a-1/2}(m_e/a)}{\cosh\left[\pi\left(\omega - \Delta m\right)/a\right]}$$

- Result (tree level): $\Gamma_{in}^{p\to n} = \Gamma_{com}^{p\to n}$.

Conclusion[*]

The Unruh effect is **mandatory** for the General Covariance of QFT

- Generalization to 4D with **massive neutrino**[†]: similar (analytical) results.

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[*] D. A. T. Vanzella and G. E. A. Matsas, Phys. Rev. Lett. (2001).

[†] H. Suzuki and K. Yamada, Phys. Rev. D (2003).

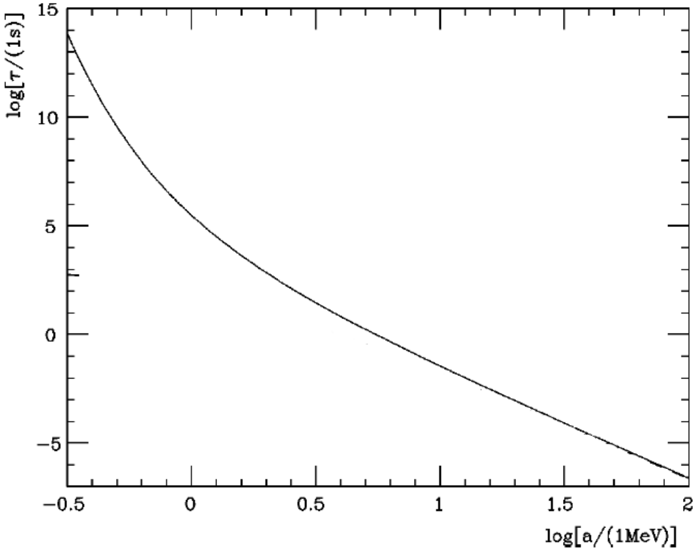


Figure 3: The mean proper lifetime τ of proton versus its proper acceleration a.

Inverse β-decay and neutrino mixing

– Neutrino mixing in the inverse β-decay *:

"*In the laboratory frame, the interaction is the electroweak vertex, hence neutrinos are in* ***flavor eigenstates****. In the comoving frame, the proton interacts with neutrinos in Rindler states, which display an effective thermal weight and are* ***mass eigenstates***".

"*...if charge eigenstates were the asymptotic states also in the accelerating frame, the* ***thermality*** *of the Unruh effect would be* ***violated****".*

"*...we conclude that the rates in the two frames* ***disagree*** *when taking into account neutrino mixings*".

*D. V. Ahluwalia, L. Labun and G. Torrieri, Eur. Phys. J. A (2016)

Non-thermal Unruh effect for mixed neutrinos*

When flavor mixing for an accelerated observer is considered, the two Bogoliubov transformations involved:

$$\phi_{\mathcal{R}} \overset{\text{thermal Bogol. } (a)}{\longrightarrow} \phi_{\mathcal{M}} \Rightarrow \text{condensate in } |0_{\mathcal{M}}\rangle$$

$$\phi_1, \phi_2 \overset{\text{mixing Bogol. } (\theta)}{\longrightarrow} \phi_e, \phi_\mu \Rightarrow \text{condensate in } |0_{e,\mu}\rangle$$

combine with each other.

*M. B., G. Lambiase and G. Luciano, Phys. Rev. D (2017)

The Unruh spectrum for mixed neutrinos

$$ {}_{\rm M}\langle 0|\mathcal{N}(\theta,\omega)|0\rangle_{\rm M} \;=\; \underbrace{\frac{1}{e^{a\,\omega/T_{\rm U}}+1}}_{Thermal\ spectrum} \;+\; \underbrace{\sin^2\theta\left\{\mathcal{O}\left(\frac{\delta m}{m}\right)^2\right\}}_{Non-thermal\ corrections} $$

acquires non-thermal corrections.

Inverse β-decay and neutrino mixing[†]

Working with **flavor neutrinos**, we compute

$$\mathcal{A}^{p\rightarrow n} = \langle n| \otimes \langle e^{+}_{k_e\,\sigma_e}, \nu_{k_\nu\,\sigma_\nu}|\widehat{S}_I\,|0\rangle \otimes |p\rangle$$

in the *laboratory frame...*

$$\Gamma^{p\rightarrow n}_{in} \;=\; \cos^4\theta\,\Gamma^{p\rightarrow n}_1 \,+\, \sin^4\theta\,\Gamma^{p\rightarrow n}_2 \,+\, \cos^2\theta\sin^2\theta\,\Gamma^{p\rightarrow n}_{12}$$

$$\Gamma^{p\rightarrow n}_i \;\equiv\; \frac{1}{T}\sum_{\sigma_\nu,\sigma_e} {G_F}^2 \int d^3k_\nu \int d^3k_e \left|\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_i},\omega_e)\right|^2, \quad i=1,2,$$

$$\Gamma^{p\rightarrow n}_{12} \;\equiv\; \frac{1}{T}\sum_{\sigma_\nu,\sigma_e} {G_F}^2 \int d^3k_\nu \int d^3k_e \left[\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_1},\omega_e)\,\mathcal{I}^*_{\sigma_\nu\sigma_e}(\omega_{\nu_2},\omega_e) \,+\, \text{c.c.}\right]$$

[†] M. B., G. Lambiase, G. Luciano and L.Petruzziello, Phys. Rev. D (2018)

. . . and in the *comoving frame*

$$\Gamma^{p\to n}_{com} = \cos^4\theta\,\widetilde{\Gamma}^{p\to n}_{1} + \sin^4\theta\,\widetilde{\Gamma}^{p\to n}_{2} + \cos^2\theta\sin^2\theta\,\widetilde{\Gamma}^{p\to n}_{12}$$

$$\begin{aligned}
\widetilde{\Gamma}^{p\to n}_{12} \;=\;& \frac{2\,G_F^2}{a^2\,\pi^7\,\sqrt{l_{\nu_1} l_{\nu_2}}\,e^{\pi\Delta m/a}}\int_{-\infty}^{+\infty} d\omega \left\{\int d^2k_e\, l_e \left|K_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)\right|^2\right.\\
&\times \int d^2k_\nu\,\left(\kappa_\nu^2 + m_{\nu_1} m_{\nu_2} + l_{\nu_1} l_{\nu_2}\right)\\
&\times \mathrm{Re}\left\{K_{i(\omega-\Delta m)/a+1/2}\left(\frac{l_{\nu_1}}{a}\right) K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\}\\
&+ m_e \int d^2k_e \int d^2k_\nu\,\left(l_{\nu_1} m_{\nu_2} + l_{\nu_2} m_{\nu_1}\right)\\
&\times \mathrm{Re}\left\{K^2_{i\omega/a+1/2}\left(\frac{l_e}{a}\right) K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_1}}{a}\right)\right.\\
&\left.\left.\times\, K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\}\right\}, \quad \kappa_\nu \equiv (k_\nu^x, k_\nu^y)
\end{aligned}$$

Comparing the rates

Laboratory vs comoving decay rates

$$\Gamma_{in}^{p\to n} \;=\; \cos^4\theta\,\Gamma_1^{p\to n} \,+\, \sin^4\theta\,\Gamma_2^{p\to n} \,+\, \cos^2\theta\sin^2\theta\,\Gamma_{12}^{p\to n},$$

$$\Gamma_{com}^{p\to n} = \cos^4\theta\,\widetilde{\Gamma}_1^{p\to n} \,+\, \sin^4\theta\,\widetilde{\Gamma}_2^{p\to n} \,+\, \cos^2\theta\sin^2\theta\,\widetilde{\Gamma}_{12}^{p\to n}$$

$$\Gamma_i^{p\to n} = \widetilde{\Gamma}_i^{p\to n}, \qquad i = 1,2$$

What about the "off-diagonal" terms?

$$\Gamma_{12}^{p\to n} \stackrel{?}{=} \widetilde{\Gamma}_{12}^{p\to n}$$

Non-trivial calculations…

…for $\frac{\delta m}{m} \ll 1$

$$\Gamma_{12}^{p\to n} = \widetilde{\Gamma}_{12}^{p\to n} \quad \text{up to } \mathcal{O}\left(\frac{\delta m}{m}\right)$$

Result

$$\Gamma_{in}^{p\to n} = \Gamma_{com}^{p\to n} \quad \text{up to } \mathcal{O}\left(\frac{\delta m}{m}\right)$$

General Covariance and mass states?

Solving the problem with **mass eigenstates** [‡] ?

"[...] a physical Fock space for flavor neutrinos cannot be constructed. Flavor states are only phenomenological since their definition depends on the specific considered process."

" We should view the neutrino states with well defined mass as the fundamental ones. [...] The decay rates calculated in this way are perfectly in agreement".

[‡] G.Cozzella, S.A.Fulling, A.G.S.Landulfo, G.E.A.Matsas and D.A.T.Vanzella, Phys.Rev. (2018)

Why not mass states?

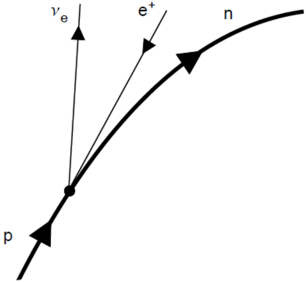
- A physical Fock space for flavor neutrinos can be rigorously defined
- The use of mass eigenstates wipes mixing out of calculations

$$\Gamma^{p\rightarrow n+\bar{\ell}_\alpha+\nu_i} = |U_{\alpha,i}|^2\Gamma_i, \qquad i=1,2$$

- Inconsistency with the asymptotic occurrence of *flavor oscillations*[§]

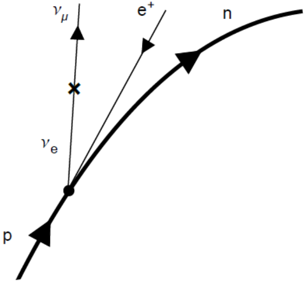
[§] M. B., G.Lambiase, G. Luciano and L.Petruzziello, arXiv:1903.03382

Neutrino oscillations (inertial frame)



a) Without oscillations

$$\Gamma_{in}^{(\nu_e)} = c_\theta^4\,\Gamma_1^{p\to n} + s_\theta^4\,\Gamma_2^{p\to n} + c_\theta^2\,s_\theta^2\,\Gamma_{12}^{p\to n}$$



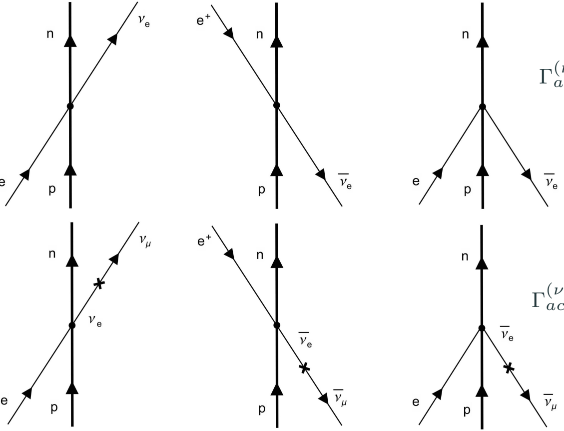
b) With oscillations

$$\Gamma_{in}^{(\nu_\mu)} = c_\theta^2\,s_\theta^2\left(\Gamma_1^{p\to n} + \Gamma_2^{p\to n} - \Gamma_{12}^{p\to n}\right)$$

Total decay rate

$$\Gamma_{in}^{tot} \equiv \Gamma_{in}^{(\nu_e)} + \Gamma_{in}^{(\nu_\mu)} = \cos^2\theta\,\Gamma_1^{p\to n} + \sin^2\theta\,\Gamma_2^{p\to n}$$

Neutrino oscillations (comoving frame)



a) Without oscillations

$$\Gamma^{(\nu_e)}_{acc} = c_\theta^4\, \widetilde{\Gamma}_1^{p\to n} + s_\theta^4\, \widetilde{\Gamma}_2^{p\to n} + c_\theta^2\, s_\theta^2\, \widetilde{\Gamma}_{12}^{p\to n}$$

b) With oscillations

$$\Gamma^{(\nu_\mu)}_{acc} = c_\theta^2\, s_\theta^2 \left(\widetilde{\Gamma}_1^{p\to n} + \widetilde{\Gamma}_2^{p\to n} - \widetilde{\Gamma}_{12}^{p\to n}\right)$$

Total decay rate

$$\Gamma^{tot}_{acc} = \cos^2\theta\, \widetilde{\Gamma}_1^{p\to n} + \sin^2\theta\, \widetilde{\Gamma}_2^{p\to n} = \Gamma^{tot}_{in}$$

Conclusion

- Asymptotic neutrinos must be in **flavor eigenstates** in order to preserve the General Covariance of QFT in curved background.

	Ahluwalia's approach	Matsas's approach	Our approach
Asympt. neutrinos in the laboratory frame	Flavor	Mass	Flavor
Asympt. neutrinos in the comoving frame	Mass	Mass	Flavor
Agreement between the decay rates	✗	✓	✓
Consistency with neutrino oscillations	✗	✗	✓

Flavor–Energy Uncertainty Relations for flavor neutrinos

Motivations

- Time-energy uncertainty relations (TEUR) in the Mandelstam–Tamm form, furnish lower-bounds on neutrino energy uncertainty in order to measure flavor oscillations[*]

- QFT formulation of neutrino oscillations suggests that these bounds can be read as flavor-energy uncertainty relations (FEUR)[†]. Moreover this energy uncertainty is connected with the intrinsic unstable nature of flavor neutrinos.

[*] S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

[†] M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)

Time-energy uncertainty relations

Mandelstam–Tamm TEUR is*:

$$\Delta E \, \Delta t \geq \frac{1}{2}$$

where

$$\Delta E \equiv \sigma_H \qquad \Delta t \equiv \sigma_O / \left| \frac{\mathrm{d}\langle O(t)\rangle}{\mathrm{d}t} \right|$$

Here $\langle \ldots \rangle \equiv \langle \psi | \ldots | \psi \rangle$ and $O(t)$ represents the "clock observable" whose dynamics quantifies temporal changes in a system.

*L. Mandelstam and I.G. Tamm, J. Phys. USSR (1945)

TEUR for unstable particles

Consider the projector on an unstable particle state $|\phi(t)\rangle$:

$$P_\phi(t) \ = \ |\phi(t)\rangle\langle\phi(t)| \, ,$$

Taking $|\phi\rangle = |\phi(0)\rangle$ we get the TEUR[†]

$$\left| \frac{\mathrm{d}\mathcal{P}_\phi(t)}{\mathrm{d}t} \right| \leq \ 2\Delta E \, \sqrt{\mathcal{P}_\phi(t)(1-\mathcal{P}_\phi(t))}$$

Here $\mathcal{P}_\phi(t)$ is the survival probability

$$\mathcal{P}_\phi(t) \ = \ |\langle\phi(t)|\phi\rangle|^2$$

[†] K. Bhattacharyya, J. Phys. A (1983)

$$\Delta E \ \geq \ \left|\frac{\mathrm{d}\mathcal{P}_\phi(t)}{\mathrm{d}t}\right|$$

$\mathcal{P}_\phi(t)$ is monotonically decreasing and we can integrate both sides between 0 and T:

$$\Delta E\, T \geq \frac{1}{2}\left[\frac{\pi}{2} - \arcsin\left(2\mathcal{P}_\phi(T) - 1\right)\right]$$

From this, one can derive:

$$\Delta E\, T_h \ \geq \ \frac{\pi}{4}$$

where $\mathcal{P}_\phi(T_h) = 1/2$.

TEUR for neutrino oscillations

Neutrino fields with definite masses can be expanded as:

$$\hat{\nu}_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i}(t) \hat{\alpha}^r_{\mathbf{k},i} \ + \ v^r_{-\mathbf{k},i}(t) \hat{\beta}^{r\dagger}_{-\mathbf{k},i} \right] e^{i\mathbf{k}\cdot\mathbf{x}} \,, \quad i = 1,2$$

A neutrino mass-eigenstate is defined as:

$$|\nu^r_{\mathbf{k},i}\rangle \ = \ \hat{\alpha}^{r\dagger}_{\mathbf{k},i}|0\rangle \,, \quad i = 1,2$$

In the ultra-relativistic limit we can define:

$$\begin{pmatrix} \tilde{\alpha}^r_{\mathbf{k},e} \\ \tilde{\alpha}^r_{\mathbf{k},\mu} \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha^r_{\mathbf{k},1} \\ \alpha^r_{\mathbf{k},2} \end{pmatrix}$$

Pontecorvo flavor states:

$$|\nu^r_{\mathbf{k},\sigma}\rangle_P \ \equiv \ \tilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}|0\rangle \,,$$

The number operator is:

$$\tilde{N}_\sigma(t) \;=\; \sum_{\mathbf{k},r} \tilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)\tilde{\alpha}^{r}_{\mathbf{k},\sigma}(t)\,, \qquad \sigma = e,\mu$$

The QM oscillation formula can be found by taking the expectation value of the number operator over the corresponding Pontecorvo flavor state:

$$\mathcal{P}_{\sigma\to\sigma}(t) = \langle \tilde{N}_\sigma(t)\rangle_\sigma = 1 - \sin^2(2\theta)\sin^2\left(\frac{\omega_{\mathbf{k},1}-\omega_{\mathbf{k},2}}{2}t\right)$$

where $\langle\cdots\rangle_\sigma = {}_P\langle \nu^r_{\mathbf{k},\sigma}|\cdots|\nu^r_{\mathbf{k},\sigma}\rangle_P$.

TEUR for neutrino oscillations[*]

$$\left|\frac{\mathrm{d}\mathcal{P}_{\sigma\to\sigma}(t)}{\mathrm{d}t}\right| \leq 2\Delta E\,\sqrt{\mathcal{P}_{\sigma\to\sigma}(t)\,(1-\mathcal{P}_{\sigma\to\sigma}(t))}$$

Integrating and using the triangular inequality:

$$\Delta E\,T \geq \mathcal{P}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho\,,$$

with $\mathcal{P}_{\sigma\to\rho}(t) \;=\; 1-\mathcal{P}_{\sigma\to\sigma}(t)$.

For $T = T_h$ we get

$$\Delta E\,T_h \geq \frac{1}{2}$$

[*]S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

Flavor-energy uncertainty relations

The (non-conserved) flavor charges:

$$Q_{\nu_\sigma}(t) \;\equiv\; \int d^3x\, \nu_\sigma^\dagger(x)\, \nu_\sigma(x)\,, \qquad \sigma = e, \mu$$

Flavor charges are not conserved: $[Q_{\nu_\sigma}(t)\,,\, H] \neq 0$

$\Rightarrow$ flavor-energy uncertainty relation[†]:

$$\langle \Delta H \rangle \, \langle \Delta Q_{\nu_\sigma}(t) \rangle \;\geq\; \frac{1}{2} \left| \frac{d\langle Q_{\nu_\sigma}(t) \rangle}{dt} \right|$$

Taking the state $|\psi\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$:

$$\Delta Q_{\nu_\sigma}(t) \;=\; \sqrt{\mathcal{Q}_{\sigma\to\sigma}(t)(1-\mathcal{Q}_{\sigma\to\sigma}(t))}$$

we obtain:

$$\Delta E\, T \geq \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho$$

[†]M. B., P. Jizba and L.Smaldone, Phys. Rev. D (2019)

Neutrino oscillation condition

When $m_i/|\mathbf{k}| \to 0$:

$$\Delta E \;\geq\; \frac{2\,\sin^2 2\theta}{L_{osc}}$$

This relation is usually interpreted as neutrino oscillation condition.

The situation is similar to that of unstable particles:

$$\Delta E \;\approx\; \frac{1}{2\tau}$$

where the τ is the particle life-time.

– As for unstable particles only energy distribution are meaningful. The width of the distribution is related to the oscillation length.

Corrections beyond ultra-relativistic limit:

$$\Delta E \;\geq\; \frac{2\,\sin^2 2\theta}{L_{osc}}\;\left[1-\varepsilon(\mathbf{k})\,\cos^2\left(\frac{|\mathbf{k}|L_{osc}}{2}\right)\right]$$

with $\varepsilon(\mathbf{k}) \equiv (m_1-m_2)^2/(4|\mathbf{k}|^2)$. When $|\mathbf{k}| = \tilde{k} = \sqrt{m_1 m_2}$:

$$\Delta E \;\geq\; \frac{2\sin^2 2\theta}{\tilde{L}_{osc}}\;(1-\chi)$$

where

$$\chi \;=\; \xi\,\sin\left(\frac{\tilde{\omega}_1\tilde{L}_{osc}}{4}\right)\sin\left(\frac{\tilde{\omega}_2\tilde{L}_{osc}}{4}\right)+\cos\left(\frac{\tilde{\omega}_1\tilde{L}_{osc}}{4}\right)\cos\left(\frac{\tilde{\omega}_2\tilde{L}_{osc}}{4}\right)\;.$$

and $\xi = 2\sqrt{m_1 m_2}/(m_1+m_2)$.

Finally note that

$$\begin{aligned}\sigma_Q^2 &= \langle Q_{\nu_\sigma}^2(t)\rangle_\sigma \; - \; \langle Q_{\nu_\sigma}(t)\rangle_\sigma^2 \\ &= \mathcal{Q}_{\sigma\to\sigma}(t)\,(1-\mathcal{Q}_{\sigma\to\sigma}(t))\ .\end{aligned}$$

quantifies dynamical (flavor) entanglement for neutrino states[‡] since it concides with the linear entropy in terms of the flavor qubits:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu \equiv |01\rangle_f,$$

[‡] M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009)

Conclusions

• Mixing transformations are not trivial in Q.F.T. (not just a rotation!) $\Rightarrow$ inequivalent representations.

• The vacuum for mixed fields has the structure of a $SU(N)$ generalized coherent state (condensate of particle-antiparticle pairs).

• Flavor neutrino states can be consistently defined. They are necessary for the general covariance of the theory;

• Flavor states can be formally regarded as unstable states, obeying flavor-energy uncertainty relations;

• Lorentz invariance (?)