

Novel Diagrammatic Method for Computing Transport Coefficients

— Beyond the Boltzmann approximation —



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Collaboration with Teiji Kunihiro

Goal

Derive a self-consistent equation
for transport coefficients.

Keywords

Beyond the Boltzmann equation,
Pinch singularity,
Eliashberg's method.

Hydrodynamics

as a Low energy effective theory

Conservation law:

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu J^\mu = 0$$

Hydrodynamic equation is universal.

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Details of theory are reflected by

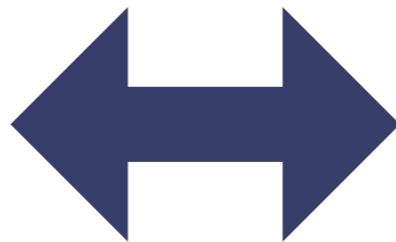
Equation of state : $P = P(\epsilon, n)$

Transport coefficients : η, ζ, \dots

Kubo Formula

Kubo and Tomita('54), Nakano('56), Kubo('57)

Transport
coefficient



Green function

Kubo Formula

Kubo and Tomita('54), Nakano('56), Kubo('57)

Transport
coefficient



Green function

Shear Viscosity:

$$\eta = \frac{1}{20} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \int d^4x e^{i\omega t} \theta(t) \langle [\pi_{ij}(x), \pi^{ij}(0)] \rangle$$

Bulk Viscosity:

$$\zeta = \frac{1}{2} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \int d^4x e^{i\omega t} \theta(t) \langle [\mathcal{P}(x), \mathcal{P}(0)] \rangle$$

where $\mathcal{P}(x) = -T^i_i(x)/3$, $\pi_{ij}(x) = T_{ij}(x) + g_{ij}\mathcal{P}(x)$

Why diagram?

Can start with **FUNDAMENTAL THEORY**.

In principle, **EXACT**.

Can apply **FIELD THEORETICAL TECHNICS**.

Why diagram?

Can start with **FUNDAMENTAL THEORY**.

In principle, **EXACT**.

Can apply **FIELD THEORETICAL TECHNICS**.

Our aim

Develop **SYSTEMATIC METHOD**
for calculating transport coefficients.

Apply **ELIASHBERG's METHOD ('62)**
to relativistic quantum field theory.

Diagrammatic Method

PREVIOUS WORKS

Fermi liquid:

Eliashberg('62), ...

Leading order:

Scalar theory: Jeon('95), Jeon and Yaffe('96),
Carrington, Hou and R. Kobes ('00),
Basagoiti ('02), Wang and Heinz ('03)

Chiral perturbation theory: Fernandez-Fraile and Nicola('06)

O(N) model, 2PI expansion: Aarts and Martinez Resco ('03), ('04), ('05)

QED: Gagnon and Jeon ('07)

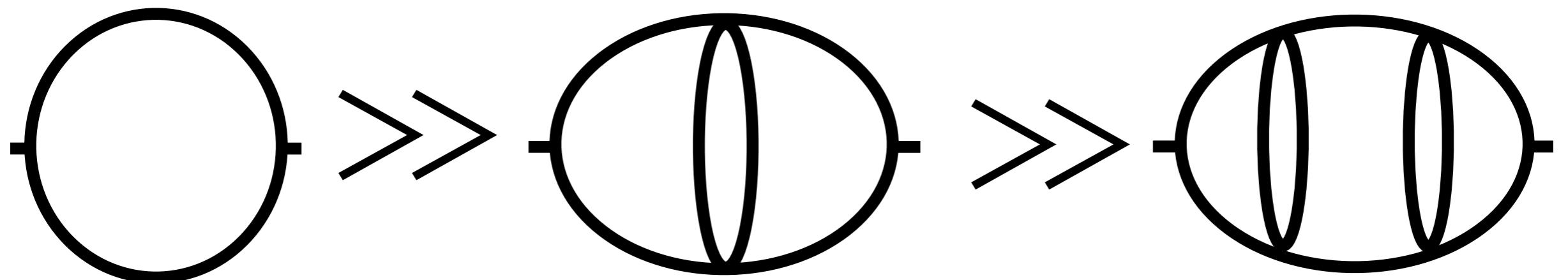
QCD, 3PI expansion: Carrington and Kovalchuk ('07), ('08), ('09)

Next to leading order:

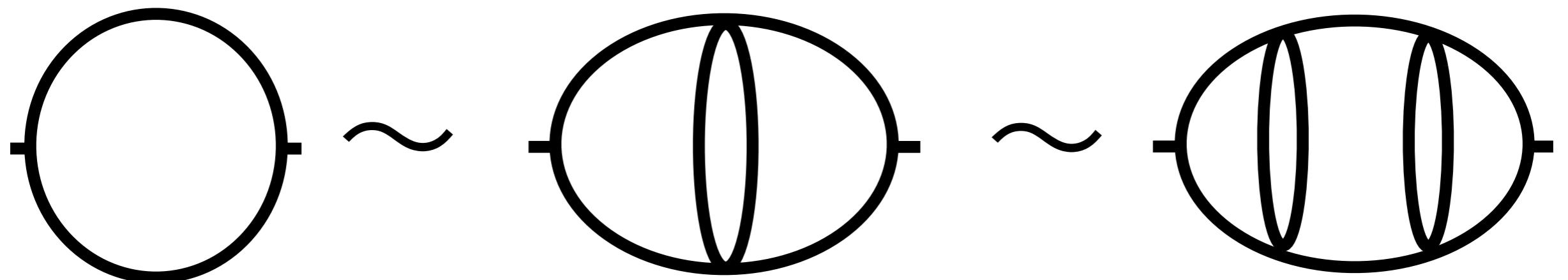
Scalar theory: Moore ('07)

4PI expansion: Carrington and Kovalchuk ('09)

Perturbation theory in vacuum



Perturbation theory in medium

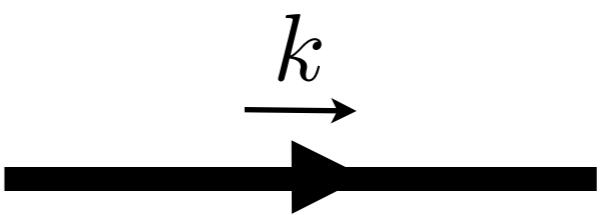


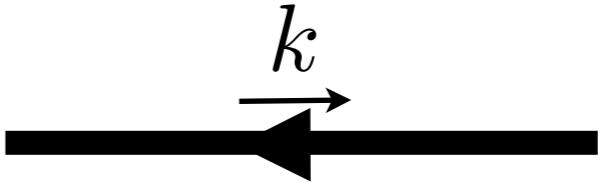
Real time formalism in R/A basis

R/A basis

Aurenche and Becherraw('92)

Propagator: $D^{\alpha\beta}(k) = \begin{pmatrix} 0 & -iD_R(k) \\ -iD_A(k) & 0 \end{pmatrix}$

Retarded propagator: $D_R(k)$ 

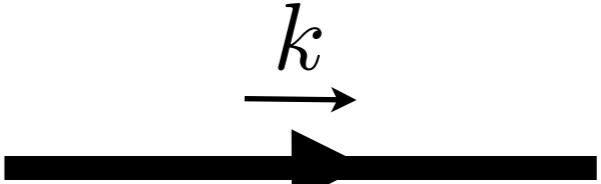
Advanced propagator: $D_A(k)$  $= D_R(-k)$

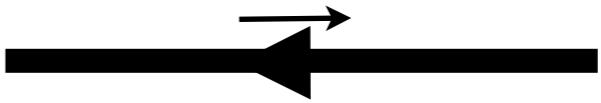
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R/A basis

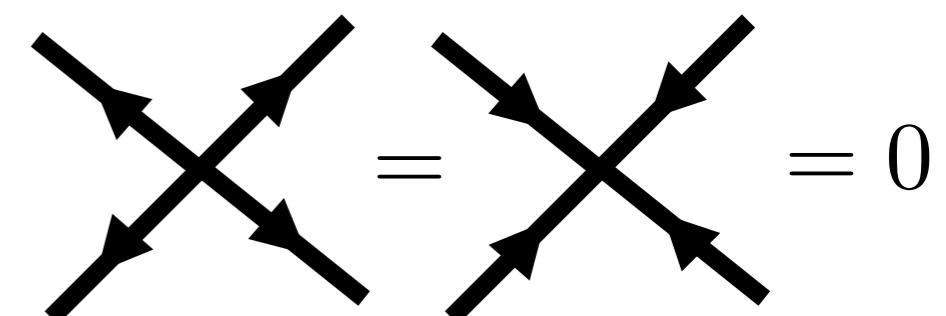
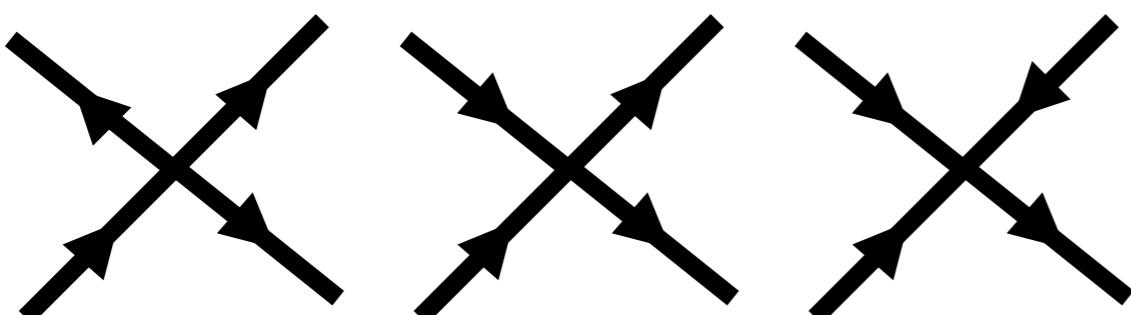
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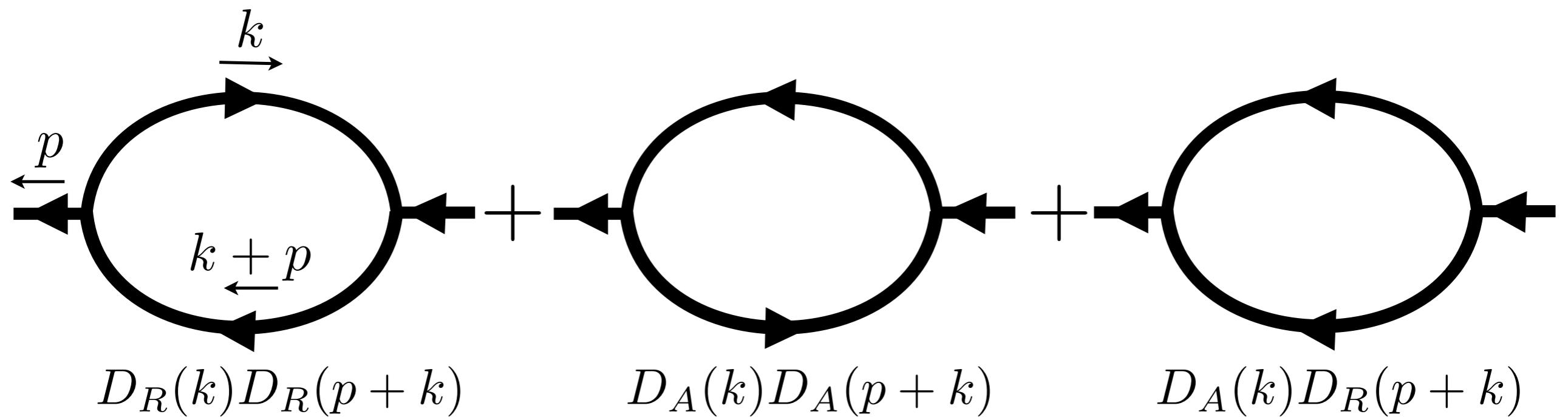
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Vertex: Φ^4 theory for simplicity

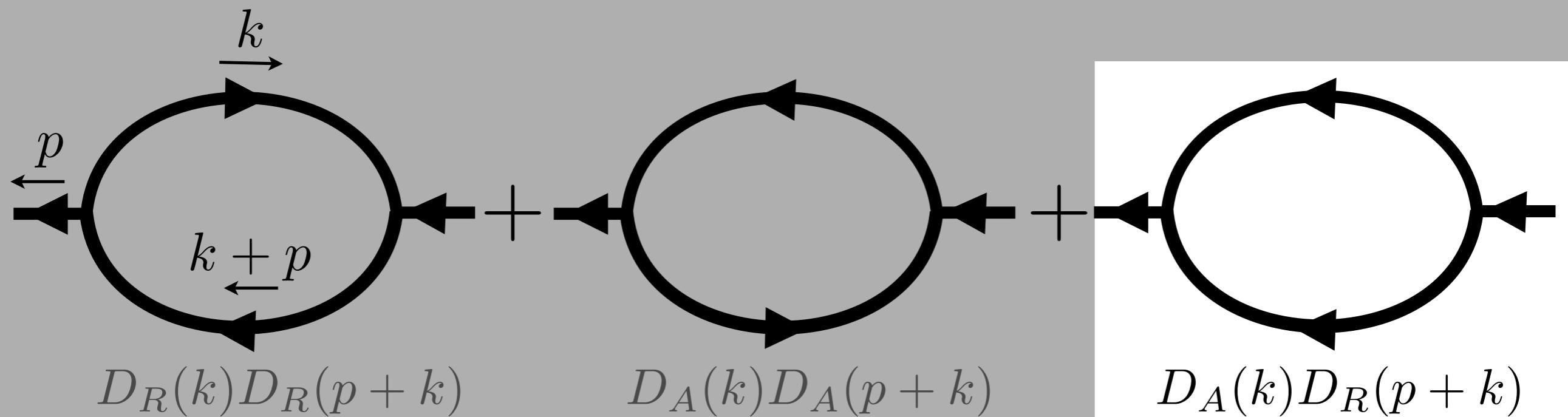


because of causality.

One loop diagram



One loop diagram

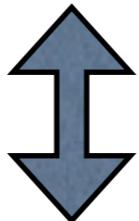


Pinch singularity

Quasi particle approximation

$$D_R(p+k)D_A(k) = \frac{2\pi i z_k^2}{p_0 - \mathbf{p} \cdot \mathbf{v}_k + 2i\gamma_k} \delta(k_0 - \epsilon_k) + \dots$$

for small p



$$(\partial_t - \mathbf{v}_k \cdot \boldsymbol{\partial}) f(x, p) = -C[f]$$

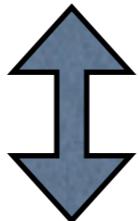
Inverse of LHS in Boltzmann equation

Pinch singularity

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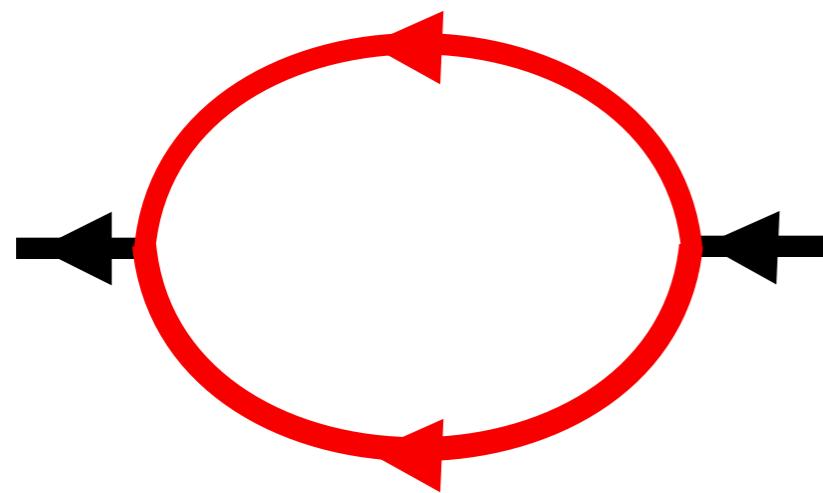
Inverse of LHS in Boltzmann equation

At $p = 0$,

$$D_R(k)D_A(k) = \frac{1}{|k^2 - m^2 - \Pi(k)|^2} = \frac{1}{2\text{Im } \Pi(k)} \rho(k)$$

$\rho(k)$:spectral function

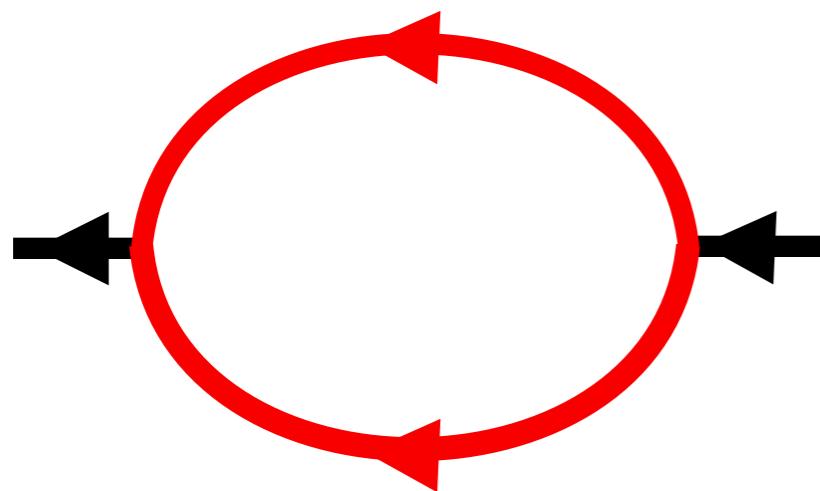
Shear viscosity at one-loop



A one-loop diagram representing a closed loop of red arrows. Two black arrows point towards the loop from the left side. The loop has two vertices where the black arrows enter, and two arrows pointing clockwise around the loop.

$$\sim \frac{1}{\text{Im } \Pi(k)} \sim \frac{1}{\lambda^2} \sim \lambda_{\text{mfp}}$$

Shear viscosity at one-loop



A red oval loop with a clockwise arrow. Two black arrows point towards each other from the left side of the loop.

$$\sim \frac{1}{\text{Im } \Pi(k)} \sim \frac{1}{\lambda^2} \sim \lambda_{\text{mfp}}$$

From transport theory

$$\rightarrow \eta \approx \frac{1}{3} \bar{p} n \lambda_{\text{mfp}}$$



Maxwell(1860)

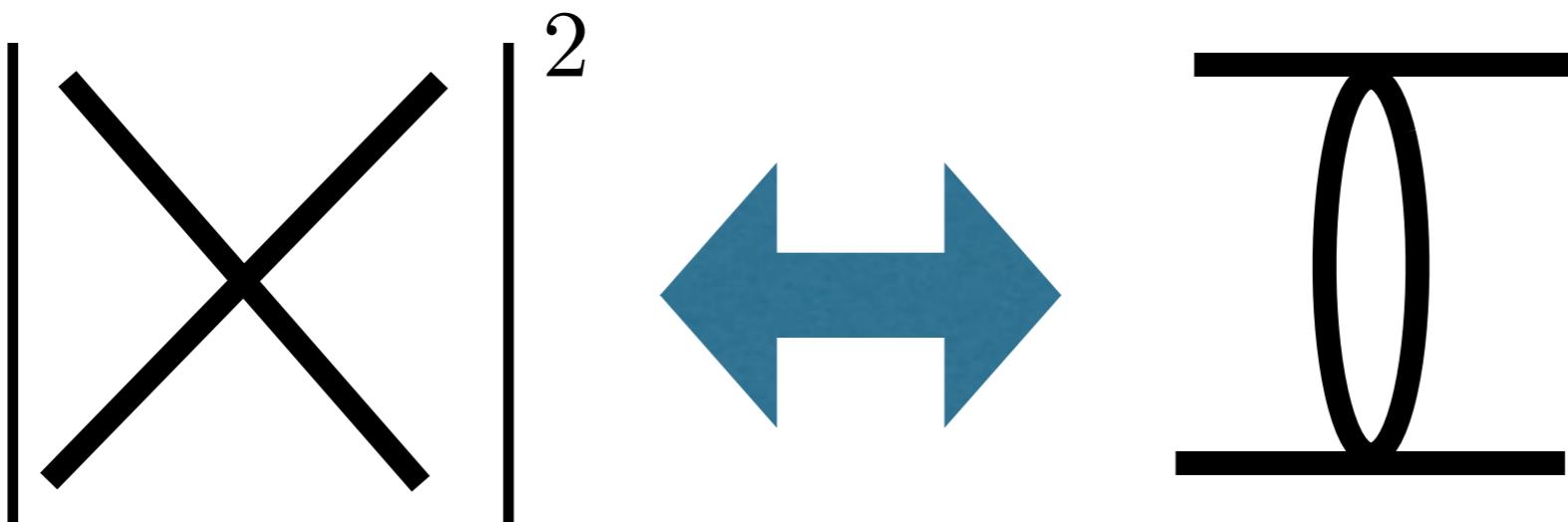
This is the same order as the one-loop diagram.

Optical theorem

$$| \times |^2$$

Squared scattering
amplitude

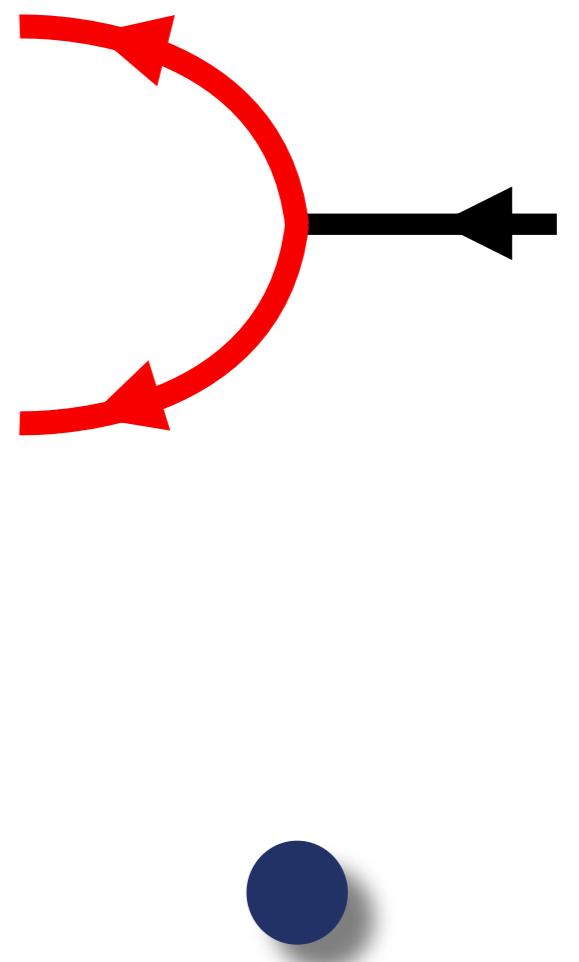
Optical theorem



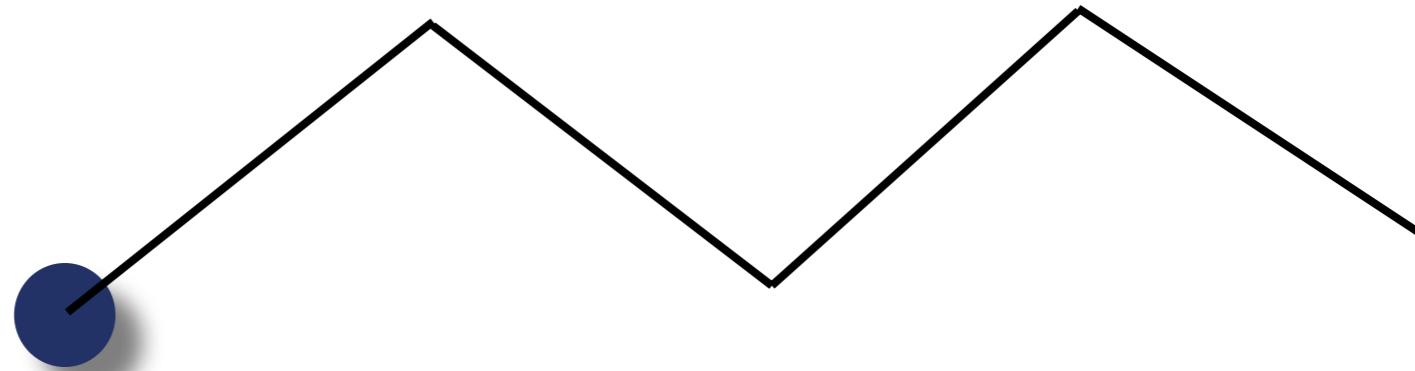
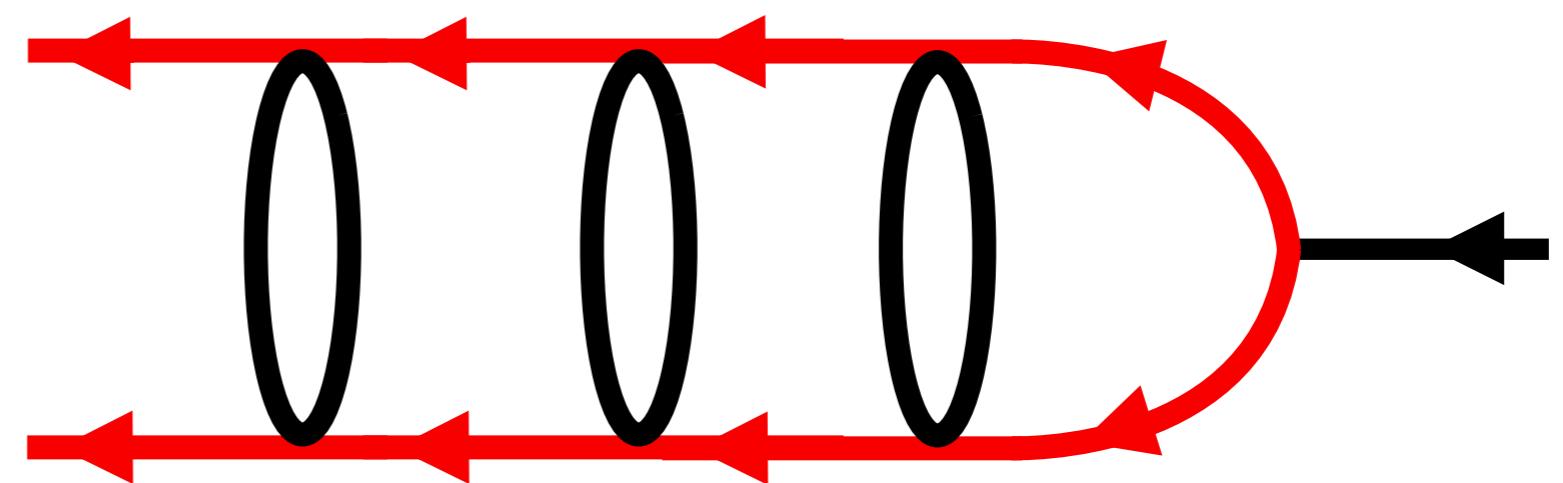
Squared scattering
amplitude

Imaginary part of
loop diagram

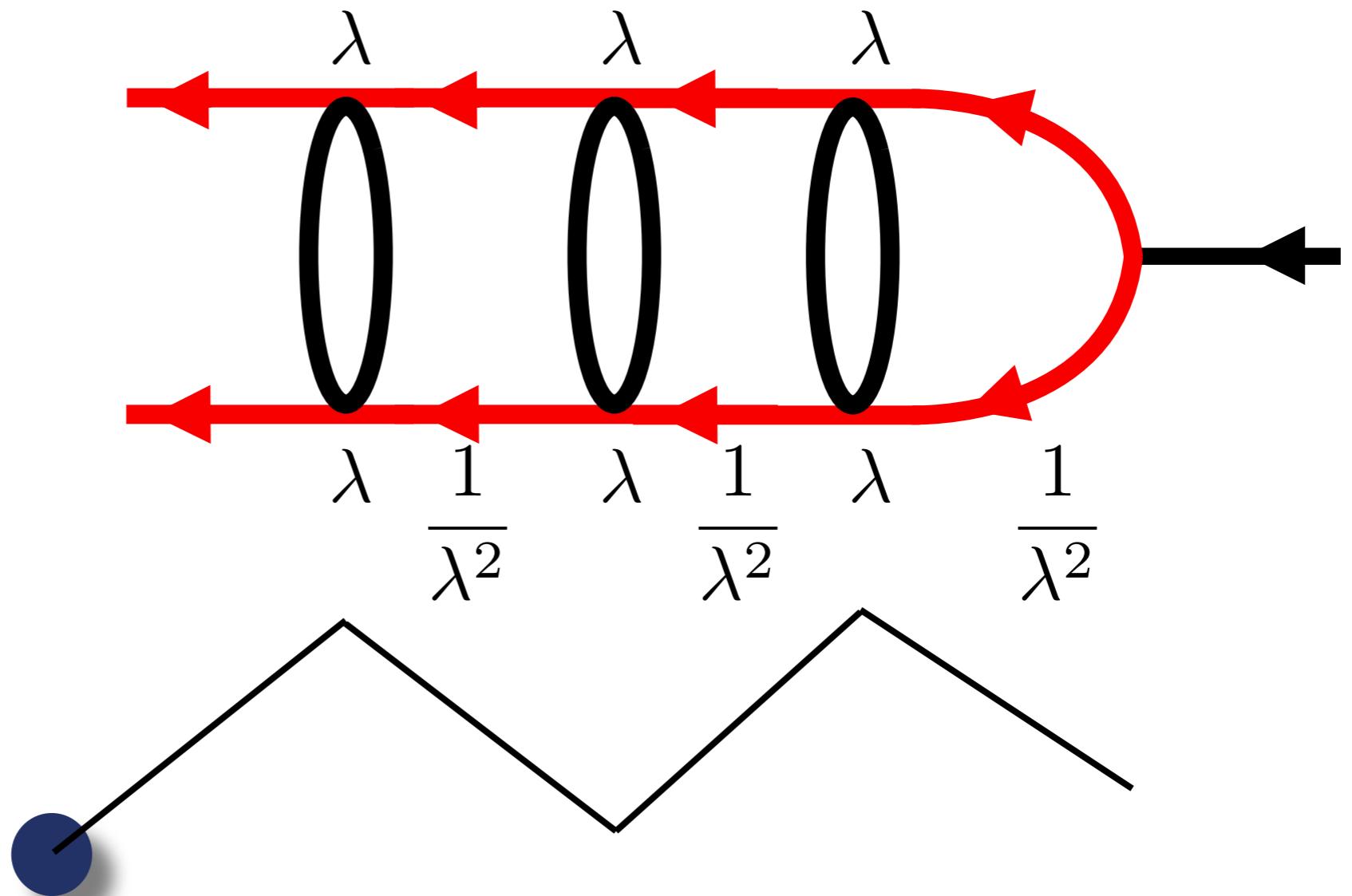
Resummation



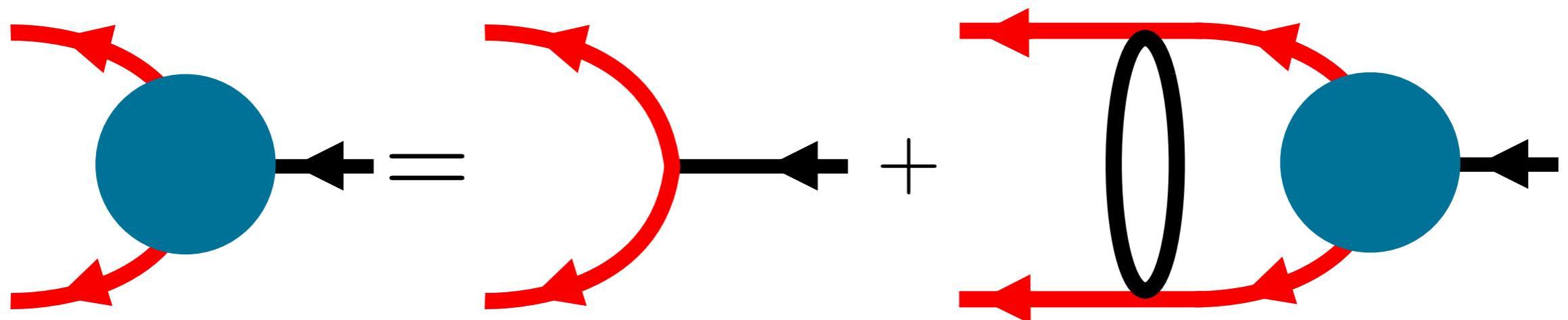
Resummation



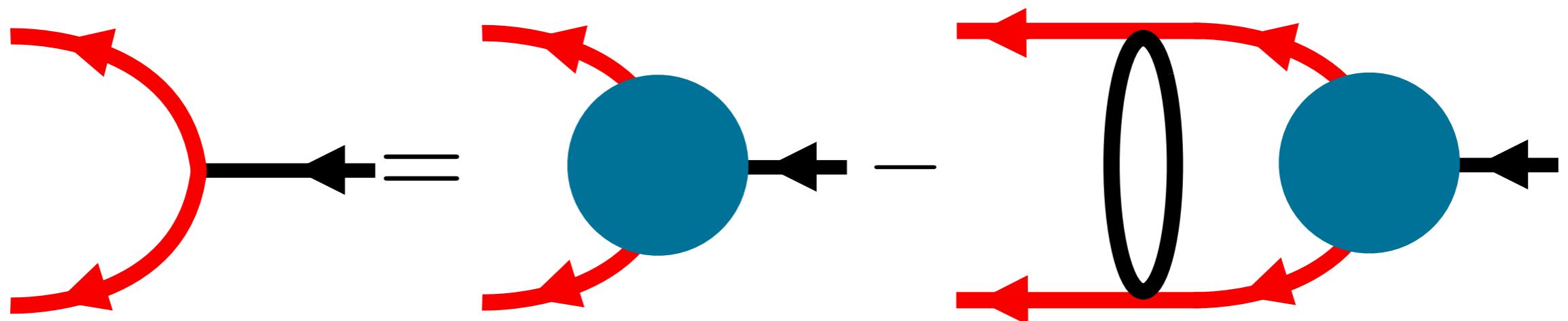
Resummation



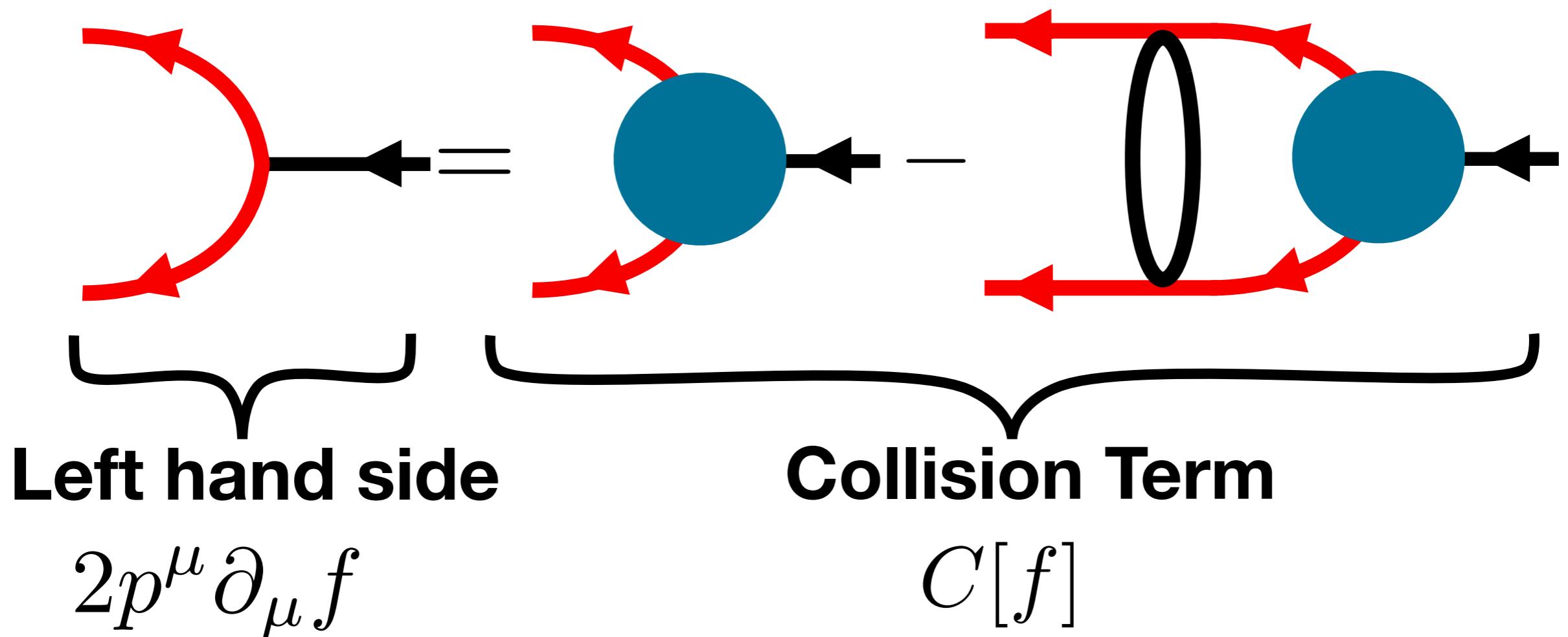
Resummation



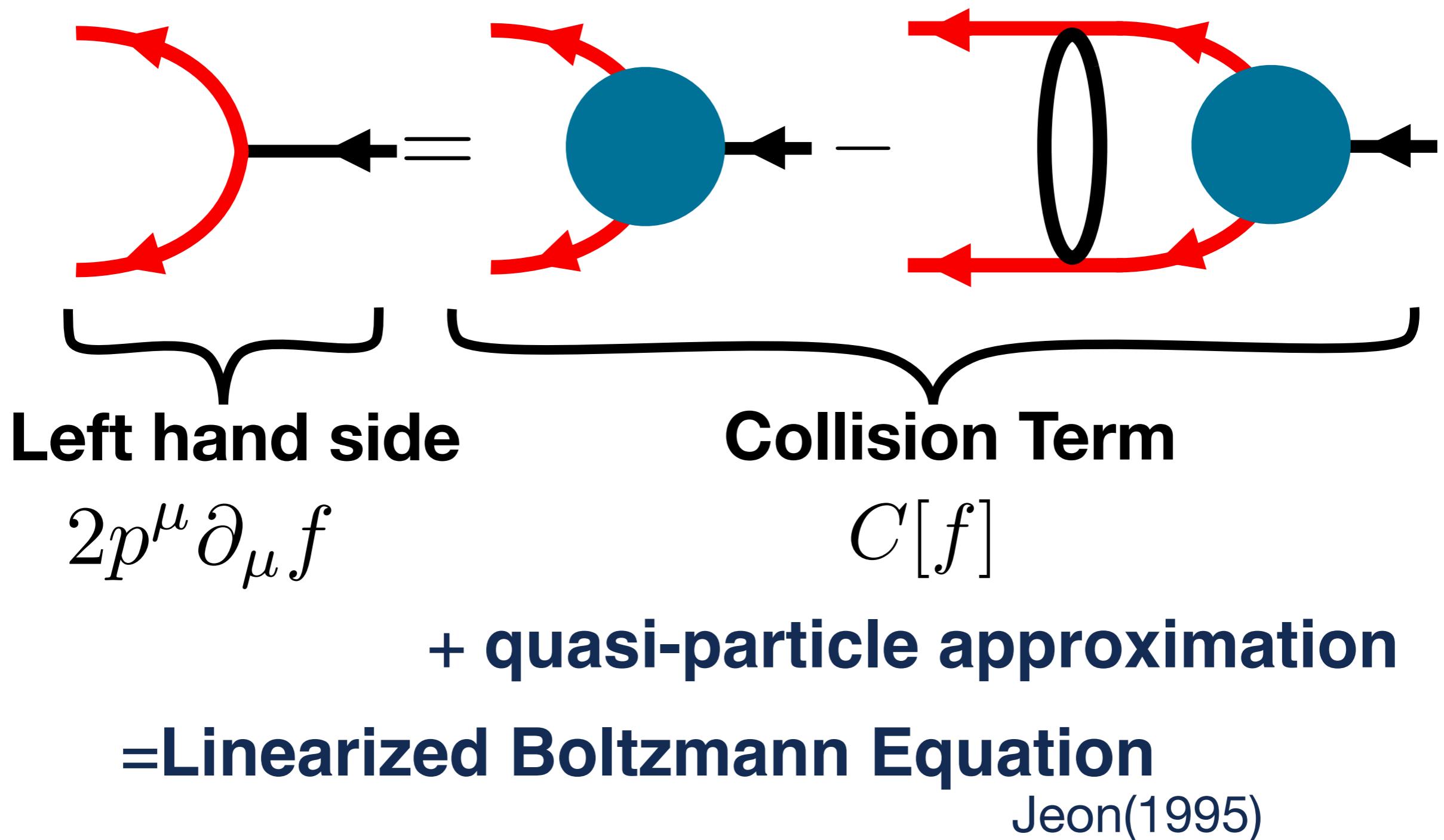
Resummation



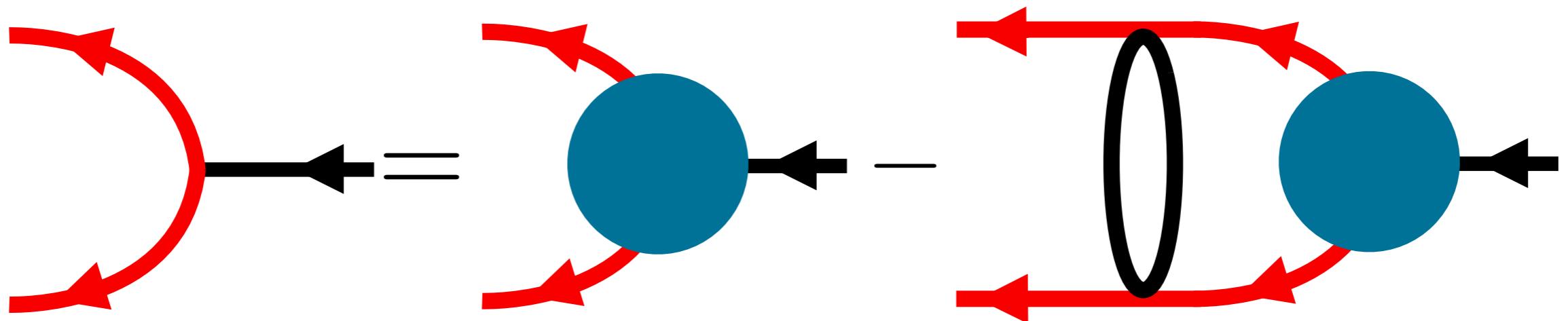
Resummation



Resummation



Resummation



Correlation function

$$\langle [T_{xy}(x), T_{xy}(0)] \rangle = \text{Diagram}$$

The diagram for the correlation function shows a blue circle with a black arrow pointing to it from the left. Attached to the top of this blue circle is a red loop with two internal arrows forming a circle around the blue circle.

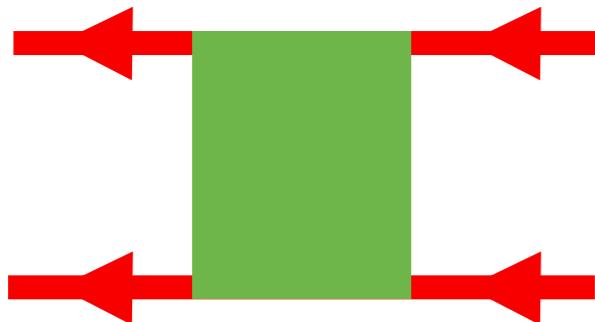
Beyond the Boltzmann Equation

YH, Kunihiro

Eliashberg's method

Eliashberg('62)

Decompose four point function to them with the pinch singularities and the others.

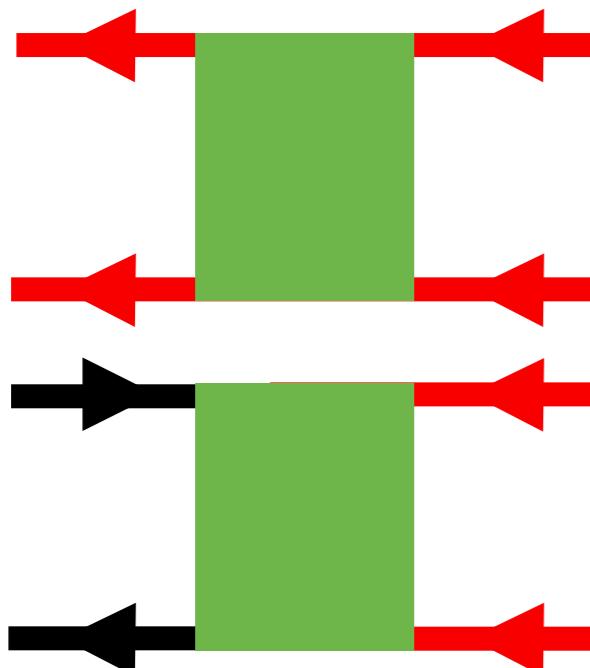


Both sides connect to pinch diagrams.

Eliashberg's method

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Decompose four point function to them with the pinch singularities and the others.



Both sides connect to pinch diagrams.

One side connects to pinch diagrams.

Eliashberg's method

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Decompose four point function to them with the pinch singularities and the others.



Both sides connect to pinch diagrams.



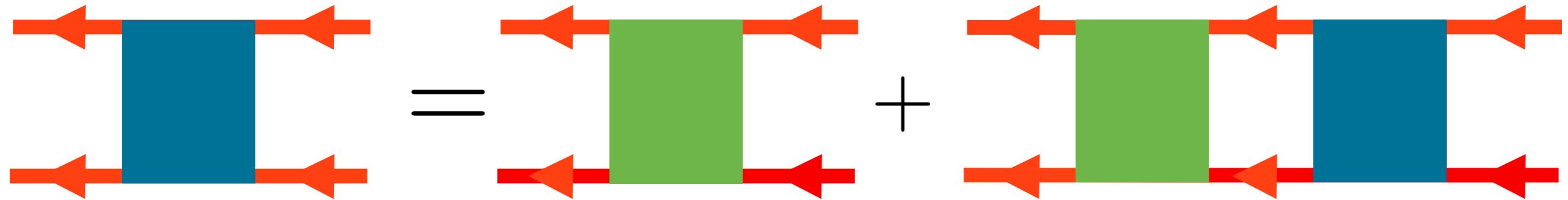
One side connects to pinch diagrams.



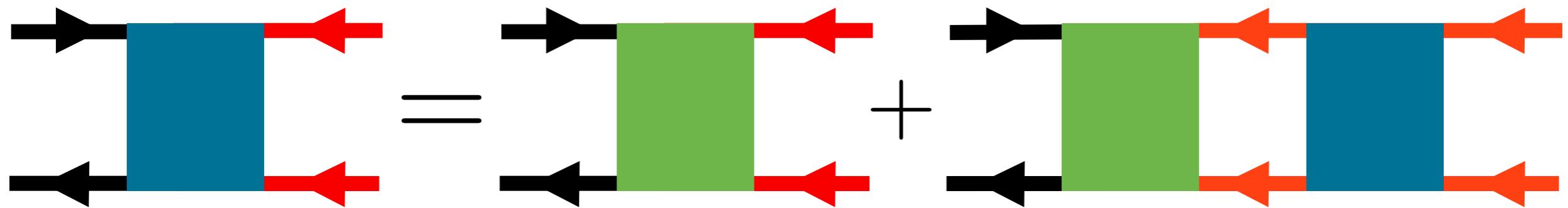
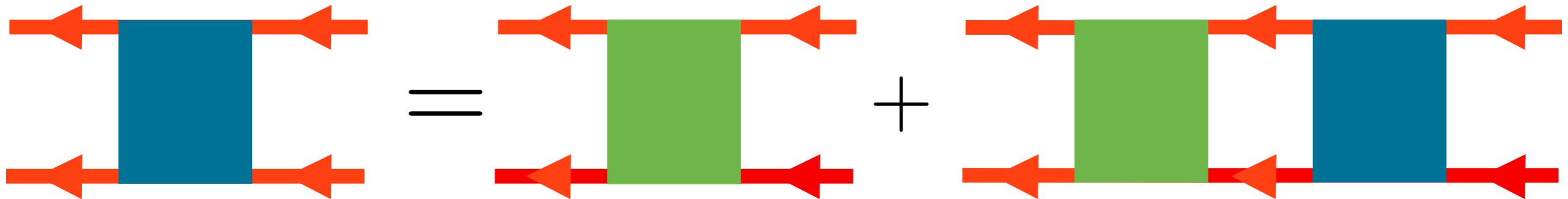
Both sides do not connect to pinch diagrams.



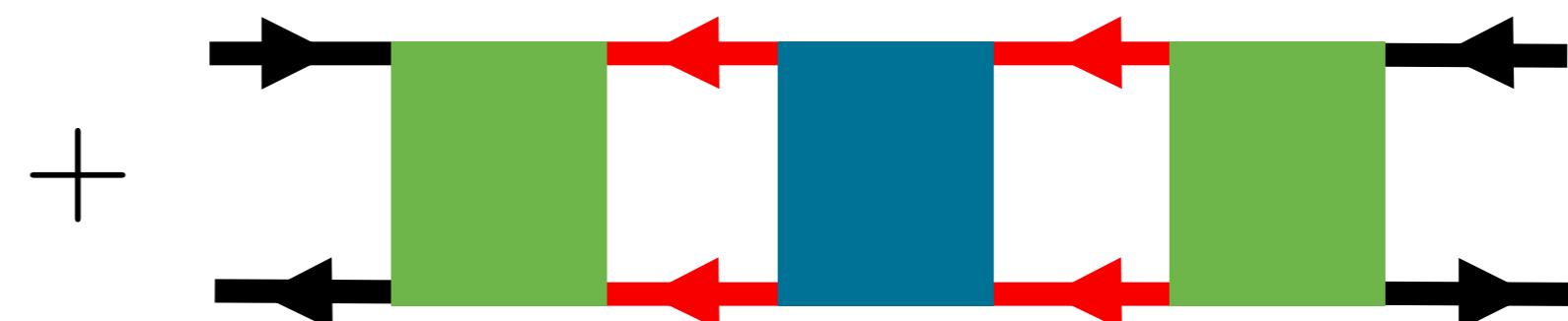
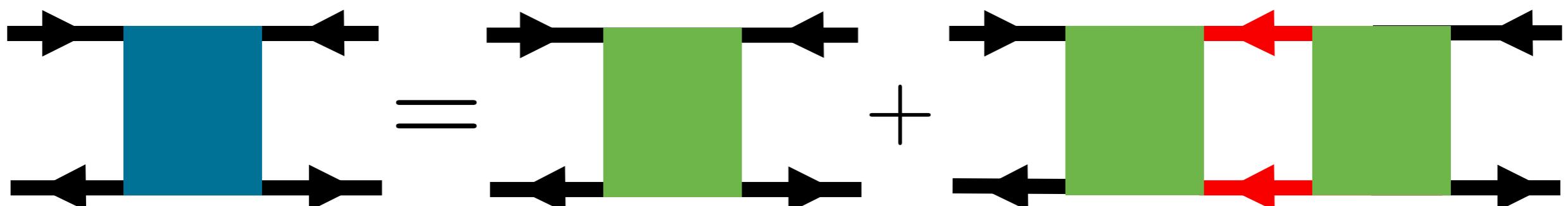
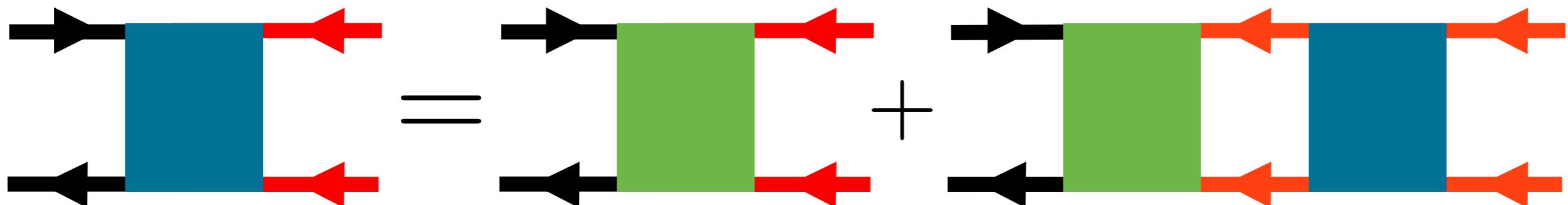
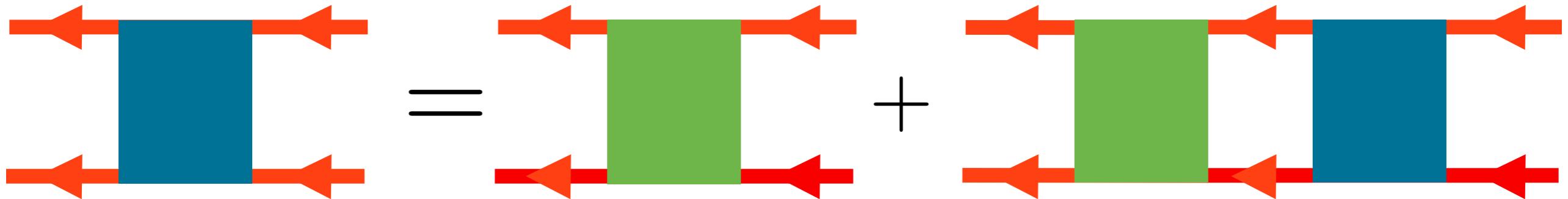
Resummation of Pinch Singularities



Resummation of Pinch Singularities



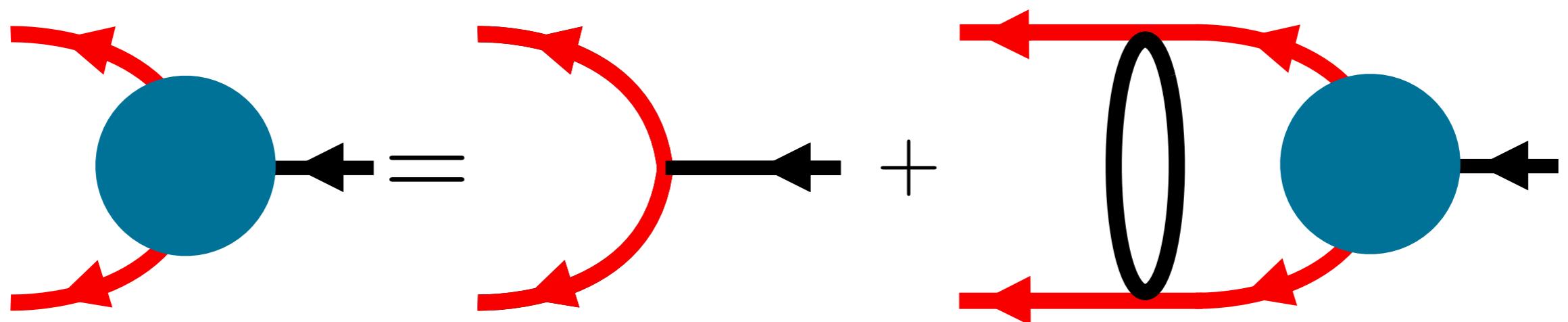
Resummation of Pinch Singularities



Beyond the Boltzmann Eq.

Leading order

YH, Kunihiro



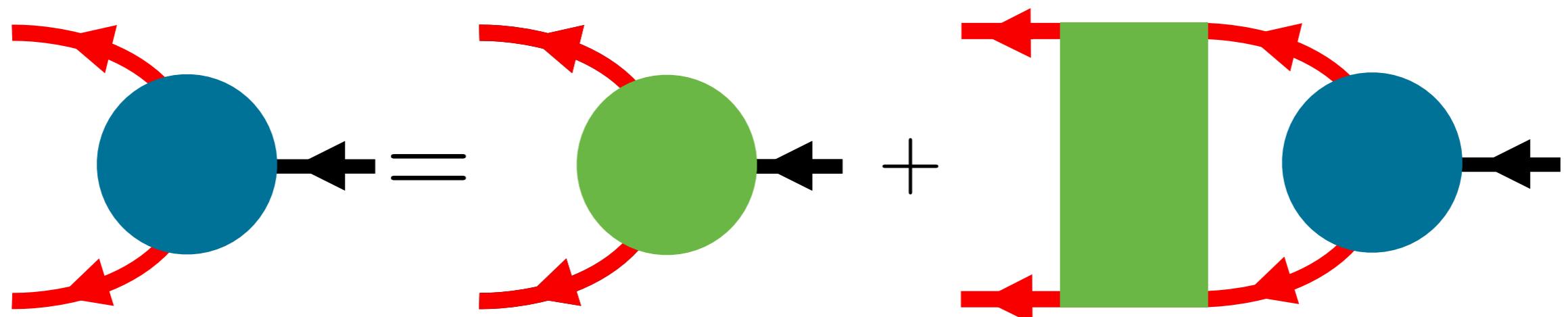
Energy Momentum Tensor

$$\langle [T_{xy}(x), T_{xy}(0)] \rangle = \quad \leftarrow \quad \text{Diagram showing a blue circle with a black arrow pointing to its left, with a red loop around it. The red loop has two red arrows, and a black arrow points to its left. The entire expression is preceded by a left-pointing arrow.}$$

Beyond the Boltzmann Eq.

Including higher orders

YH, Kunihiro



Energy Momentum Tensor

$$\langle [T_{xy}(x), T_{xy}(0)] \rangle = \text{Diagram with green and blue circles connected by a circle with red arrows}$$

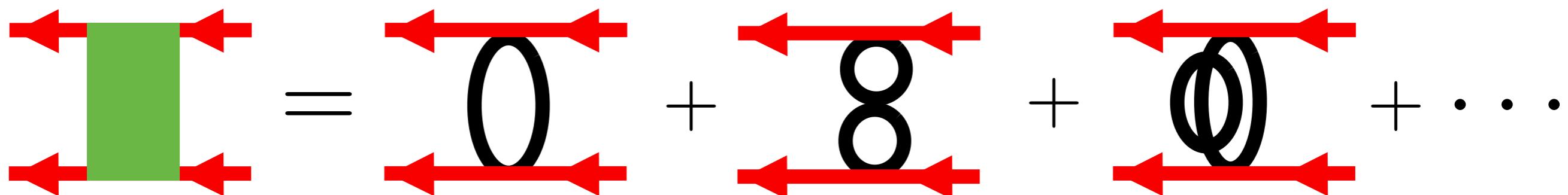
The equation shows the expectation value of the commutator of energy-momentum tensors at two different points. The left side is $\langle [T_{xy}(x), T_{xy}(0)] \rangle$. To its right is a diagram consisting of two circular vertices, one green and one blue, connected by a horizontal line. A circle surrounds them, with red arrows forming a clockwise loop around the central connection point.

+ no pinch singular diagrams

Beyond the Boltzmann Eq.

YH, Kunihiro

Correction to scattering amplitude

$$\text{Diagram with green box} = \text{Diagram with circle} + g \text{Diagram with circle} + \phi \text{Diagram with circle} + \dots$$


Beyond the Boltzmann Eq.

YH, Kunihiro

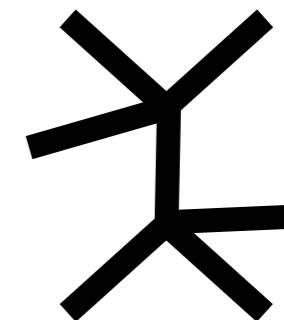
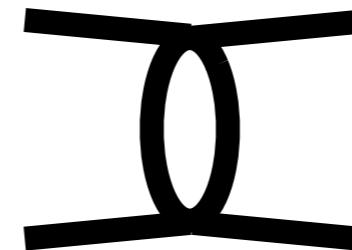
Correction to scattering amplitude

$$\text{Leading} = \text{loop correction} + \text{many-body scattering} + \dots$$

Leading

loop correction

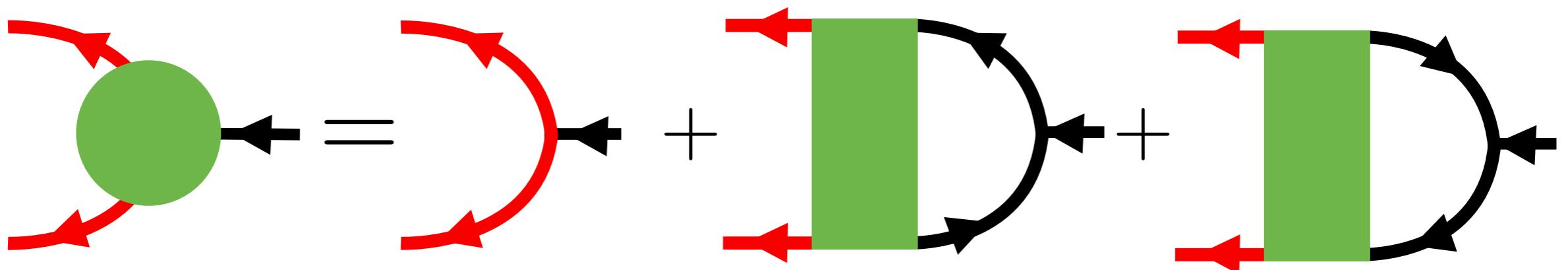
Many-body
scattering



Beyond the Boltzmann Eq.

YH, Kunihiro

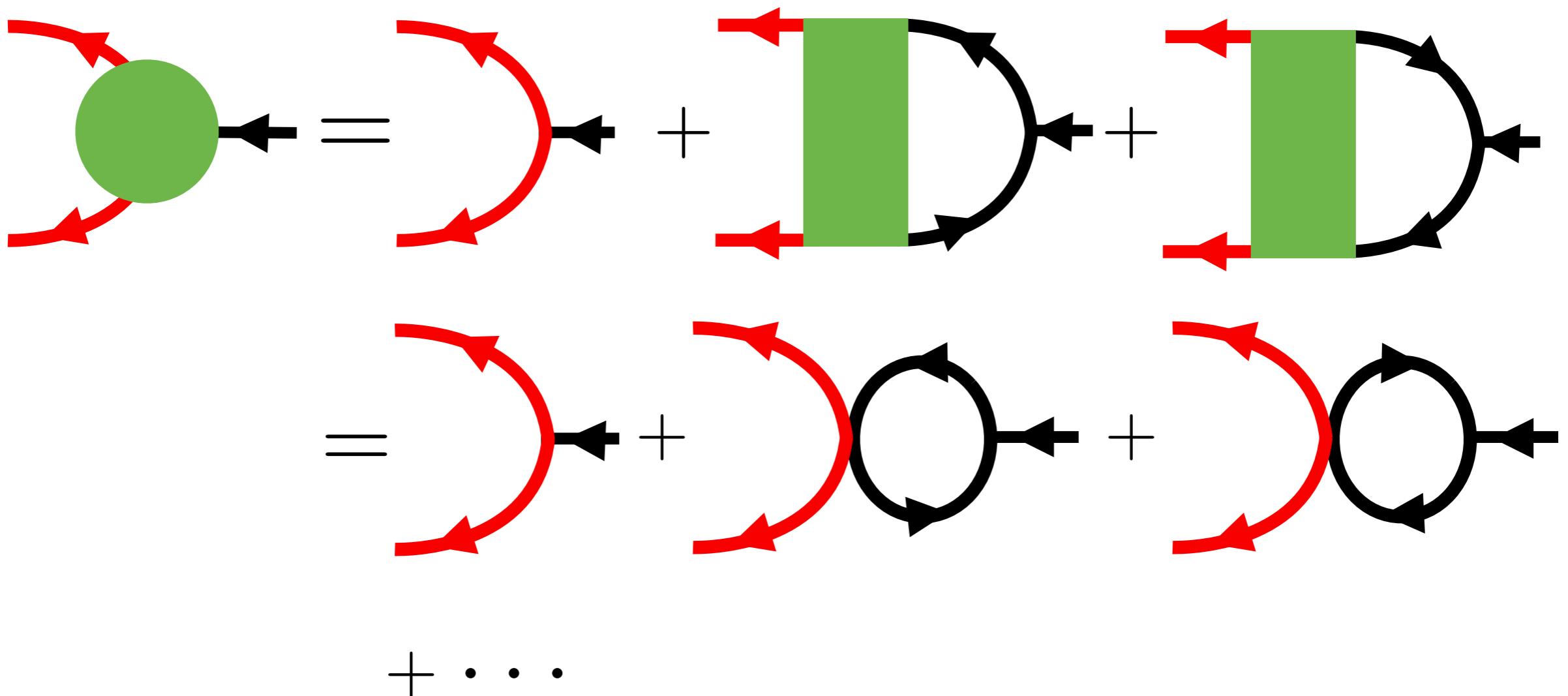
Vertex correction



Beyond the Boltzmann Eq.

YH, Kunihiro

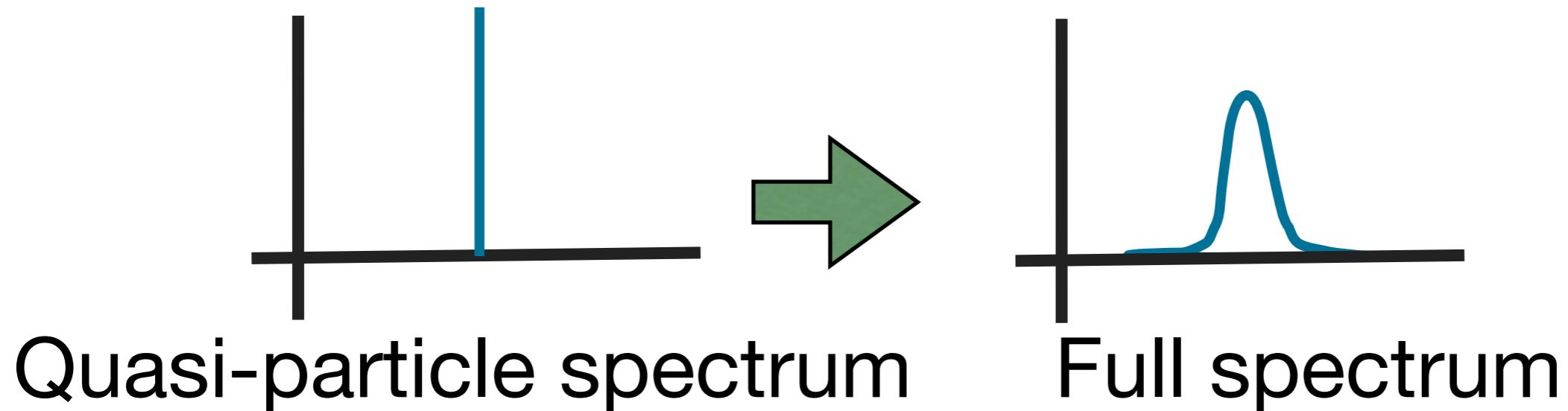
Vertex correction



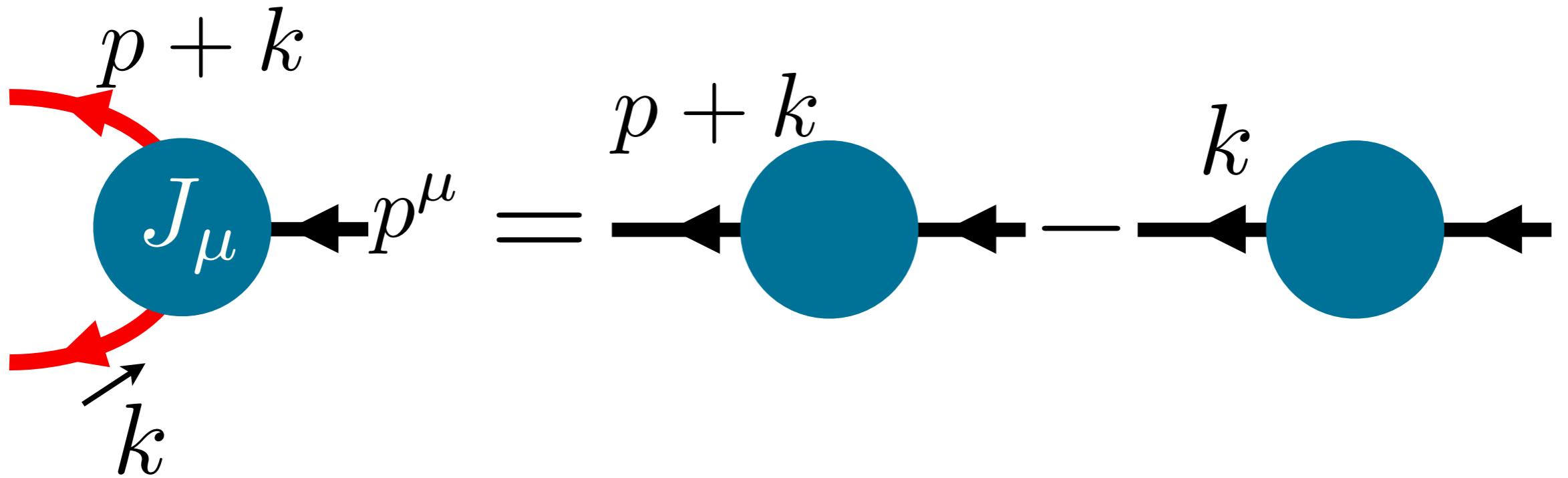
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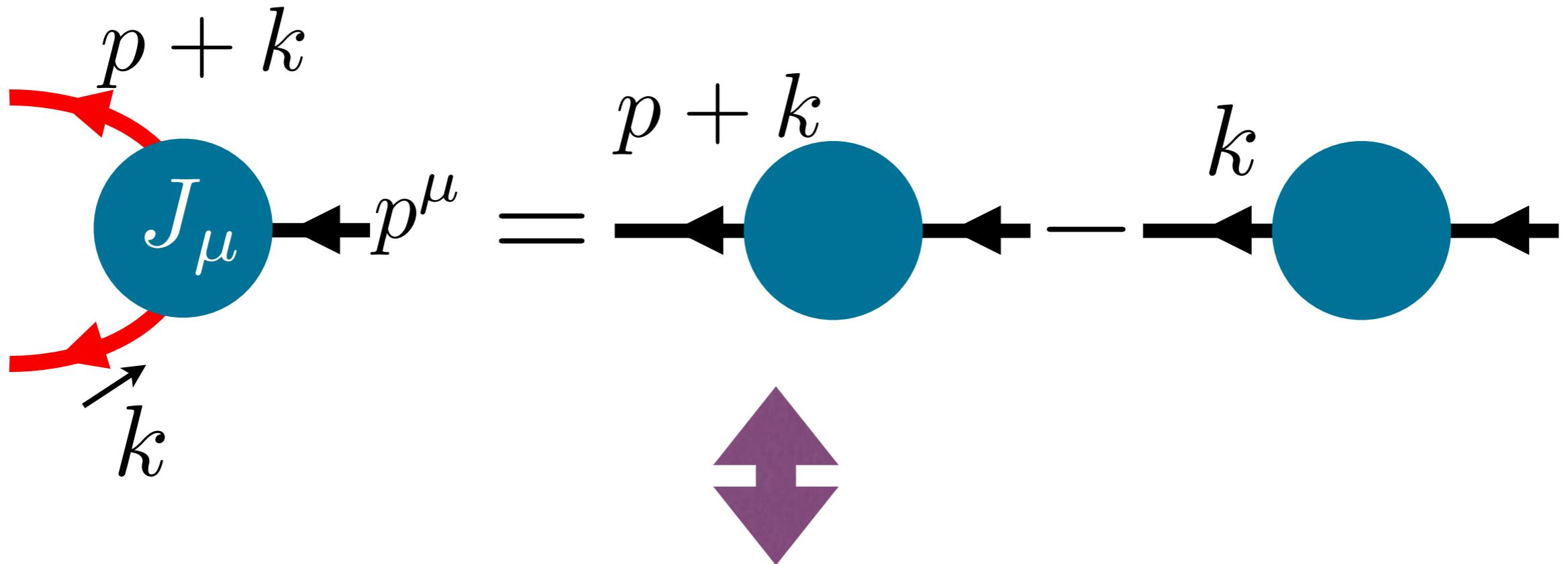
Spectrum



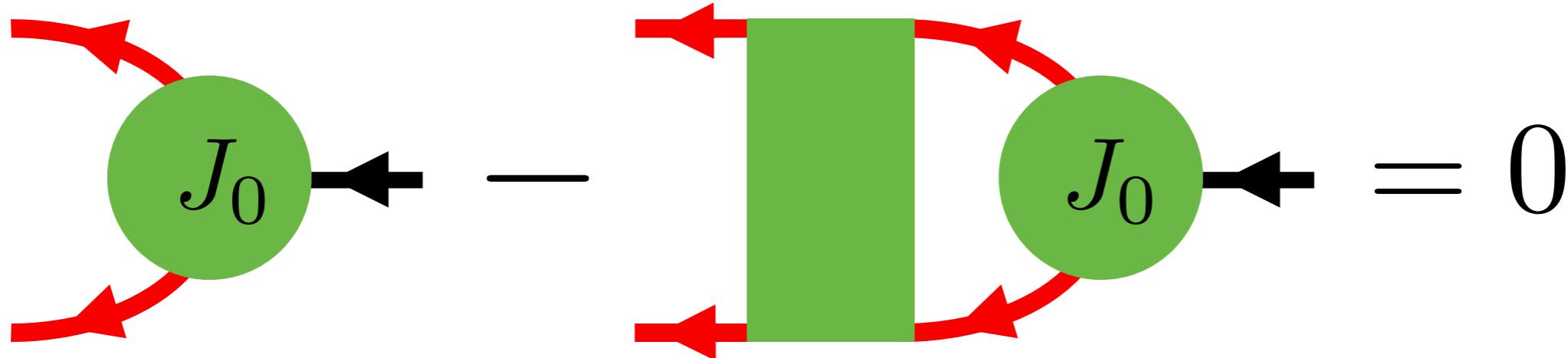
Ward-Takahashi identity



Ward-Takahashi identity



Collision invariant



Summary

- Applied Eliasberg's method to QFT.
- Leading order: Boltzmann Equation.
- Higher order: Modification of spectra, vertex renormalization, and many-body scatterings.
- Ward-Takahashi identity
↔ collision invariant

Future Plan

- Apply to QCD.
- Not only pinch singularity but also collinear singularity must be summed (Landau-Pomeranchuk-Migdal effect).
- Apply to critical phenomena.
- Can't neglect hydrodynamic modes.
Relate our formalism to mode coupling theory.