Quasicrystal in QCD

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Low Energy Regime of QCD

Spontaneous Chiral Symmetry Breaking

$$U(1)_V \times SU(N_f)_L \times SU(N_f)_R \to U(1)_V \times SU(N_f)_V$$

Nambu Goldstone (NG) boson

$$\Sigma = \exp\left(iT^a \pi^a(x)/f_\pi\right) \in SU(N_f)$$

Chiral Lagrangian (Kinetic & Mass Terms):

$$\mathcal{L}_{\text{chiral}} = \frac{f_\pi^2}{4}\text{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma\right) - \frac{b}{2}\text{Tr}\left[M(\Sigma - 1) + \text{h.c.}\right]$$

$N_f = 2$ case with approximately $m_u \approx m_d$

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \quad m_u \approx m_d \equiv m$$

Topology with Magnetic Field

Pions $\xrightarrow{\boldsymbol{B}}$ Chiral Soliton Lattice (CSL) / π^0 domain wall (DW)

Idea by T. Brauner and N. Yamamoto, JHEP 04, 132 (2017).

LLL: $n = 0$, Pion: Spin $s = 0$

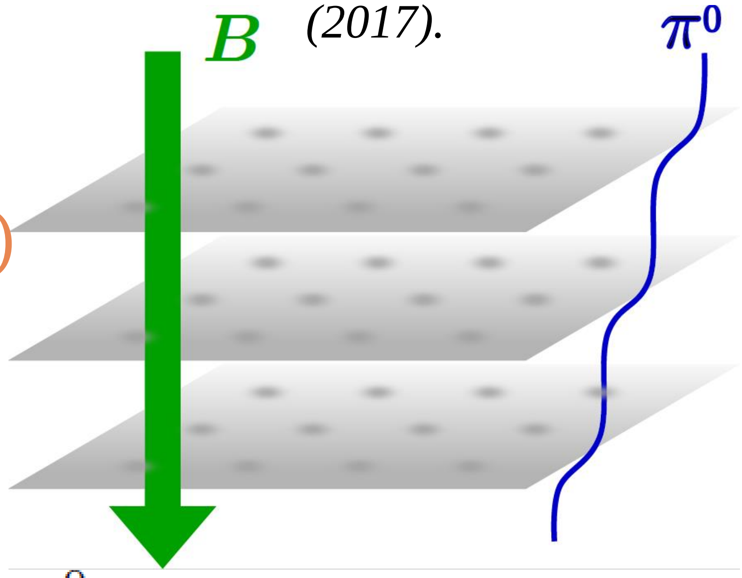
$$\varepsilon^2 = p_z^2 + 2|eB|(n + 1/2) + m^2 - 2seB$$

$\pi^{1,2} \xrightarrow{\boldsymbol{B}\uparrow}$ "massive", $\Sigma \to \exp(i\tau^3\pi^3)$

Transverse homogeneity $\partial_{x,y} = 0$ minimizes the Hamiltonian

$$\frac{f_\pi^2}{2}(\nabla\pi^0)^2 + m_\pi^2 f_\pi^2 (1 - \cos\pi^0) - \frac{\mu_B B}{4\pi^2}\partial_z\pi^0$$

∂_z Sine Gordon Solution



Winding $\pi_1(U(1)) = \mathbb{Z}$

Result from Triangle Anomaly

Coupling of two U(1) via triangle anomaly to NG boson ϕ_i

$$\mathcal{L}_B = \frac{1}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\sum_i C_i \partial_\mu \phi_i A_\nu^B F_{\alpha\beta}$$

D. T. Son, M. A. Stephanov, and A. R. Zhitnitsky, Phys. Rev. Lett. 86, 3955 (2001);

Setup of chemical potential μ and magnetic field $\boldsymbol{B} = B\hat{z}$

$$U(1)_B : A_\nu^B = (\mu, \mathbf{0}) . \quad U(1)_{\mathrm{EM}} : A_\mu = (0, yB/2, -xB/2, 0)$$

ϕ_i be not only pions but also η mesons (also form DW)

$$\phi_3 \equiv \frac{\pi_3}{f_\pi} : \quad \mathcal{L}_{\mathrm{WZW}} = -\frac{\mu B}{4\pi^2}\partial_z \phi_3 \Rightarrow \mathrm{CSL}$$

D. T. Son and A. R. Zhitnitsky , Phys. Rev. D 70, 074018 (2004).

$$\phi_0 \equiv \frac{\eta}{f_\eta} : \quad \mathcal{L}_\eta = -\frac{\mu B}{12\pi^2}\partial_z \phi_0 \Rightarrow \eta\text{-CSL?}$$

Our motivation: mixture of the two, new ground state?

ChPT of U(2) and Reduction

$\pi^{\pm}$ decoupled, η with π^0 remained: $U = \exp\phi_0 \exp(i\tau_3\phi_3)$

Newly included η

Kinetic

Anomaly

$$H = \frac{1}{2}\left[\alpha\left(\partial_z\phi_3\right)^2 + \left(\partial_z\phi_0\right)^2\right] - \frac{\gamma}{2\pi}\left(\partial_z\phi_3 + \frac{1}{3}\partial_z\phi_0\right)$$

$$+ \sin\beta\left(1 - \cos 2\phi_0\right) + \cos\beta\left(1 - \cos\phi_0\cos\phi_3\right)$$

Mass

Redefine parameters to have dimensionless quantities

$$\alpha \equiv \frac{f_\pi^2}{f_\eta^2}, \quad \beta \equiv \arctan\frac{a}{2mb}, \quad \gamma \equiv \frac{\mu B}{2\pi}\frac{\left[a^2 + (4mb)^2\right]^{1/4}}{f_\eta}$$

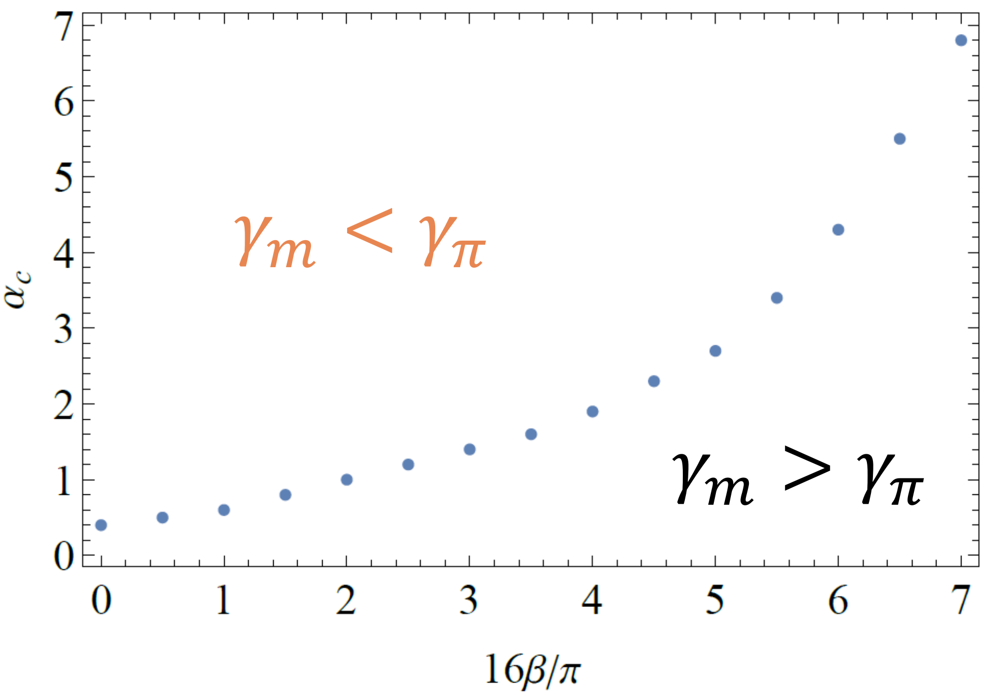
Boundary conditions for periodic lattices with both mesons

$$\left(\phi_3, \phi_0\right)\Big|_{z=0} = (0,0)\,, \quad \left(\phi_3, \phi_0\right)\Big|_{z=d} = (p\pi, q\pi)\,, \quad \frac{p \pm q}{2} \in \mathbb{Z}$$

Mixed Soliton Lattice

Irrelevant $\gamma_\eta \gg \gamma_{\pi,m}$. Duel is between γ_π and γ_m.

(p,q)	Soliton Type	Critical magnetic field
(2,0)	π^0	γ_π
(0,2)	η	γ_η
(1,1)	Mixed	γ_m



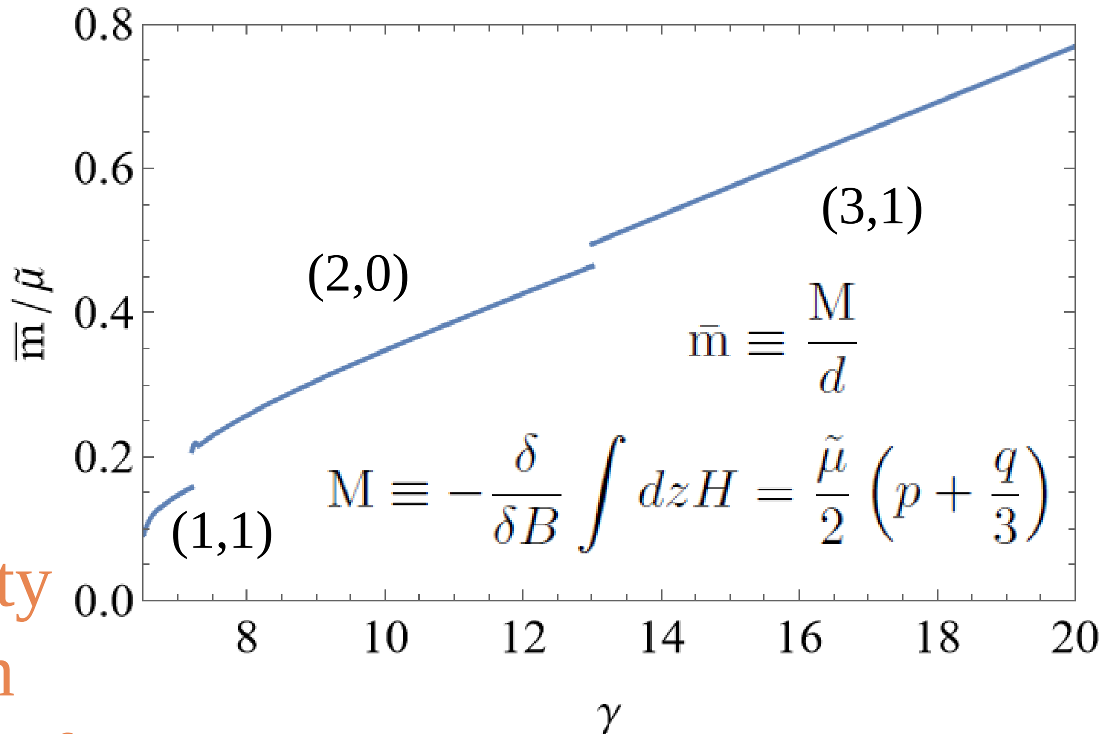
γ_m can be lowest! for $\alpha > \alpha_c(\beta)$

Ground State Alternation

Up to $p+q=4$, under $\alpha = 0.7$ and $\beta = \pi/16$: competitive configurations are (1,1), (2,0), and (3,1) with critical $\gamma_m = 6.5$, $\gamma_\pi = 6.7$ and $\gamma_{31} = 7.3$.

Ground State

$$= \begin{cases} (1,1) & \gamma \in [6.4,\ 7.2) \\ (2,0) & \gamma \in [7.2,\ 13.0) \\ (3,1) & \gamma \in (13.0,\ \ldots) \end{cases}$$



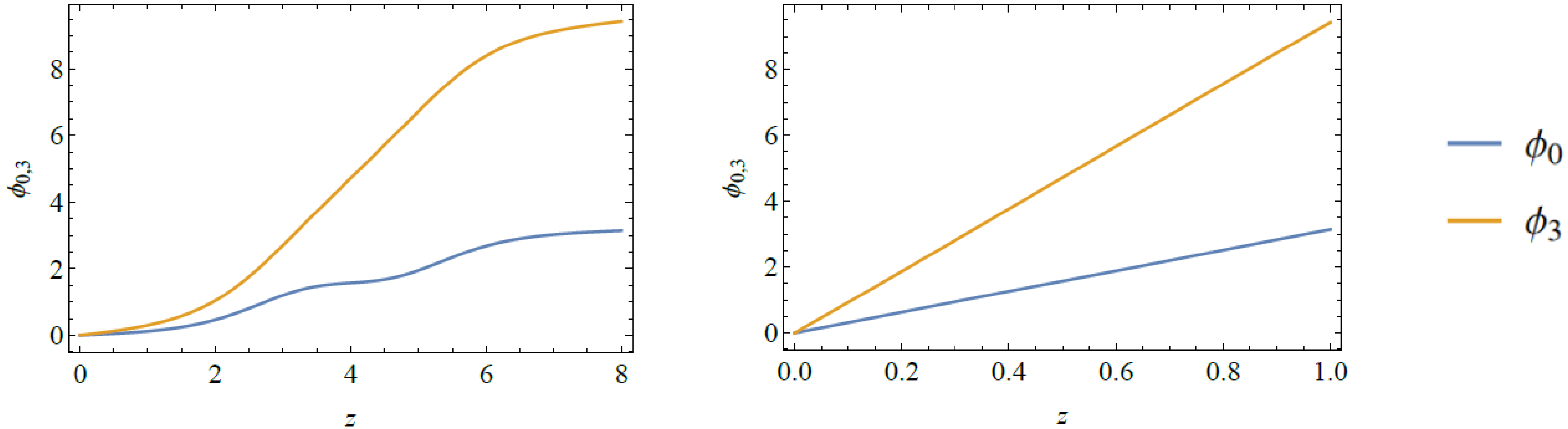
Magnetic moment density quantifies the alternation as a piece-wise function of μB

Strong Magnetic Field Limit

Conclusion: in $\gamma \to \infty$ limit, the ground state satisfies

$$r \equiv \frac{p}{q} = \frac{3}{\alpha}$$

Profiles: $\gamma \uparrow$, $d \downarrow$, $\phi_{3(0)} \to p(q)\pi z/d$; (almost linear)



The near linearity means mass (β) terms are negligible.

Semi-analytical Derivation

Kinetic and WZW terms remain:

$$H\left(\gamma\to\infty\right)\simeq\frac{1}{2}\left(\alpha\phi_3'^2+\phi_0'^2\right)-\frac{\gamma}{2\pi}\left(\phi_3'+\frac{1}{3}\phi_0'\right)\equiv H_\infty.$$

Simple trick of total square finds us the minimum:

$$H_\infty=\frac{1}{2}\left[\left(\sqrt{\alpha}\phi_3'-\frac{\gamma}{2\sqrt{\alpha}\pi}\right)^2+\left(\phi_0'-\frac{\gamma}{6\pi}\right)^2\right]-\frac{\gamma^2}{8\pi^2}\left(\frac{1}{\alpha}+\frac{1}{9}\right)$$

$$\geq-\frac{\gamma^2}{8\pi^2}\left(\frac{1}{\alpha}+\frac{1}{9}\right)\equiv E_{\rm min}.$$

Lattice period d_L and ratio p/q minimizing the energy

$$d_L=\frac{2\pi^2}{\gamma}\cdot p\alpha=\frac{2\pi^2}{\gamma}\cdot 3q\quad\Rightarrow\quad\frac{p}{q}=\frac{3}{\alpha},$$

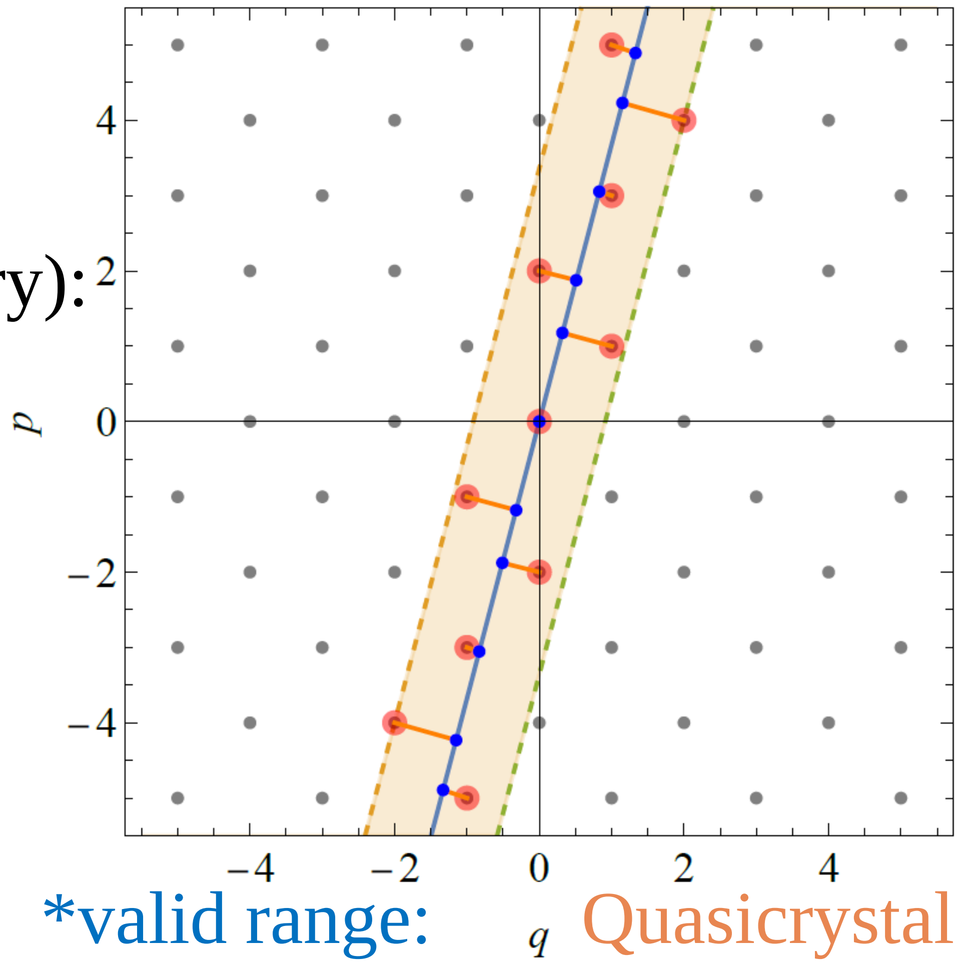
Quasicrystal for irrational α

Grey dots: mixed soliton lattice of different *p*, *q*.

Blue solid line (exemplary): $p/q = 3f_\eta^2/f_\pi^2$ (irrational)

Orange dots: adjacent *(p,q)* to the blue line.

Beige band: candidate ground states with no defined period.



*valid range: Quasicrystal!

$\mu B \ll \Lambda^3, \quad \Lambda \simeq 4\pi f_{\pi,\eta}; \quad \gamma \ll 5.6\pi^2$

Conclusion

- Depending on decay constants and effective masses, the mixed soliton lattice of η and π^0 could have lower energy / critical magnetic field than separate ones.

- The ground state is alternating among mixed soliton lattices with different p and q when changing γ.

- In strong magnetic field / density limit, the ground state ratio p/q approaches $3f_\eta^2/f_\pi^2$ which is generally irrational and unreachable. The result is a mesonic quasicrystal (perhaps the first in QCD context).