Entanglement in Few-Particle Scattering

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Motivation

- Entanglement as a measure of correlation in a system
- New perspective on nuclear systems
- Scattering of nucleons and deuterons
  $\rightarrow$ already previous work on it
- Extension to halo nuclei
  - $^{11}Be - n$
  - $^{15}C - n$
  - $^{19}C - n$

Examples - Entanglement Entropy

- Quantifying entanglement: entanglement entropies
- Von Neumann entropy $E_N(\hat{\rho}_1) = -\operatorname{Tr}[\hat{\rho}_1 \ln(\hat{\rho}_1)]$ with reduced density matrix $\hat{\rho}_1 = \operatorname{Tr}_2[|\psi\rangle\langle\psi|]$

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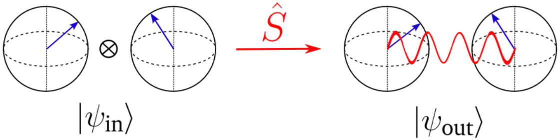
- Rényi entropy $E_R(\alpha) = \frac{1}{1-\alpha}\ln\left(\operatorname{Tr}[\hat{\rho}^\alpha]\right) = \frac{1}{1-\alpha}\ln\left(\sum_i \lambda_i^\alpha\right)$ for $\alpha > 0, \quad \alpha \neq 1$

Formalism

$S = \frac{1}{2}$:

Bloch sphere



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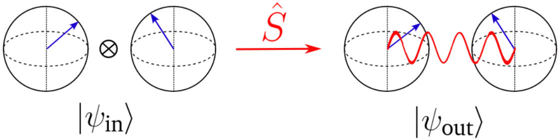

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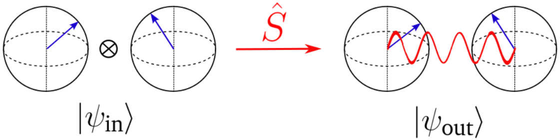

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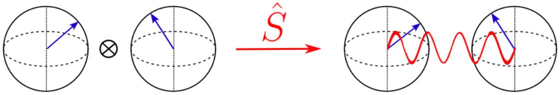

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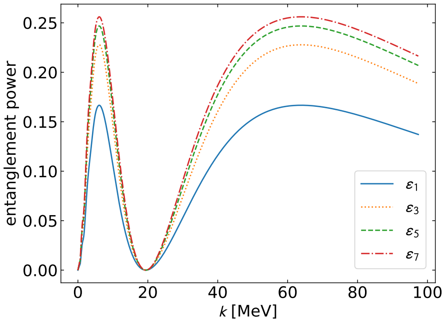
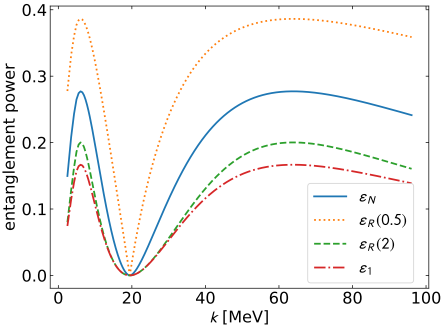
- Entanglement power = entropy averaged over initial states
- Spin $\frac{1}{2}$ and spin 1 - systems
  - np scattering
  - nd and pd scattering
  - dd scattering
  - spin-$\frac{1}{2}$ $halo\ n$ scattering
    - $^{11}Be - n$, $^{15}C - n$ and $^{19}C - n$ scattering

Proton-neutron scattering

Taylor expansions:



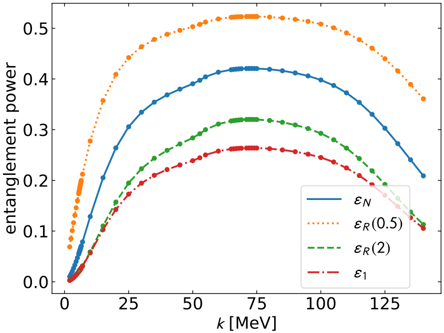
Entropies:



$$\varepsilon_1 = \frac{1}{6}\sin(2\cdot(\delta_0 - \delta_1))^2$$

[Beane et al., PRL 122, 102001 (2019)]

Neutron-deuteron scattering

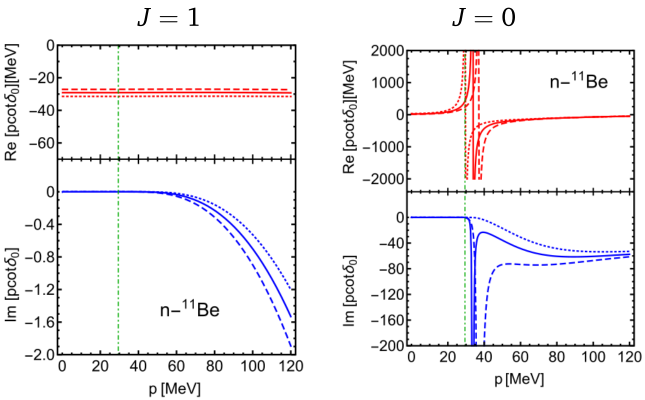


$$\varepsilon_1 = \frac{8}{243}(17 + 10\cos[2(\delta_{1/2} - \delta_{3/2})])\sin[\delta_{1/2} - \delta_{3/2}]^2$$

[TK et al., 2023]

Why study halo nuclei

- use n-halo scattering to study entanglement
- not easy to measure - I use theoretically calculated phase shift data
- $a_{\mathrm{Be}^{10}-\mathrm{n}} = 6.741$ fm



[Zhang et al., 2023]

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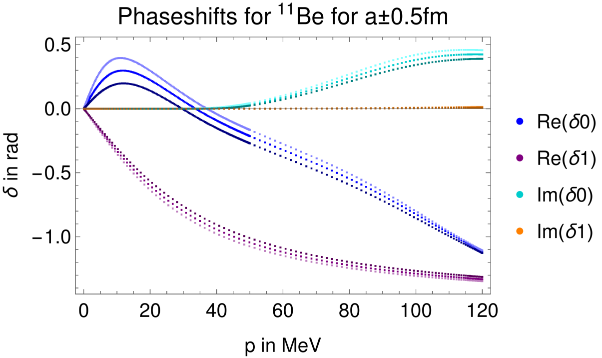
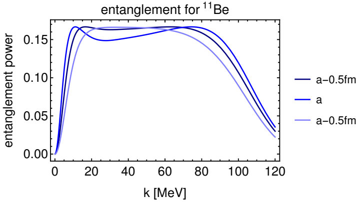


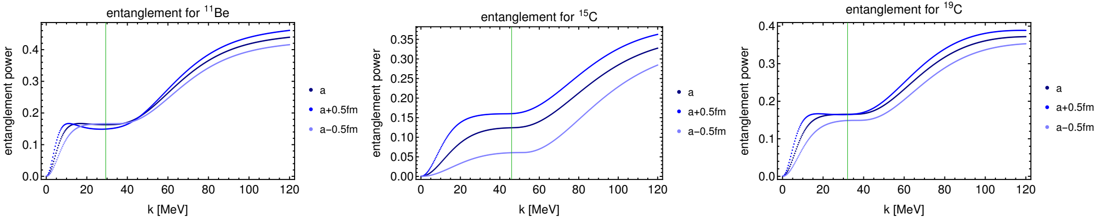
Halo-neutron scattering

- $\varepsilon_1 = \frac{1}{6}\sin(2(\delta_0 - \delta_1))^2$ for non-imaginary phase shifts



Halo-neutron scattering

- breakup of core and neutron gives new channel
  $\rightarrow$ imaginary phase shifts
- entanglement power:
- $\varepsilon_1^{nh} = 1 - \frac{1}{24}\,\mathrm{e}^{-8\delta_0^I} - \frac{13}{24}\,\mathrm{e}^{-8\delta_1^I} - \frac{1}{3}\,\mathrm{e}^{-4(\delta_0^I+\delta_1^I)} - \frac{1}{24}\,\mathrm{e}^{-4(\delta_0^I+\mathrm{i}\delta_0^R+\delta_1^I-\mathrm{i}\delta_1^R)} - \frac{1}{24}\,\mathrm{e}^{-4(\delta_0^I-\mathrm{i}\delta_0^R+\delta_1^I+\mathrm{i}\delta_1^R)}$



- green lines show breakup threshold

Halo-neutron scattering

- breakup of core and neutron gives new channel
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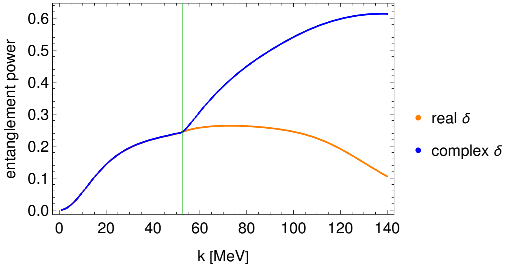
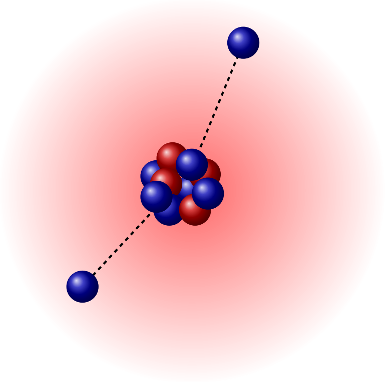
- nd scattering with deuteron breakup threshold



Outlook

- Study of two particle scattering with spin $\frac{1}{2}$ and spin 1
  $\rightarrow$ First Taylor expansion valid quantity
- Compare with other halo nuclei
  $\rightarrow$ Rise in entanglement after breakup threshold
- Only spin entanglement so far
  $\rightarrow$ Interesting approach: investigate spatial entanglement for halo nuclei



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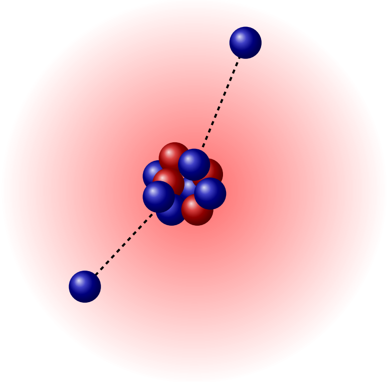
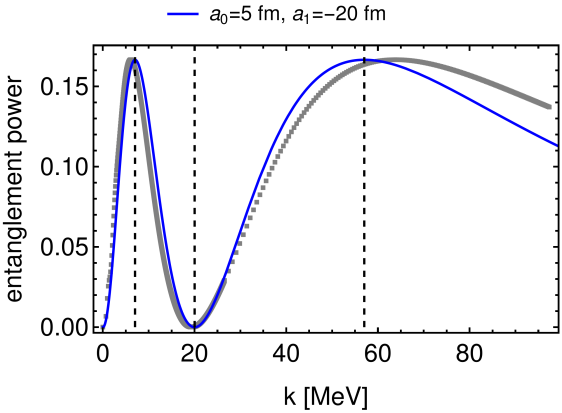


Proton-neutron scattering

- $\varepsilon_1 = \frac{1}{6}\sin(2\cdot(\delta_0 - \delta_1))^2$
- $k\cot(\delta(k)) = -\frac{1}{a} + \frac{r}{2}k^2 + \dots$
- $a_0 \approx 5$ fm and $a_1 \approx -20$ fm
- results in leading order:
  - $k_{\mathrm{min},1} = 0$ and $k_{\mathrm{min},2} = \frac{1}{\sqrt{-a_0 \cdot a_1}} \approx 20$ MeV
  - $k_{\mathrm{max},\pm} = \frac{\pm(a_0 - a_1) - \sqrt{a_0^2 - 6a_0a_1 + a_1^2}}{2a_0a_1}$
  - $k_{\mathrm{max},+} \approx 7$ MeV and $k_{\mathrm{max},-} \approx 57$ MeV

[Beane et al., 2021]



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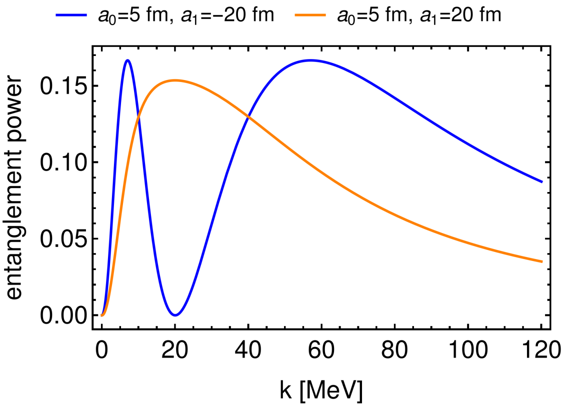
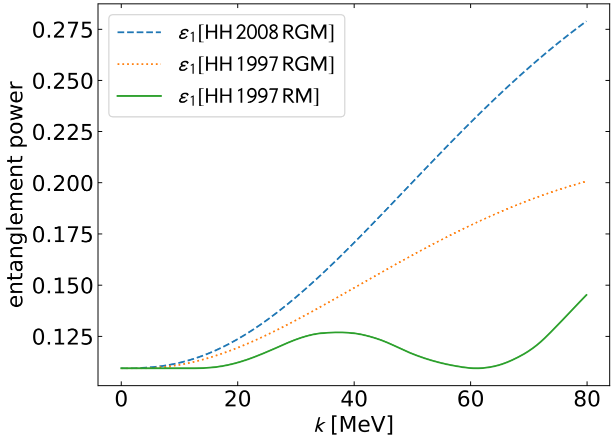


Coulomb influence

- $\delta_{\text{tot}}(k) = -\eta_k \log(2kr) + \sigma(k) + \delta_N(k)$
- $\sigma(k) = \arg\Gamma(1 + \mathrm{i}\eta_k)$ pure s-wave Coulomb phase shift
  - $\eta_k = \alpha e Z_1 Z_2 \mu / k$ Sommerfeld Parameter
  - $\alpha = e^2/(4\pi)$ electromagnetic fine structure constant in Heaviside-Lorentz units
  - μ reduced mass, Z_i charge numbers of both particles
- δ_N Coulomb-modified nuclear part (Coulomb-subtracted phase shift)
- influence cancels out as long as entanglement depends on phase shift differences
  $\rightarrow$ in all our systems the case

Deuteron-deuteron scattering



$$\varepsilon_1 = \frac{1}{576}\Big(153 - 70\cos[2(\delta_0 - \delta_2)] - 20\cos[4(\delta_0 - \delta_2)]\Big)$$

- Datasets from [Hofmann et al., 1997] and [Hofmann et al., 2008]
- calculated with "R-matrix analysis" and "Resonating Group Model"

Deuteron Deuteron scattering

- spin channel $S = 1$ forbidden due to Bose symmetry
- $\left\langle S = 1 \middle| \hat{S}_{dd} \middle| S = 1 \right\rangle = 0$
- $\hat{S}_{dd} = -\frac{1}{3}\left(\mathrm{e}^{2\mathrm{i}\delta_0} - \mathrm{e}^{2\mathrm{i}\delta_2}\right)\mathbb{1} + \frac{1}{2}\,\mathrm{e}^{2\mathrm{i}\delta_2}\,\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{6}\left(2\,\mathrm{e}^{2\mathrm{i}\delta_0} + \mathrm{e}^{2\mathrm{i}\delta_2}\right)(\vec{S}_1 \cdot \vec{S}_2)^2$

  $\rightarrow$ forbids the free propagation in the $S = 1$ channel
- $\varepsilon_1 = \frac{1}{576}(153 - 70\cos[2(\delta_0 - \delta_2)] - 20\cos[4(\delta_0 - \delta_2)])$ depends only on phase shift differences