Hadron spectra and properties with functional methods

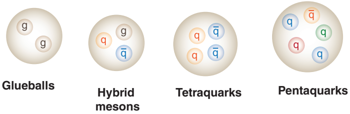
Gernot Eichmann
University of Graz

International School of Nuclear Physics, 44th Course:
From quarks and gluons to hadrons and nuclei

Erice, Sicily, Sep 21, 2023

Many open questions

- **Understanding exotic hadrons:**
  Hadron spectroscopy at LHC, Belle II, BES III, PANDA, JLab, ELSA, ...

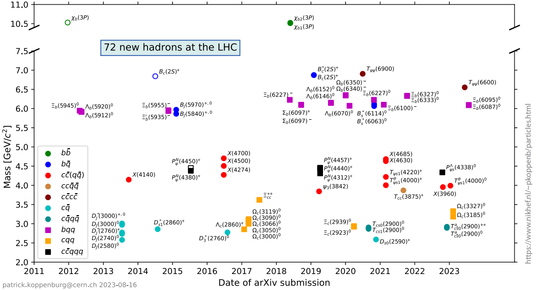


Glueballs

Hybrid mesons

Tetraquarks

Pentaquarks

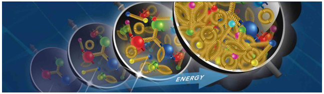


patrick.koppenburg@cern.ch 2023-08-16

https://www.nikhef.nl/~pkoppenb/particles.html

- **Mass generation and confinement?**

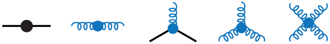
- **Quark-gluon structure** of hadrons and nuclei: Hadron tomography at EIC, JLab, COMPASS/AMBER, ...



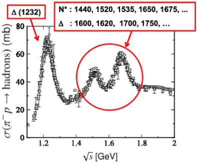
Brookhaven National Laboratory

Theory tools

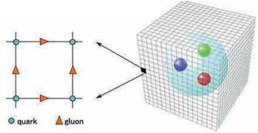
Functional methods
(DSEs & BSEs, FRG, ...)



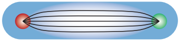
Amplitude analyses



Lattice QCD



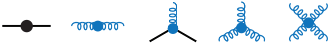
Phenomenological models



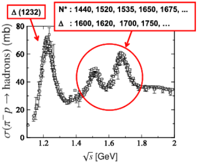
Effective theories
(ChPT, ...)

Theory tools

Functional methods
(DSEs & BSEs, FRG, ...)

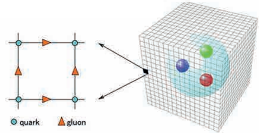


Amplitude analyses

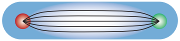


ExoHAD
Exotic Hadrons Topical Collaboration

Lattice QCD



Phenomenological models

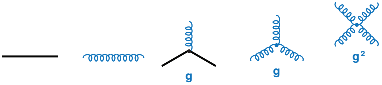


Effective theories
(ChPT, ...)

Functional methods

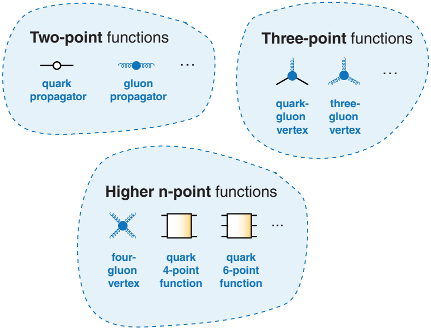
Classical Lagrangian of **QCD:**

$$\mathcal{L} = \bar{\psi}\,(\,i\not\partial + g\not{A} - M\,)\,\psi - \tfrac{1}{4}\,F^{a}_{\mu\nu}\,F^{\mu\nu}_{a}$$



Quantum field theory

Correlation functions in **QCD:**



Two-point functions

quark propagator

gluon propagator

…

Three-point functions

quark-gluon vertex

three-gluon vertex

…

Higher n-point functions

four-gluon vertex

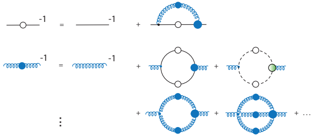
quark 4-point function

quark 6-point function

…

Can be calculated ...

- in **perturbation theory**
- in **lattice QCD**
- with **functional methods:**
  DSEs (Dyson-Schwinger equations),
  FRG (functional renormalization group)

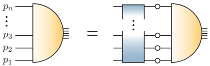


Quark DSE

Gluon DSE

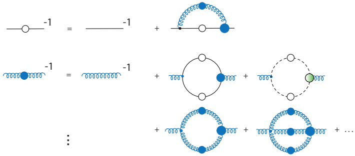
Functional methods

- Hadronic **bound-state equations** (Bethe-Salpeter & Faddeev eqs)



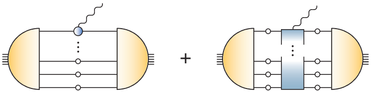
"QFT analogue of Schrödinger eq."
→ hadron masses & "wave functions"
→ **spectroscopy calculations**

- Ingredients: **QCD's n-point functions,** Satisfy quantum eqs. of motion (DSEs)



→ Dynamical mass generation, gluon mass gap, confinement, ...

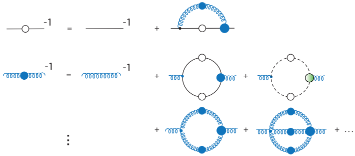
- **Structure calculations:** form factors, PDFs, GPDs, TMDs, two-photon processes, ...



Functional methods

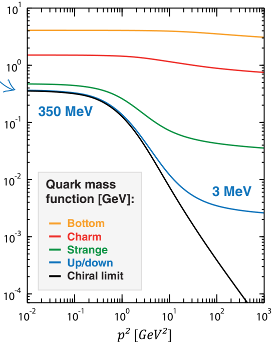
- Ingredients: **QCD's n-point functions,** Satisfy quantum eqs. of motion (DSEs)

Quark mass



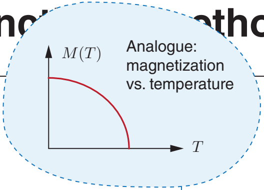
→ Dynamical mass generation, gluon mass gap, confinement, ...

"constituent-quark mass"



current-quark mass

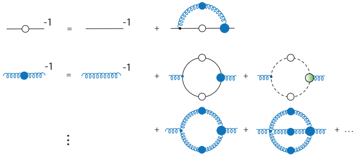
Func thods



Analogue:
magnetization
vs. temperature

- Ingredients: **QCD's n-point functions,** Satisfy quantum eqs. of motion (DSEs)

Quark mass



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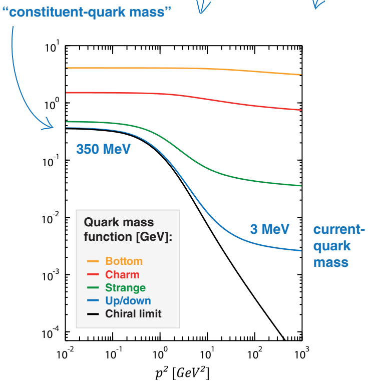
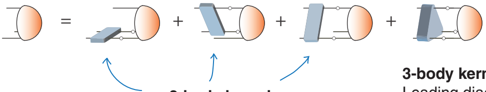


Baryons

Three-quark BSE (Faddeev equation) for baryons:

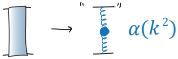
GE, Alkofer, Nicmorus, Krassnigg, PRL 104 (2010)

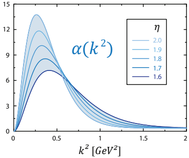


2-body kernel:
fixed in meson sector

3-body kernel:
Leading diagram (3-gluon vertex)
vanishes by color trace,
higher-order diagrams small (?)
2-quark correlations dominant?

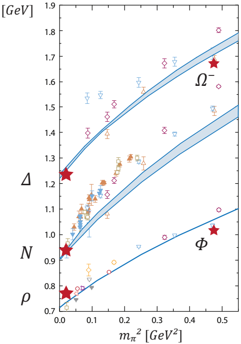
Rainbow-ladder





Scale set by f_π ,
shape parameter → bands

Maris, Tandy, PRC 60 (1999)



- Analogous results for many **form factors**

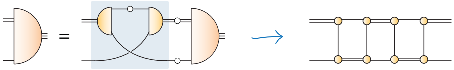
  Review: GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 91 (2016)

- Relativistically, nucleon also has **p waves!**

see also:
Qin, Roberts, Schmidt,
PRD 97 (2018)

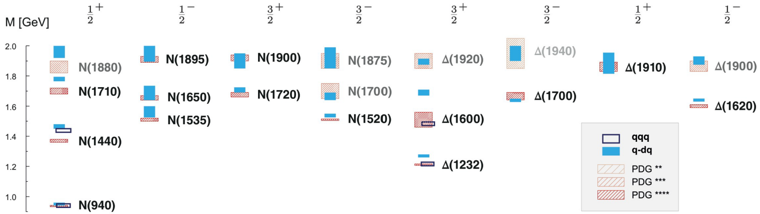
Diquark correlations

- **Quark-diquark** (two-body) equation
  Oettel et al., PRC 58 (1998), GE et al., Ann. Phys. 323 (2008), Cloet et al., FBS 46 (2009), Segovia et al., PRL 115 (2015), Chen et al., PRD 97 (2018)



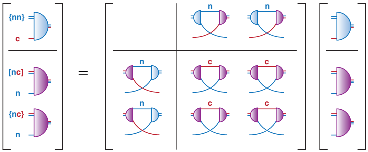
Diquark clustering in baryons?
Barabanov et al., Prog. Part. Nucl. Phys. 116 (2021)

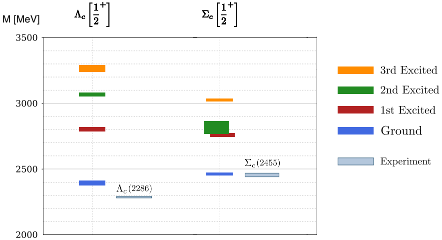
- **Three-quark** and **quark-diquark** results very similar
  GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



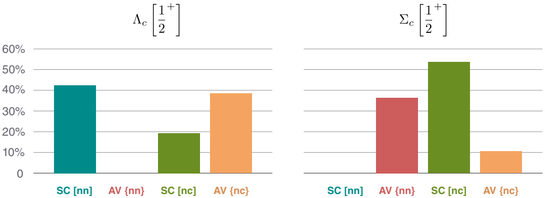
Heavy baryons

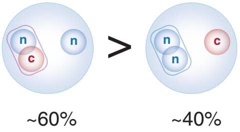
- Coupled quark-diquark equations for Σ_c, Λ_c
  Torcato, Arriaga, GE, Peña, FBS 64 (2023)





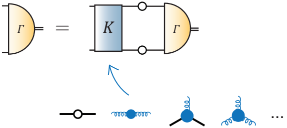
- **(nc)n** components dominane over **(nn)c**





Towards ab-initio

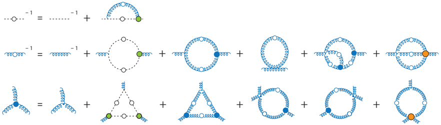
- **Goal:** go towards ab-initio calculations by calculating **higher n-point functions**



- Lots of activity with **DSEs, FRG, lattice QCD**

  . . . , Williams, Fischer, Heupel, PRD 93 (2016), Cyrol et al., PRD 97 (2018), Oliveira, Silva, Skullerud, Sternbeck, PRD 99 (2019), Aguilar et al., EPJ C 80 (2020), Qin, Roberts, Chin. Phys. Lett. 38 (2021), . . .

- Coupled **Yang-Mills DSEs** Huber, PRD 101 (2020), GE, Pawlowski, Silva, PRD 104 (2021)



Towards ab-initio

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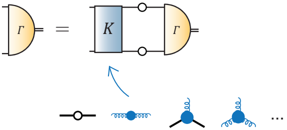

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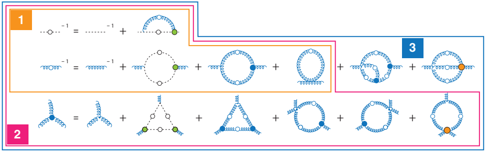
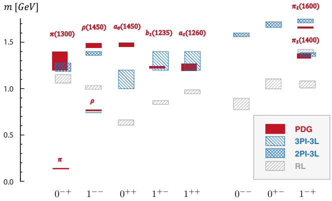


truncation error:

1 60% 2 10% 3 4%

Towards ab-initio

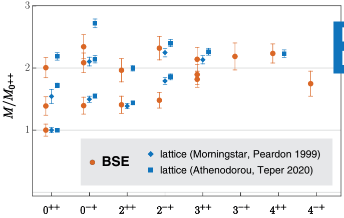
- Beyond rainbow-ladder calculations improve **light-meson spectrum**
  Williams, Fischer, Heupel, PRD 93 (2016)



truncation error:

1 60% **2** 10% **3** 4%

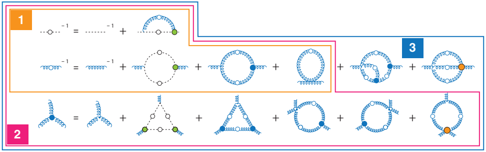
- **Glueball spectrum** agrees with lattice QCD
  Huber, Fischer, Sanchis-Alepuz, EPJ C 80 (2020), EPJ C 81 (2021)



Markus Huber

- Coupled **Yang-Mills DSEs**

Huber, PRD 101 (2020),
GE, Pawlowski, Silva, PRD 104 (2021)



Towards ab-initio

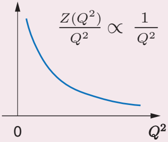
Gluon propagator:

transverse dressing

longitudinal dressing = 1

$$D^{\mu\nu}(Q) = \boxed{\frac{Z(Q^2)}{Q^2}} \left(\delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + \xi \, \frac{\boxed{L(Q^2)}}{Q^2} \, \frac{Q^\mu Q^\nu}{Q^2}$$

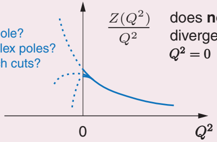
- **Perturbation theory:**
  Massless gluon pole



$$\frac{Z(Q^2)}{Q^2} \propto \frac{1}{Q^2}$$

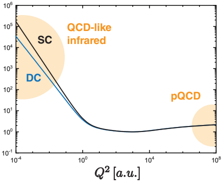
- **Nonperturbative calculations:**
  Massless pole disappears!

real pole?
complex poles?
branch cuts?

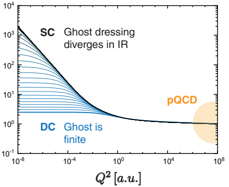


$\frac{Z(Q^2)}{Q^2}$ does **not** diverge at $Q^2 = 0$

Gluon $1/Z(Q^2)$



Ghost $G(Q^2)$



Family of **"decoupling"** solutions, also seen in lattice QCD

Cucchieri, Maas, Mendes, PRD 77 (2008)
Boucaud et al., JHEP 06 (2008)
Bogolubsky et al., PLB 676 (2009)
Fischer, Maas, Pawlowski, Ann. Phys. 324 (2009)
Duarte, Oliveira, Silva, PRD 94 (2016)
Aguilar et al., EPJ C 80 (2020)

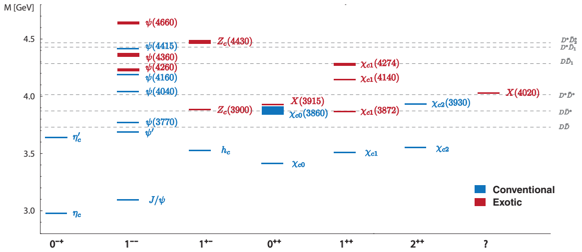
Endpoint is **"scaling"** solution, confinement manifest

Lerche, Smekal, PRD 65 (2002)
Fischer, Alkofer, PLB 536 (2002)
Alkofer, Fischer, Llanes-Estrada, MPLA 23 (2008)

All solutions show **gluon mass gap**

$$\lim_{r\to\infty} \int \frac{d^3Q}{(2\pi)^3} \, \frac{Z(\boldsymbol{Q}^2)}{\boldsymbol{Q}^2} \, e^{i\boldsymbol{x}\cdot\boldsymbol{Q}} \propto e^{-m_{\text{gap}}\, r}$$

Exotic mesons



- Several tetraquark candidates in **charmonium spectrum**: X(3872), X(3915), Zc(3900), ...
- Z states cannot be $c\bar{c}$ since they carry charge
- Recent additions: all-charm X(6900), open-charm T_{cc}^+, ...
- Oldest tetraquark candidates: **light scalar mesons**

Reviews:

Chen, Chen, Liu, Zhu,
Phys. Rept. 639 (2016), 1601.02092

Lebed, Mitchell, Swanson
PPNP 93 (2017), 1610.04528

Esposito, Pilloni, Polosa,
Phys. Rept 668 (2017), 1611.07920

Guo, Hanhart, Meißner et al.,
Rev. Mod. Phys. 90 (2018), 1705.00141

Ali, Lange, Stone,
PPNP 97 (2017), 1706.00610

Olsen, Skwarnicki, Zieminska,
Rev. Mod. Phys. 90 (2019), 1708.04012

Liu, Chen, Chen, Liu, Zhu,
PPNP 107 (2019), 1903.11976

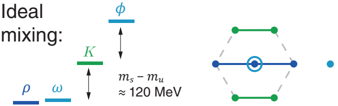
Brambilla, Eidelman, Hanhart et al.,
Phys. Rept. 873 (2020)

. . .

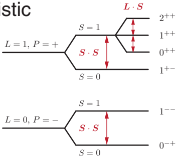
Light exotic mesons

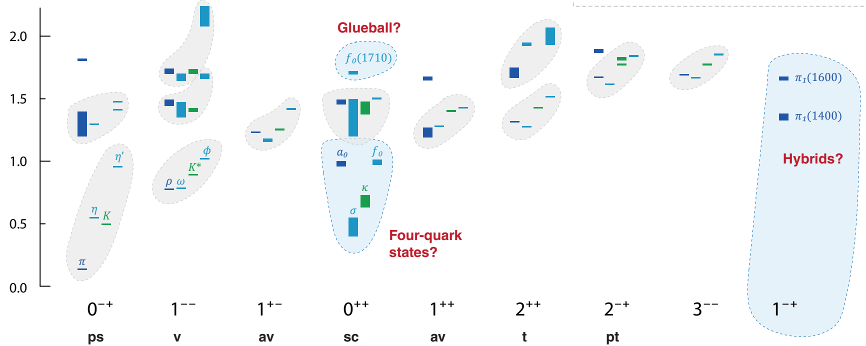
Light meson spectrum
(PDG 2020)

Ideal mixing:



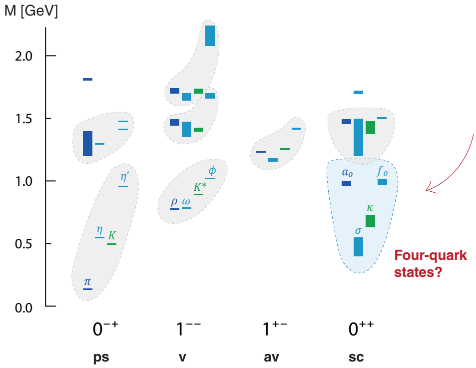
Non-relativistic level ordering





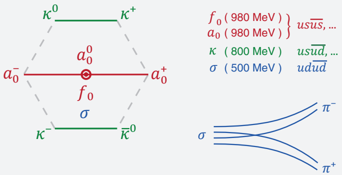
Light exotic mesons

Light meson spectrum
(PDG 2020)



Four-quark states?

- **Diquark-antidiquark**?
  Explains mass ordering & decay widths
  Jaffe 1977, Close, Tornqvist 2002,
  Maiani, Polosa, Riquer 2004

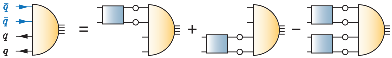


- **Meson molecules?**
  Weinstein, Isgur 1982, 1990; Close, Isgur, Kumano 1993

- **Non-$q\bar{q}$ nature** supported by various approaches
  Pelaez, Phys. Rept. 658 (2016)

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states:**
  GE, Fischer, Heupel, PLB 753 (2016)



+ perm.

$$\Gamma(p,q,k,P)=\sum_i f_i\left(p^2,q^2,k^2,\{\omega_j\},\{\eta_j\}\right)\,\tau_i(p,q,k,P)\quad\otimes\quad\text{Color}\quad\otimes\quad\text{Flavor}$$

9 Lorentz invariants:

$p^2,\quad q^2,\quad k^2,\quad P^2=-M^2$

$\omega_1=q\cdot k\qquad \eta_1=p\cdot P$
$\omega_2=p\cdot k\qquad \eta_2=q\cdot P$
$\omega_3=p\cdot q\qquad \eta_3=k\cdot P$

256 Dirac-Lorentz tensors

2 Color tensors:

$3\otimes\overline{3},\quad 6\otimes\overline{6}$ or
$1\otimes 1,\quad 8\otimes 8$
(Fierz-equivalent)

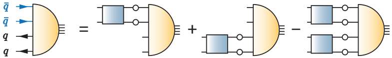
$$K\,\psi_i=\lambda_i\,\psi_i$$

	dim K	memory
Mesons	10^3	20 MB
Baryons	10^8	10^7 GB
Tetraquarks	10^{13}	10^{18} GB

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states:**

  GE, Fischer, Heupel, PLB 753 (2016)



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$$\Gamma(p,q,k,P) = \sum_i f_i\left(p^2, q^2, k^2, \{\omega_j\}, \{\eta_j\}\right) \tau_i(p,q,k,P) \quad \otimes \quad \text{Color} \quad \otimes \quad \text{Flavor}$$

9 Lorentz invariants:

$p^2, \quad q^2, \quad k^2, \quad P^2 = -M^2$

$\omega_1 = q\cdot k \qquad \eta_1 = p\cdot P$
$\omega_2 = p\cdot k \qquad \eta_2 = q\cdot P$
$\omega_3 = p\cdot q \qquad \eta_3 = k\cdot P$

256 Dirac-Lorentz tensors

2 Color tensors:

$3\otimes\bar{3}$, $6\otimes\bar{6}$ or
$1\otimes 1$, $8\otimes 8$
(Fierz-equivalent)

- Group momentum variables into multiplets of **permutation group S4:** can switch off groups of variables without destroying symmetries

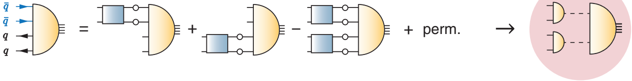
  GE, Fischer, Heupel, PRD 92 (2015)

$$f_i\left(\mathcal{S}_0, \triangledown, \triangle, \bigcirc\right)$$

Four-quark states

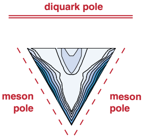
- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states:**

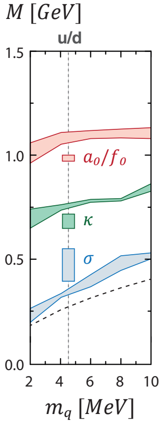
GE, Fischer, Heupel, PLB 753 (2016)



- BSE dynamically generates **meson poles** in BS amplitude:

$f_i\,(\,\mathcal{S}_0\,,$ ▽ , △ , ○ $)$ → 1500 MeV
$f_i\,(\,\mathcal{S}_0\,,$ ▽ , △ , ● $)$ → 1500 MeV
$f_i\,(\,\mathcal{S}_0\,,$ ▽ , ▲ , ○ $)$ → 1200 MeV
$f_i\,(\,\mathcal{S}_0\,,$ ▼ , △ , ○ $)$ → **350 MeV !**



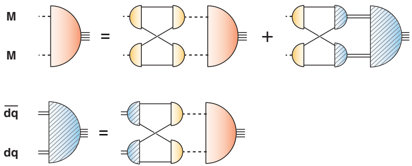


- "Light scalar mesons" look like **meson molecules,** diquark-antidiquark components almost negligible. Lightness is inherited from pseudoscalar Goldstone bosons!

Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark,** follows from four-quark eq. (analogue of quark-diquark for baryons)

Heupel, GE, Fischer, PLB 718 (2012)

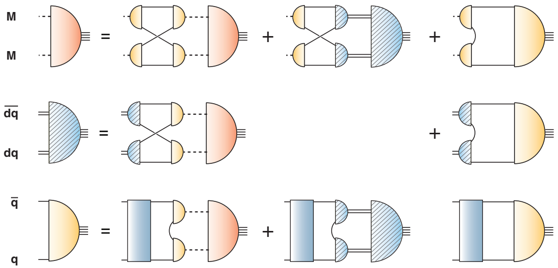


- Interaction by **quark exchange**
- System ‘wants’ to be **meson-meson**-like (no diagonal dq-$\overline{\text{dq}}$ term)
- Similar results as in 4-quark approach: m_σ ~ 400 MeV, etc.

Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark,**
follows from four-quark eq. (analogue of quark-diquark for baryons)

Heupel, GE, Fischer, PLB 718 (2012)



Include **mixing with $\mathbf{q\bar{q}}$:**
$\pi\pi$ still dominant

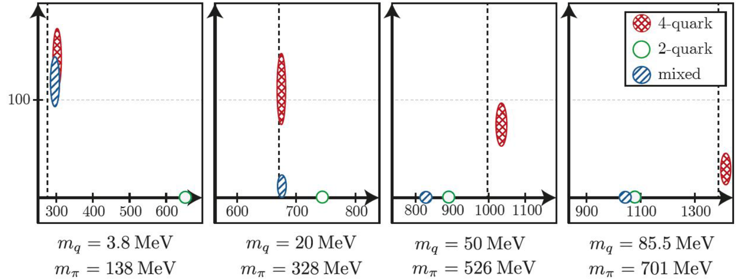
Santowsky, GE, Fischer, Wallbott, Williams, PRD 102 (2020)

[MeV]	ground state mass	first excitation
$\pi\pi$	416 ± 26	970 ± 130
$\pi\pi + 0^+0^+$	416 ± 26	970 ± 130
$q\bar{q}$	667 ± 2	1036 ± 8
$\pi\pi + q\bar{q}$	472 ± 22	1080 ± 280
$\pi\pi + 0^+0^+ + q\bar{q}$	456 ± 24	1110 ± 110

Four-quark states

Four-quark vs. $q\bar{q}$ dominance

Santowsky, Fischer, PRD 105 (2022)



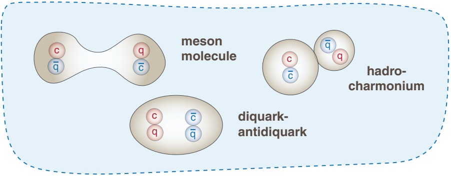
Light quarks:
$\pi\pi$ dominant

Strange quarks:
$q\bar{q}$ dominant

Four-quark states

- Heavy-light **four-quark states:** what is their internal decomposition?

$cq\bar{q}\bar{c}$ →



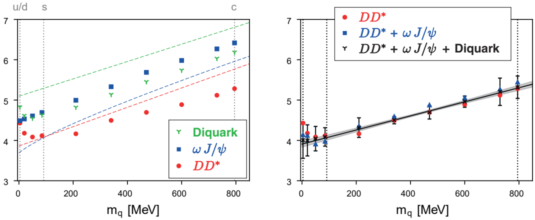
- **Four-quark BSE:** all mix together

$M_{cq\bar{q}\bar{c}}$ [GeV] in X(3872) channel

$m_S + m_A$

$m_\omega + m_{J/\psi}$

$m_D + m_{D^*}$

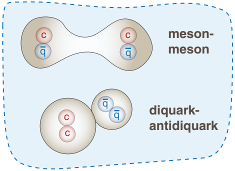


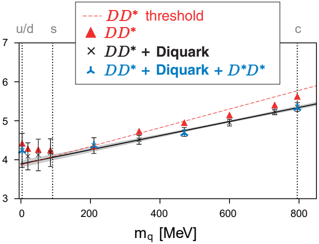
Wallbott, GE, Fischer,
PRD 100 (2019),
PRD 102 (2020)

$cq\bar{q}\bar{c}$ → strong meson-meson component: DD* for X(3872), Zc(3900)

Four-quark states

- **Open-charm states:** $cc\bar{q}\bar{q}$





Wallbott, GE, Fischer, PRD 102 (2020)

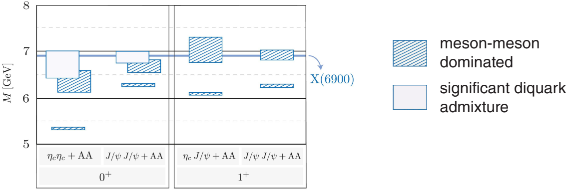
Experimental candidate:
T_{cc}^{+}, $0(1^{+})$, 3875 MeV

Joshua Hoffer

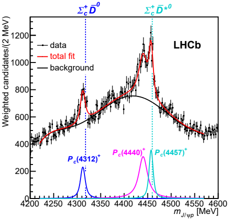
- **All-charm state:** $cc\bar{c}\bar{c}$
  X(6900)

  Results so far available in two-body approach, 1st radial excitation?

  Santowsky, Fischer, EPJC 82 (2022)

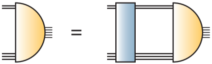


Pentaquarks?

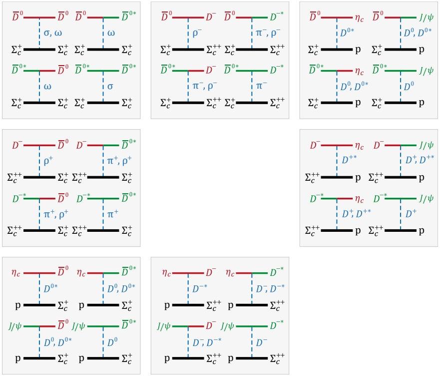


Aaij et al., PRL 112 (2019)

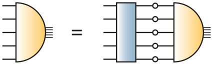
- Meson-baryon equation with hadronic exchanges
  GE, Lourenco, Peña, Stadler, Torres, in preparation



... all couplings calculated dynamically

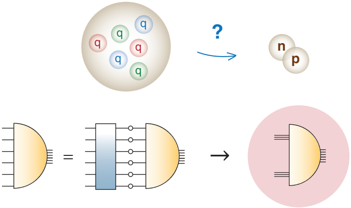


- Next up: **5-body equation**
  GE, Peña, Torres, in preparation

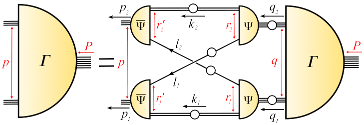


Nucleons in nuclei?

Transition from quarks & gluons to **light nuclei:**

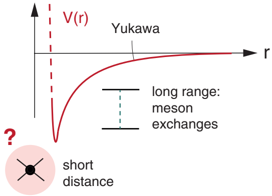


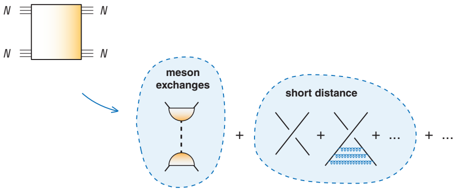
- Relativistic structure of **deuteron?**
  Arriaga, GE, Nunes, in preparation



- Exotic dibaryons, hypernuclei, short-range correlations, EMC effect ...

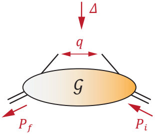
Microscopic origins of **short-range nuclear force?**





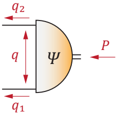
GE, Mirtl, in progress

Hadron structure



Hadron-to-hadron correlator

$$\mathcal{G}(z,P,\Delta) = \langle P_f |\, \mathsf{T}\, \Phi(z)\, \mathcal{O}\, \Phi(0)\, | P_i \rangle$$



Bethe-Salpeter WF:
vacuum-to-hadron correlator

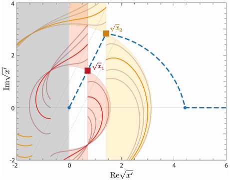
$$\Psi(z,P) = \langle 0 | \mathsf{T}\, \Phi(z)\, \Phi(0) | P \rangle$$

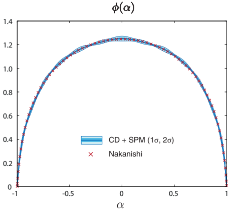
	$\mathcal{G}(q,P,\Delta=0)$	$\mathcal{G}(q,P,\Delta)$	$\Psi(q,P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2\boldsymbol{q}_\perp \int dq^-$	PDF	GPD	PDA

Diehl, Phys. Rept. 388 (2003)
Belitsky, Radyushkin,
Phys. Rept. 418 (2005)
Lorcé, Pasquini, Vanderhaeghen,
JHEP 05 (2011)
. . .

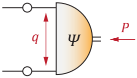
Novel method to compute
light-front wave functions
via contour deformations

Editors' Suggestion:
GE, Ferreira, Stadler, PRD 105 (2022)



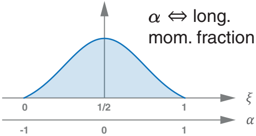


Light-front wave function



LFWF = BSWF integrated over q^-

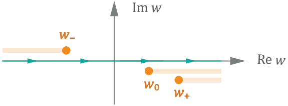
$$\psi(\alpha,\boldsymbol{k}_\perp) = \mathcal{N} P^+ \int\limits_{-\infty}^{\infty} \frac{dq^-}{2\pi}\,\Psi(q,P)\Big|_{q^+=\frac{\alpha}{2}P^+,\,\boldsymbol{q}_\perp=\boldsymbol{k}_\perp}$$



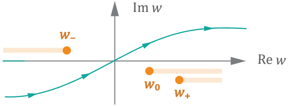
α ⇔ long. mom. fraction

How is this related to an **analytic function**? Set $x = \frac{\boldsymbol{k}_\perp^2}{m^2}$, $w = \frac{M}{2m}\,q^-$, $t = -\frac{M^2}{4m^2}$

$$\int\limits_{-\infty}^{\infty} dw$$

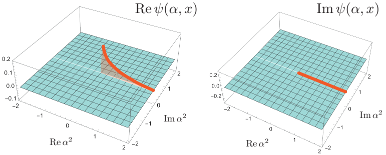


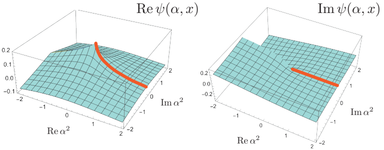
$$\int\limits_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw$$



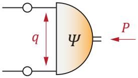
⇒ support only for $-1 < \alpha < 1$,
not an analytic function

⇒ analytic function in α^2 for any x, t ∈ ℂ,
for $-1 < \alpha < 1$ result is the same





Light-front wave function



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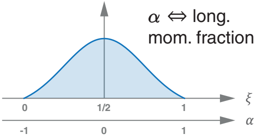

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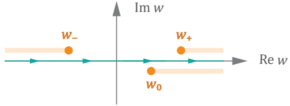

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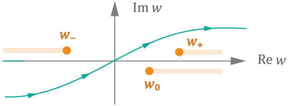

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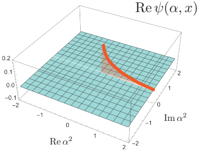
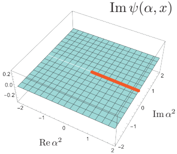
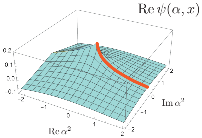
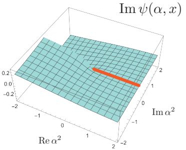
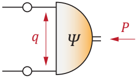








Light-front wave function



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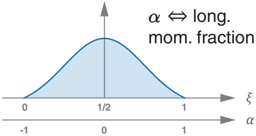
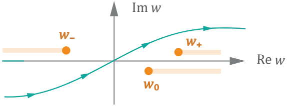
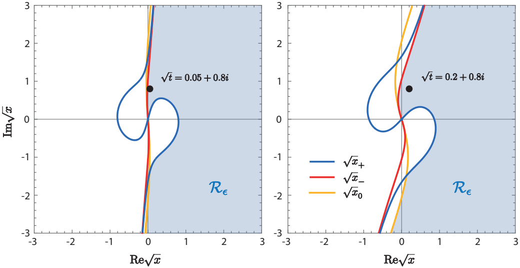


α ⇔ long. mom. fraction

How is this related to an **analytic function**? Set $x = \frac{\boldsymbol{k}_\perp^2}{m^2}$, $w = \frac{M}{2m}\, q^-$, $t = -\frac{M^2}{4m^2}$

After integration ⇒ **branch cuts** in complex $\sqrt{x}$ plane



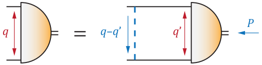


— Propagator poles

— Pole in BS amplitude

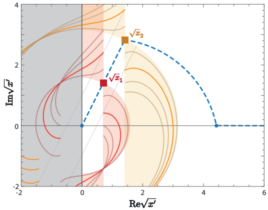
- Correct result for LFWF inside $\mathcal{R}_\epsilon$
- Physical region 0 < M < 2m is imaginary axis: $\sqrt{t} = \frac{iM}{2m}$
  Not possible ⇒ complex $\sqrt{t}$

Light-front wave function

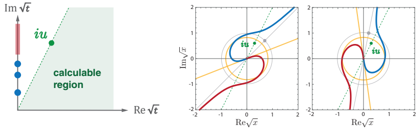


$$\Gamma(x,\omega,t,\alpha) = \int_0^\infty dx' \int_{-1}^{1} d\omega' \int_{-1}^{1} dy\, K(x,x',\Omega)\, G_0(x',\omega',t,\alpha)\, \Gamma(x',\omega',t,\alpha)$$

- BSE must be solved for $\sqrt{x} \in \mathcal{R}_\epsilon$
- BSE is **integral equation** ⇒ must be solved along path $\sqrt{x}$ that coincides with **integration path** $\sqrt{x'}$, must lie inside $\mathcal{R}_\epsilon$ ⇒ need contour deformations
- **Kernel** has pole ⇒ after integrating over y and ω', becomes branch cut in $\sqrt{x'}$, automatically avoided if Re$\sqrt{x'}$ and $|\sqrt{x'}|$ increase along integration path

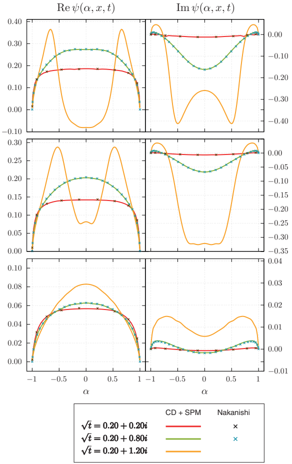


- **Propagators** have poles ⇒ branch cuts from before, automatically avoided if path within $\mathcal{R}_\epsilon$
- **BS amplitude** may dynamically generate singularities ⇒ avoided as long as arg $\sqrt{t}$ < arg(iu)

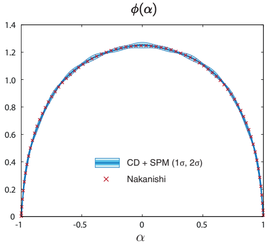


Light-front wave function

Light-front wave function
(IR, mid-momentum, UV)



Parton distribution amplitude



On the physical axis:
$\sqrt{t} = 0.5i$

use Schlessinger method in t

GE, Ferreira, Stadler, PRD 105 (2022)

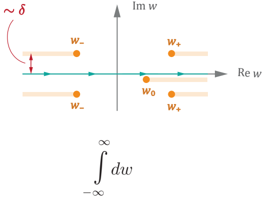
- Results agree with Nakanishi method
- LFWF vanishes at endpoints $\alpha = \pm 1$
- No expansion in moments involved, plain numerical result
- Also works above threshold (unphysical, no poles on 1st sheet, no resonances either)

Frederico, Salmè, Viviani, PRD 85 (2012)

Complex conjugate singularities

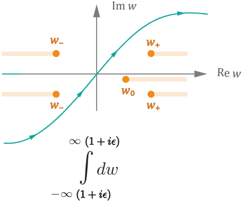
Straightforward to implement in scalar model,
contour deformations work in same way

$$D(q^2) = \frac{1}{2}\left(\frac{1}{q^2+m^2\,(1+i\delta)} + \frac{1}{q^2+m^2\,(1-i\delta)}\right) = \frac{q^2+m^2}{(q^2+m^2)^2+m^4\,\delta^2}$$



$$\int_{-\infty}^{\infty} dw$$

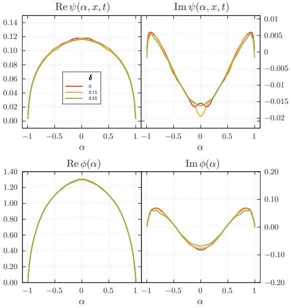
does not give correct
limit for $\delta \to 0$



$$\int_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw$$

correct limit for $\delta \to 0$,
proper analytic continuation

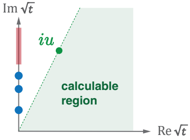
LFWF and PDA not sensitive to δ :



GE, Ferreira, Stadler, PRD 105 (2022)

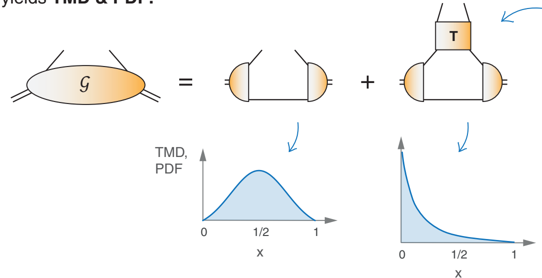
- One does not even need to know pole positions!

  ⇒ works for general singularities in n-point functions!

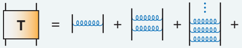


TMDs

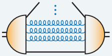
Same technique applied to hadron-to-hadron correlator yields **TMD & PDF:**

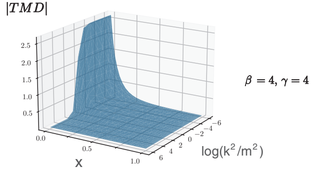


- Four-point **T matrix** solved from BSE ⇒ all **ladder exchanges**



- Analogue in **light-front formalism**: include infinitely many Fock states



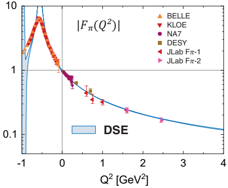


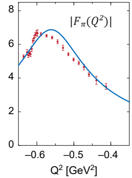
GE, Ferreira, Stadler, in preparation

	$\mathcal{G}(q,P,\Delta=0)$	$\mathcal{G}(q,P,\Delta)$	$\Psi(q,P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2\boldsymbol{q}_\perp \int dq^-$	PDF	GPD	PDA

Meson form factors

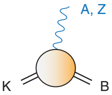
- **Pion & kaon form factors** constrain theory uncertainty for **muon g-2**
  GE, Fischer, Williams, PRD 101 (2020)





- **Timelike properties of hadrons**: pion form factor dynamically develops ρ pole with $\pi\pi$ decay channel
  Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)

- **Flavor matrix elements:** quantify QCD contributions
  GE, Petit, Stadler, Torres, in preparation



Summary & outlook

- **Baryons:**
  GE, Sanchis-Alepuz, Williams, Fischer, Alkofer, PPNP 91 (2016), arXiv:1606.09602
- **Four-quark states:**
  GE, Fischer, Heupel, Santowsky, Wallbott, FBS 61 (2020), arXiv:2008.10240

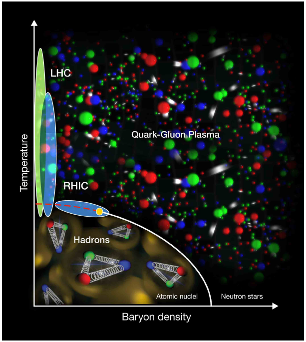
- Towards **ab-initio calculations:**
  higher n-point functions, gluon mass generation, resonances
- **Exotic hadrons:** glueballs, hybrids, tetraquarks, pentaquarks
- **Hadron structure:** PDFs, GPDs, TMDs
- **Nuclei** from quarks and gluons

EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION

Thank you!

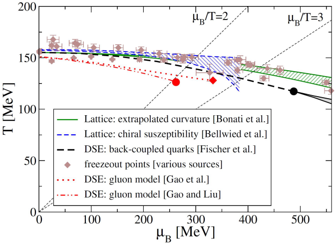
Backup slides

QCD phase diagram



Brookhaven National Laboratory

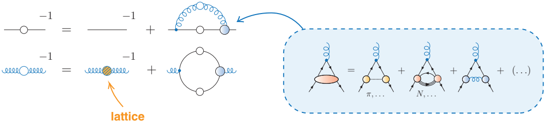
Search for **critical endpoint (CEP)** from DSEs & lattice:



Fischer, Prog. Part. Nucl. Phys. 105 (2019)

Location of CEP sensitive to **baryons**?

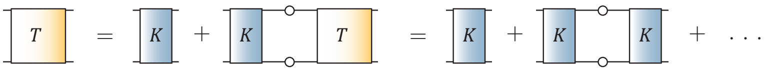
GE, Fischer, Welzbacher, PRD 93 (2016)



Bethe-Salpeter equations

Write down **inhomogeneous BSE**:

T-matrix = connected, amputated part of G



2-body irreducible kernel

Analogy: **geometric series**

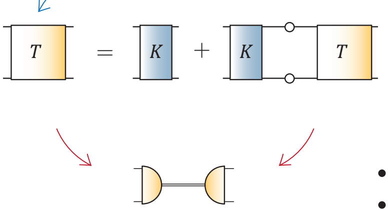
$$\left.\begin{aligned} f(x) &= 1 + x\, f(x) \\ &= 1 + x + x^2\, f(x) \\ &= 1 + x + x^2 + x^3\, f(x) \end{aligned} \qquad \Rightarrow \quad f(x) = \frac{1}{1-x} \quad \right\} \quad \text{“non-perturbative”}$$

$$\left. f(x) \approx 1 + x + x^2 + x^3 + \dots \qquad \text{only for } \; |x| < 1 \quad \right\} \quad \text{“perturbative”}$$

Bethe-Salpeter equations

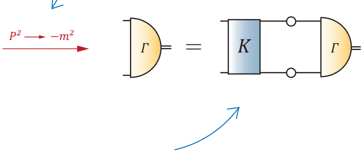
Write down **inhomogeneous BSE**:

T-matrix = connected, amputated part of G



Homogeneous BSE at pole:

compare pole residues



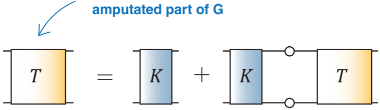
- $q\bar{q}$ irreducible kernel
- chiral symmetry constraints (V + AV WTI)
- can be systematically derived from effective action, depends on QCD's n-point functions

- Analogue of Schrödinger equation in QFT!
- Γ = **Bethe-Salpeter amplitude**

Bethe-Salpeter equations

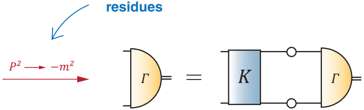
Write down **inhomogeneous BSE**:

T-matrix = connected, amputated part of G



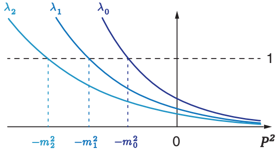
Homogeneous BSE at pole:

compare pole residues



BSE = eigenvalue equation,
pole in T ⇔ **eigenvalue = 1**

$$K G_0 \, \Gamma_i = \lambda_i \, \Gamma_i$$



Explicitly:

$$\Gamma(p,P) = \int \frac{d^4q}{(2\pi)^4} \, \mathbf{K}_{\alpha\gamma,\delta\beta}(p,q,P) \, [S(q_+) \, \Gamma(q,P) \, S(q_-)]_{\gamma\delta}$$

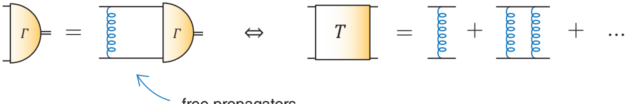
Basis decomposition:

$$\Gamma(p,P) = \sum_{i=1}^{n} f_i(p^2, \hat{p}\cdot\hat{P}, P^2) \, \tau_i(p,P)$$

⇒ Coupled Lorentz-invariant equations
for the dressing functions f_i

Ladder

Simplest attempt:

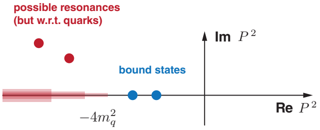


free propagators

$$\frac{-i\not p + m}{p^2+m^2}$$

- breaks chiral symmetry (**free propagators** ⇔ **contact interaction**) ↯
- generates bound-state poles in G and T, possibly also resonances
- but also quark thresholds & cuts: ↯ "hadrons" decay into quarks, no confinement

Analytic structure of G, T, etc. would look like this:

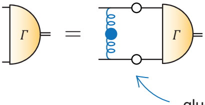


would be ok if elementary d.o.f. were not quarks but **hadrons** (→ EFTs)

Rainbow-ladder

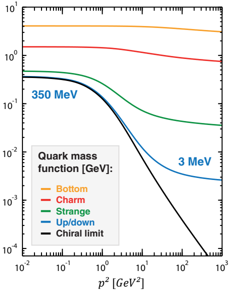
Better: **rainbow-ladder truncation**

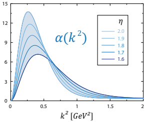
Maris, Roberts, PRC 56 (1997), Maris, Tandy, PRC 60 (1999)

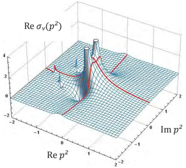


gluon = effective interaction,
dressed propagators
from quark DSE

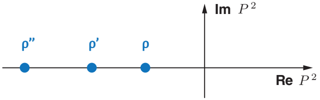
$$\frac{1}{A(p^2)}\,\frac{-i\not p + M(p^2)}{p^2+M(p^2)^2}$$







Analytic structure of G, T, etc.
would look like this:

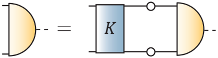


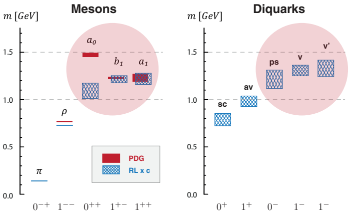
- chiral symmetry ✓
- dynamical propagators do not have real poles ⇒ no quark thresholds ✓
- but no resonances: **bound states** (need to go beyond rainbow-ladder)

Diquark correlations

Mesons and diquarks closely related through BSE

Maris, FBS 32 (2002)





Lowest-lying diquarks are dominant for ground-state octet & decuplet baryons

pseudoscalar mesons ⇔ **scalar diquarks** (~0.8 GeV)
vector mesons ⇔ **axialvector diquarks** (~1 GeV)

Higher-lying diquarks are subleading, but contribute to excited states & remaining channels

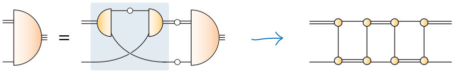
scalar mesons ⇔ **pseudoscalar diquarks** (~1.2 GeV)
axialvector mesons ⇔ **vector diquarks** (~1.3 GeV)

In RL, these are too strongly bound; simulate beyond-RL effects by (one) strength parameter c

Roberts, Chang, Cloet, Roberts, FBS 51 (2011)
GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)

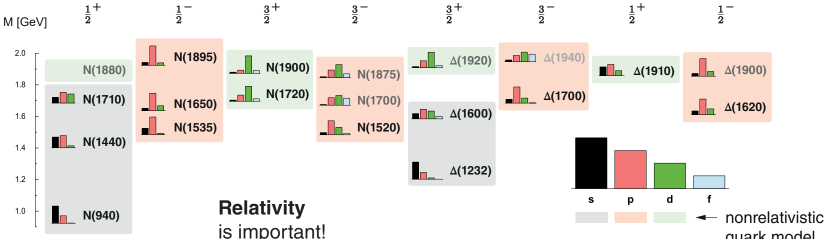
Diquark correlations

- **Quark-diquark** (two-body) equation
  Oettel et al., PRC 58 (1998), GE et al., Ann. Phys. 323 (2008), Cloet et al., FBS 46 (2009), Segovia et al., PRL 115 (2015), Chen et al., PRD 97 (2018)



Diquark clustering in baryons?
Barabanov et al., Prog. Part. Nucl. Phys. 116 (2021)

- **Three-quark** and **quark-diquark** results very similar
  GE, Fischer, Sanchis-Alepuz, PRD 94 (2016), GE, FBS 63 (2022)

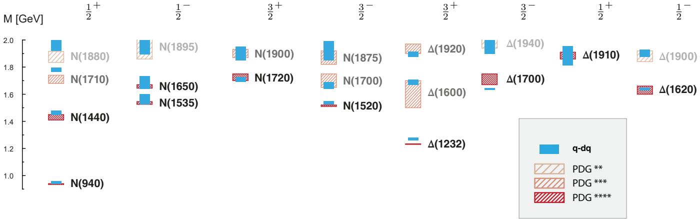


Relativity
is important!

Diquark correlations

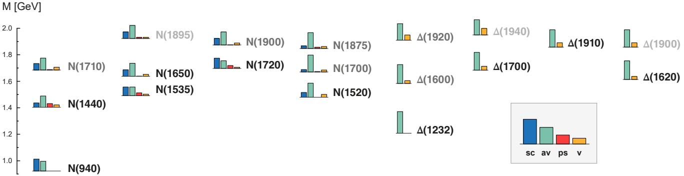
Light baryon spectrum

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



Diquark content:

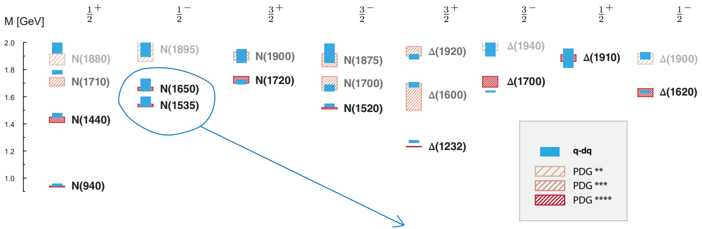
Barabanov et al., PPNP 116 (2021)



Diquark correlations

Light baryon spectrum

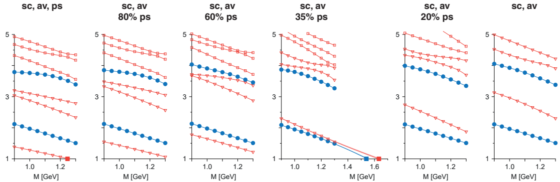
GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



RL, all diquarks:
"N(1535)" too low

"Beyond RL":
N(1535), N(1650)

RL, sc+av only:
"N(1650)" too high

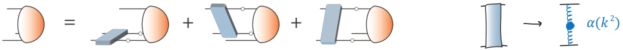


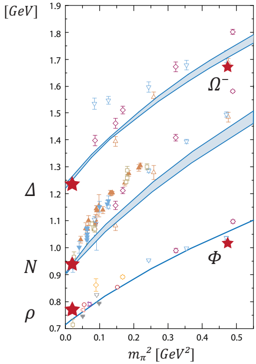
- **Level ordering** determined by diquark dynamics
- Diquarks are not pointlike, also here **rich spectrum!**

Barabanov et al., PPNP 116 (2021)

Baryons

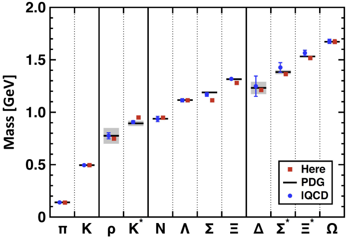
Baryon spectrum from three-body equation (in rainbow-ladder)





GE, Alkofer, Nicmorus,
Krassnigg, PRL 104 (2010)

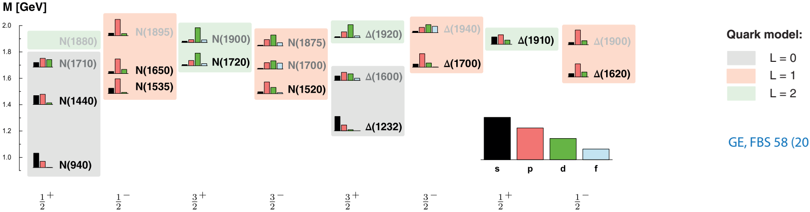
GE, Sanchis-Alepuz,
Williams, Alkofer, Fischer,
Prog. Part. Nucl. Phys. 91 (2016)



Qin, Roberts, Schmidt,
PRD 97 (2018), FBS 60 (2019)

Relativistic effects

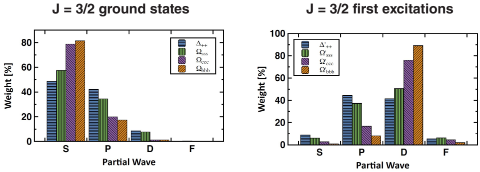
Orbital angular momentum: clear traces of nonrelativistic quark model, but strong relativistic effects (in some cases even dominant)



GE, FBS 58 (2017)

Relativistic contributions
even up to bottom baryons!

Qin, Roberts, Schmidt, PRD 97 (2018)

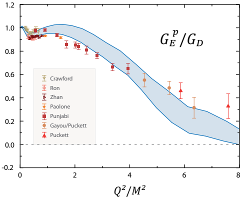

J = 3/2 ground states

J = 3/2 first excitations

Baryon structure

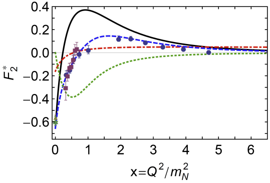
Nucleon electromagnetic FFs

GE, PRD 84 (2011)



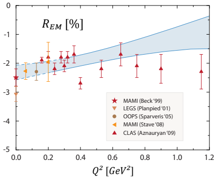
Roper em. transition FFs

Segovia et al., PRL 115 (2015)



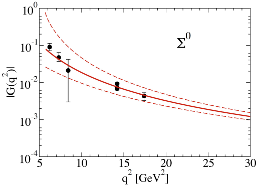
Δ em. transition FFs

GE, Nicmorus, PRD 85 (2012)



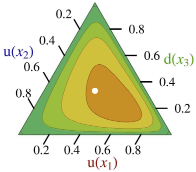
Timelike em. strangeness FFs

Ramalho, Peña, PRD 101 (2020)



Distribution amplitudes

Mezrag, Segovia, Chang, Roberts, PLB 783 (2018)

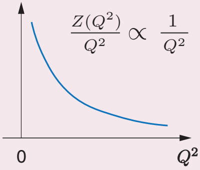


Towards ab-initio

Gluon propagator:

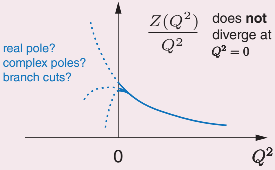
$$D^{\mu\nu}(Q) = \overset{\text{transverse dressing}}{\boxed{\frac{Z(Q^2)}{Q^2}}} \left(\delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + \xi \, \overset{\text{longitudinal dressing = 1}}{\boxed{\frac{L(Q^2)}{Q^2}}} \, \frac{Q^\mu Q^\nu}{Q^2}$$

- **Perturbation theory:** Massless gluon pole



$$\frac{Z(Q^2)}{Q^2} \propto \frac{1}{Q^2}$$

- **Nonperturbative calculations:** Massless pole disappears!



$\frac{Z(Q^2)}{Q^2}$ does **not** diverge at $Q^2 = 0$

real pole?
complex poles?
branch cuts?

Family of **"decoupling"** solutions, also seen in lattice QCD

Cucchieri, Maas, Mendes, PRD 77 (2008)
Boucaud et al., JHEP 06 (2008)
Bogolubsky et al., PLB 676 (2009)
Fischer, Maas, Pawlowski, Ann. Phys. 324 (2009)
Duarte, Oliveira, Silva, PRD 94 (2016)
Aguilar et al., EPJ C 80 (2020)

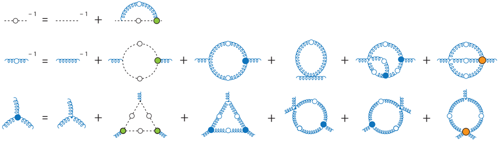
Endpoint is **"scaling"** solution, confinement manifest

Lerche, Smekal, PRD 65 (2002)
Fischer, Alkofer, PLB 536 (2002)
Alkofer, Fischer, Llanes-Estrada, MPLA 23 (2008)

All solutions show **gluon mass gap**

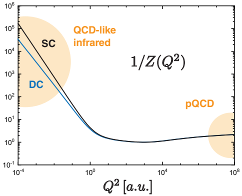
$$\lim_{r\to\infty} \int \frac{d^3Q}{(2\pi)^3} \frac{Z(\boldsymbol{Q}^2)}{\boldsymbol{Q}^2} e^{i\boldsymbol{x}\cdot\boldsymbol{Q}} \propto e^{-m_{\text{gap}}\, r}$$

Coupled **Yang-Mills DSEs**
GE, Pawlowski, Silva, PRD 104 (2021)

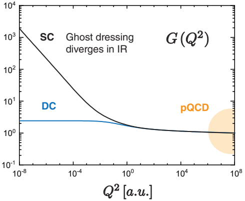


→ Test confinement in hadron observables!

Gluon mass gap



Gluon



Ghost

- What distinguishes SC + DC?
- What is the "true" solution? Are all solutions physically equivalent?

Scaling (SC) solution:

- n-point functions scale with IR power laws
  Lerche, Smekal, PRD 65 (2002), Fischer, Alkofer, PLB 536 (2002)
- Confinement
  Alkofer, Fischer, Llanes-Estrada, Mod. Phys. Lett. A 23 (2008)

Decoupling (DC) solution:

- Seen in lattice QCD
  Cucchieri, Maas, Mendes, PRD 77 (2008), Bogolubsky et al., PLB 676 (2009), Duarte, Oliveira, Silva, PRD 94 (2016), Aguilar et al., EPJ C 80 (2020)
- Functional methods: family of DC solutions with SC solution as endpoint
  Boucaud et al., JHEP 06 (2008), Fischer, Maas, Pawlowski, Ann. Phys. 324 (2009), Reinosa et al., PRD 96 (2017)

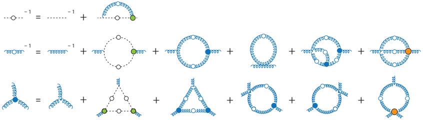
Gluon mass gap

Gluon propagator:

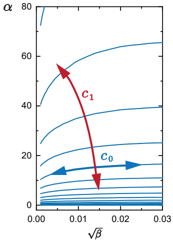
$$D^{\mu\nu}(Q) = \overset{\text{transverse dressing}}{\boxed{\frac{Z(Q^2)}{Q^2}}}\left(\delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + \xi\,\frac{\overset{\text{longitudinal dressing = 1}}{\boxed{L(Q^2)}}}{Q^2}\,\frac{Q^\mu Q^\nu}{Q^2}$$

Coupled **Yang-Mills DSEs**

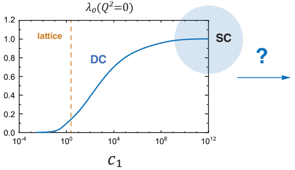
Huber, PRD 101 (2020), GE, Pawlowski, Silva, PRD 104 (2021)

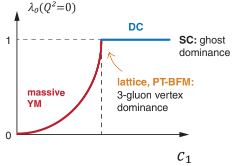


- SC & DC solutions distinguished by c_1 (combination of coupling α and mass parameter β)
- Scaling solution: $c_1 \to \infty$



Gauge consistency (L = 1) satisfied only by **SC solution**





Possible scenario for consistency of DC solutions:

Then all massless (SC, DC) solutions would be physically equivalent; distinguished from massive solutions

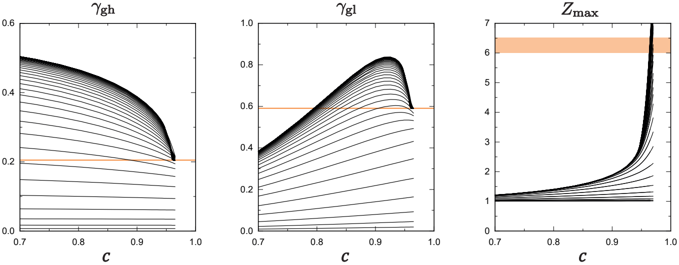
Truncation error

- Set $Z_{3g} \to c\, Z_{3g}$ ... quantifies deviation from STI
  (without truncation: $c = 1$)

- Anomalous dimensions reproduced for
  - **1** c ~ 0.4
  - **2** c ~ 0.9
  - **3** c ~ 0.96

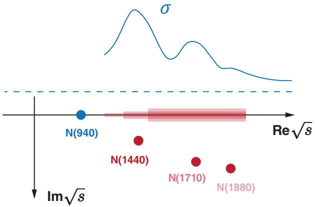
  ⇒ identifies “physical point” for each truncation

  GE, Pawlowski, Silva, PRD 104 (2021)



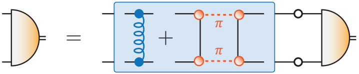
Resonances

- Most hadrons are **resonances** and decay
  ⇔ poles in complex momentum plane



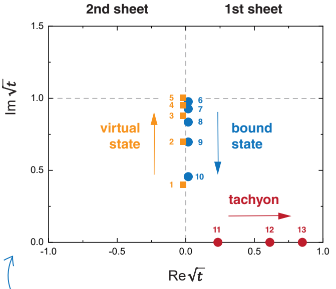
- BSE kernel must include decay channels:
  ρ meson becomes resonance

Williams, PLB 798 (2019), Miramontes, Sanchis-Alepuz, EPJA 55 (2019),
Santowsky, GE, Fischer, Wallbott, PRD 102 (2020),
Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)

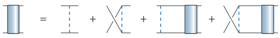


- **Contour deformations** as tool
  to go beyond thresholds

GE, Duarte, Peña, Stadler, PRD 100 (2019)

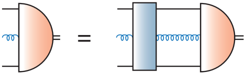


Scattering equation for 4-point function



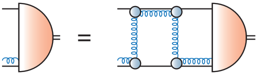
Hybrids

- **Three-body equation** (quark, antiquark, gluon)



... in progress

- **Two-body equation** [quark–gluon]–antiquark with model ansätze
  Xu, Cui, Chang, Papavassiliou, Roberts, EPJA 55 (2019)



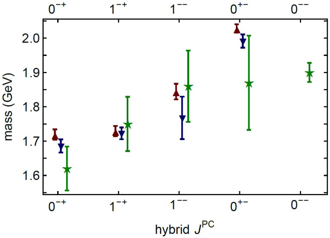
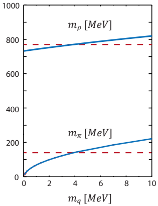


FIG. 2. Comparison between our ACM-improved spectrum (stars, green), Row 2 in Table I, and the rescaled lQCD results in Rows 3 (up-triangles, red) and 4 (down-triangles, blue).

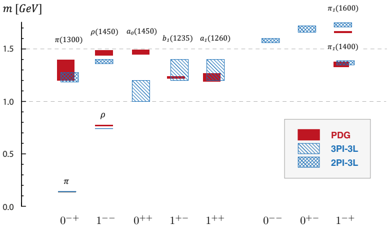
Lattice: Dudek, Edwards, Peardon, Richards, Thomas, PRD 82 (2010)

Mesons

- Pion is **Goldstone boson:** $m_\pi^2 \sim m_q$



- **Light meson spectrum** beyond rainbow-ladder

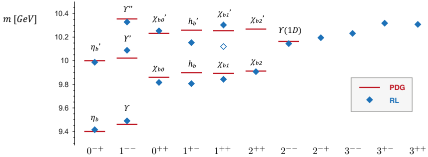


Williams, Fischer, Heupel, PRD 93 (2016)

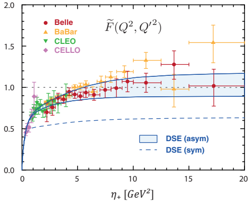
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016)

- **Bottomonium spectrum**

Fischer, Kubrak, Williams, EPJ A 51 (2015)

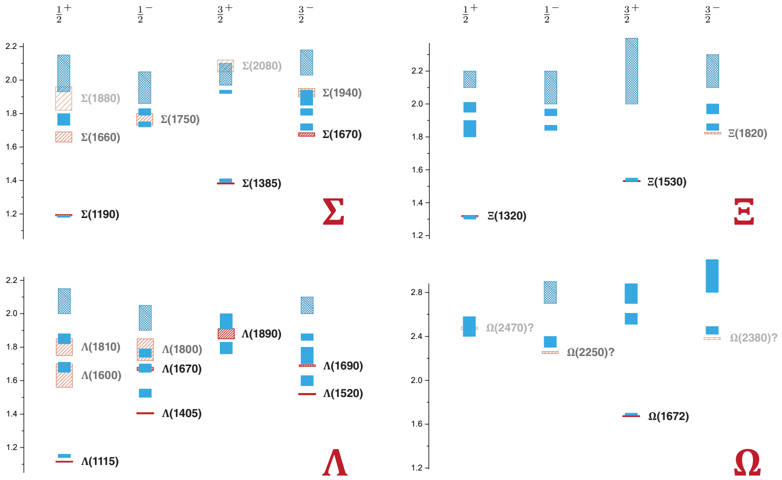


- **Pion transition form factor**



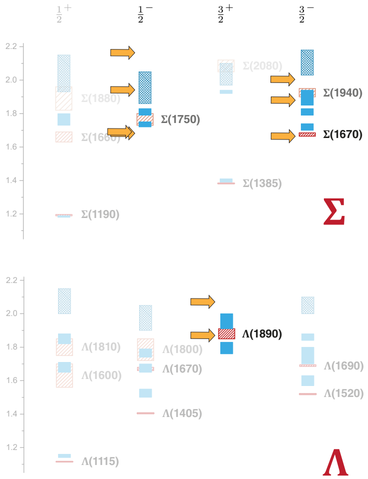
GE, Fischer, Weil, Williams, PLB 774 (2017)

Strange baryons



GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Strange baryons



New states from Bonn-Gatchina

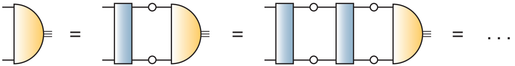
Sarantsev et al., 1907.13387 [nucl-ex]

GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

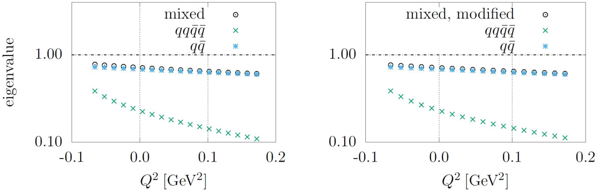
Reply to comment

- Recent comment: BSE kernel is reducible
  Blankleider & Kvinikhidze, 2102.05818

- Irrelevant for homogeneous BSE: same spectrum

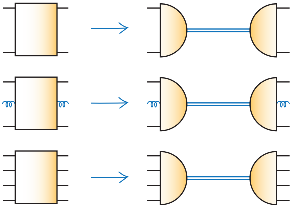


- Results are identical within numerical errors
  Santowsky, GE, Fischer, Wallbott, Williams, 2103.14673

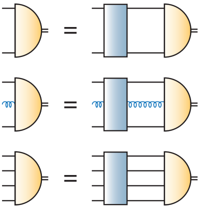


Some thoughts

- Same spectral representation for **all** correlation functions that produce $q\bar{q}$:



- **Without** truncations, BSEs for qq, $qq\overline{qq}$, $q\bar{q}g$, ... should produce **same spectrum**



→ harder to include $qq\overline{qq}$, $q\bar{q}g$

→ harder to include $q\bar{q}$ components

With truncations, $q\bar{q}$ ($qq\overline{qq}$, $q\bar{q}g$, ...) BSE should give more reliable spectrum for $q\bar{q}$ ($qq\overline{qq}$, $q\bar{q}g$, ...) dominated states

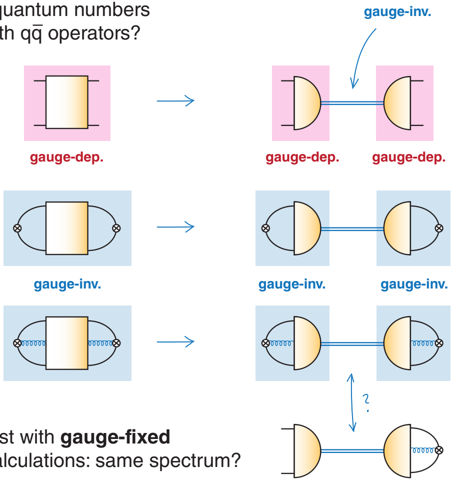
Exotic quantum numbers not excluded in principle (need gluon-rich kernel in $q\bar{q}$ BSE)

Some thoughts

- Why don't we see exotic quantum numbers in **lattice calculations** with $q\bar{q}$ operators?

$$\langle q_\alpha \, \bar{q}_\beta \, q_\gamma \, \bar{q}_\delta \rangle \; =$$

$$\langle (\bar{q}\,\Gamma q)(\bar{q}\,\Gamma q) \rangle \; =$$



= 0 for exotic quantum nrs (?)

≠ 0 for exotic quantum nrs

Could test with **gauge-fixed** lattice calculations: same spectrum?