

Lattice QCD under extreme conditions

- I Equation of State
- II Transition
- III Critical behavior ?!

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for the
RBC-Bielefeld – and
hotQCD – Collaborations

I Equation of State

the equation of state is $\epsilon(p)$

equivalent to knowing pressure $p(T)$ and energy density $\epsilon(T)$

\Rightarrow entropy density $s = (\epsilon + p)/T$ and other quantities through thermodynamic relations

start from energy-momentum tensor $\frac{\Theta^\mu_\mu(T)}{T^4} = \frac{\epsilon - 3p}{T^4} = T \frac{d}{dT}(p/T^4)$

where $\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(T, V) - \lim_{T \rightarrow 0} \frac{1}{VT^3} \ln Z(T, V)$ $T = 0$ subtraction takes care of UV divergencies

now $Z(T, V; g, m_l, m_s) = Z(N_\tau, N_\sigma, a; \beta = 6/g^2, \hat{m}_l, \hat{m}_s) \rightarrow Z(N_\tau, N_\sigma, a; \beta, m_\pi, m_K)$

and tune bare lattice parameters \hat{m}_l, \hat{m}_s with β such that $m_{\pi, K} = \text{const} \Rightarrow \hat{m}_{l,s}(\beta), a(\beta)$

\Rightarrow central quantity $\frac{\Theta^\mu_\mu(T)}{T^4} = -R_\beta(\beta) N_\tau^4 \left(\left\langle \frac{d\bar{S}}{d\beta} \right\rangle_T - \left\langle \frac{d\bar{S}}{d\beta} \right\rangle_{T=0} \right)$

with $R_\beta(\beta) = T \frac{d\beta}{dT} = -a \frac{d\beta}{da}$ (non-perturbative) β function

then $\frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_{T_0}^T dT' \frac{1}{T'^5} \Theta^\mu_\mu(T')$

action $S = \beta S_G + 2 \hat{m}_l(\beta) \bar{\psi}_l \psi_l + \hat{m}_s(\beta) \bar{\psi}_s \psi_s + \beta$ independent

such that Θ_μ^μ consists of three pieces (**asqtad** slightly more complicated: needs $du_0/d\beta$)

$$\frac{\Theta_G^{\mu\mu}(T)}{T^4} = R_\beta N_\tau^4 \Delta \langle \bar{S}_G \rangle \quad \text{where } \Delta \langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{T=0} - \langle \mathcal{O} \rangle_T$$

$$\frac{\Theta_F^{\mu\mu}(T)}{T^4} = -R_\beta R_m N_\tau^4 \{ 2 \hat{m}_l \Delta \langle \bar{\psi} \psi \rangle_l + \hat{m}_s \Delta \langle \bar{\psi} \psi \rangle_s \}$$

$$\frac{\Theta_h^{\mu\mu}(T)}{T^4} = -R_\beta R_h N_\tau^4 \hat{m}_s \Delta \langle \bar{\psi} \psi \rangle_s$$

need: non-perturbative β functions $R_\beta(\beta), R_m(\beta) = \frac{1}{\hat{m}_l(\beta)} \frac{d\hat{m}_l(\beta)}{d\beta}$

$$R_h(\beta) = \frac{1}{h(\beta)} \frac{dh(\beta)}{dh} = 0 \quad \text{at } h(\beta) = \frac{\hat{m}_l}{\hat{m}_s} = \text{const}$$

“action differences” $\Delta \bar{S}_G, \Delta \langle \bar{\psi} \psi \rangle_{l,s}$

Simulation parameters

- $N_F = 2 + 1$: two degenerate u/d quarks + strange quark
- RHMC algorithm
- lattice sizes

$T > 0$:	$N_\sigma/N_\tau = 4$	$N_\tau = 4, 6, 8$	for $m_l/m_s = 1/10$	$(m_\pi \simeq 220 \text{ MeV})$
		$N_\tau = 8, 12$	for $m_l/m_s = 1/20$	$(m_\pi \simeq 150 \text{ MeV})$

each $T > 0$ run accompanied by $T = 0$ run, for subtractions, scales

$T = 0$: $N_\sigma/N_\tau \leq 1$ same N_σ as for $T > 0$

- statistics

$\mathcal{O}(10k - 60k)$	for $T > 0$,	each $(\beta, \hat{m}_l, \hat{m}_s)$
$\mathcal{O}(5k)$	for $T = 0$,	each $(\beta, \hat{m}_l, \hat{m}_s)$
- actions

p4fat3	on $N_\tau = 4, 6, 8$	both actions improve – $\mathcal{O}((aT)^2)$ discretization effects and – taste violations (present in all staggered actions)
asqtad	on $N_\tau = 6, 8, 12$	

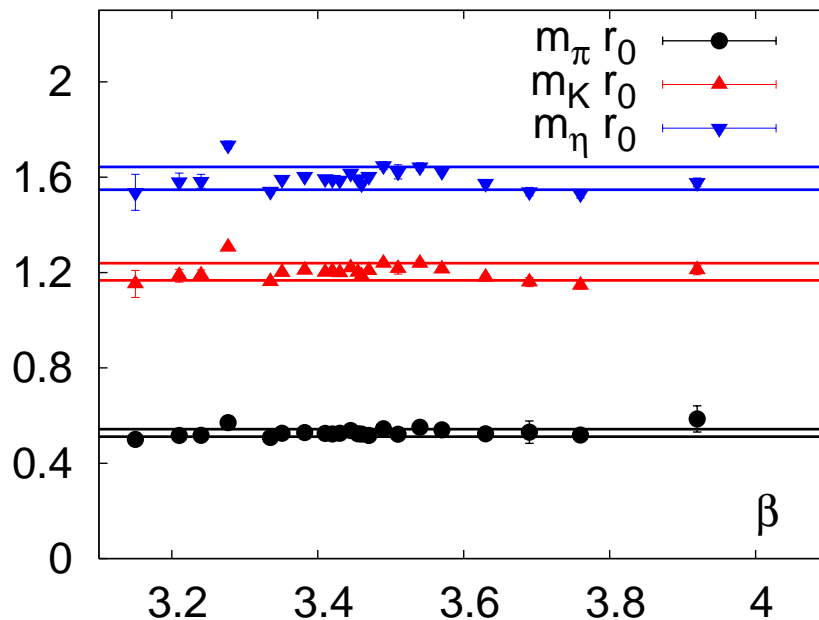
★ T = 0 scale taken from $\Upsilon 2S - 1S$ splitting [A. Gray et al.] via the heavy quark potential $V(r)$

$$r_{0,1}/a \text{ from } r^2 \frac{dV(r)}{dr} \Big|_{r=r_{0,1}} = 1.65(1.0)$$

for absolute values (in MeV) we

use $r_0 = 0.469(7)$ fm [A. Gray et al.]

★ fine tune $\hat{m}_i(\beta)$

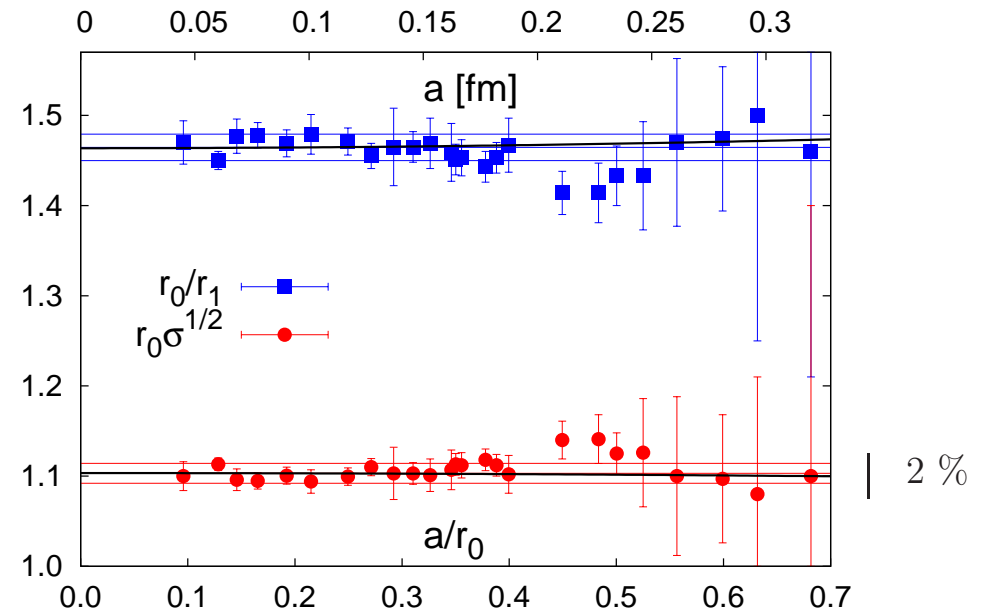


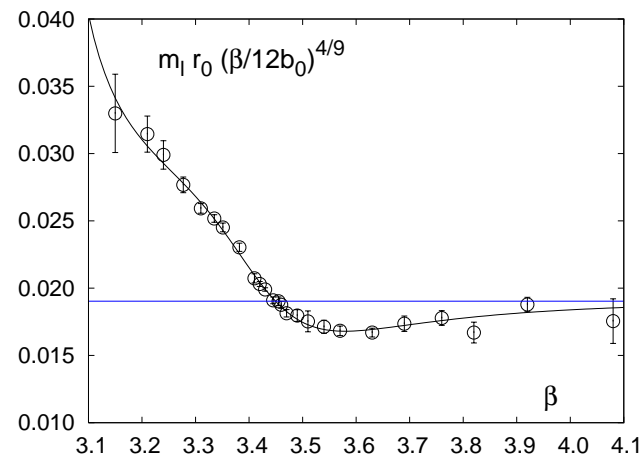
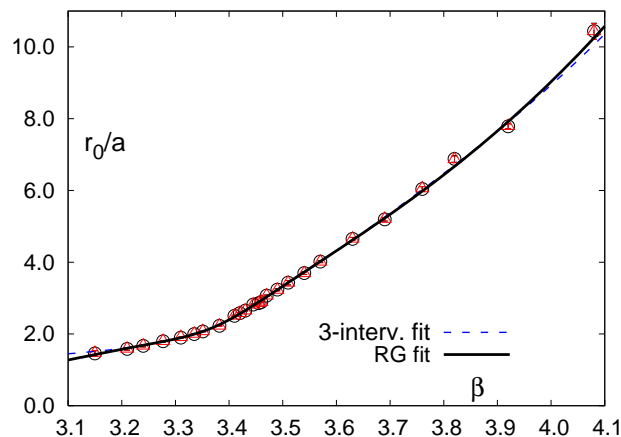
■ 3%

$$m_K \simeq m_K^{\text{phys}}$$

$$m_\pi \simeq 220 \text{ MeV}$$

p4 action, $m_l = 0.1 m_s$





p4 action

$$m_l = 0.1 m_s$$

Allton inspired parametrization with rational fct. in $\hat{a}(\beta) = R_\beta^{(2-loop)}(\beta)/R_\beta^{(2-loop)}(\beta = 3.4)$

$$\frac{a}{r_0} = a_r R_\beta^{(2-loop)} \frac{1 + b_r \hat{a}^2 + c_r \hat{a}^4 + d_r \hat{a}^6}{1 + e_r \hat{a}^2 + f_r \hat{a}^4} \quad \Rightarrow \quad R_\beta = \frac{r_0}{a} \left(\frac{dr_0/a}{d\beta} \right)^{-1}$$

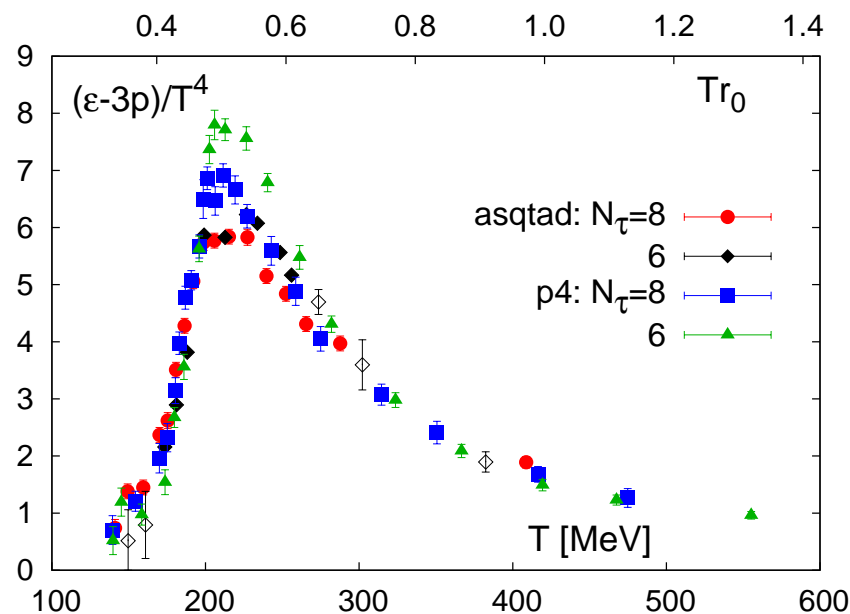
$$\hat{m}_l = a_m R_\beta^{(2-loop)} \left(\frac{12b_0}{\beta} \right)^{4/9} \frac{1 + b_m \hat{a}^2 + c_m \hat{a}^4 + d_m \hat{a}^6}{1 + e_m \hat{a}^2 + f_m \hat{a}^4 + g_m \hat{a}^6} \quad \Rightarrow \quad R_m$$

for asqtad there is a general parametrization for a large set of $T = 0$ data

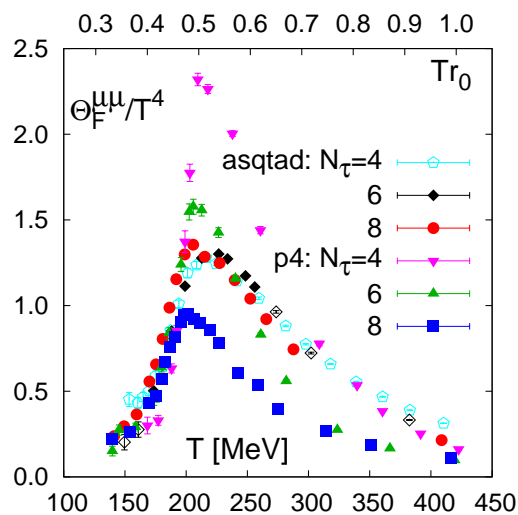
results: $m_l = 01 m_s$

$\Theta_\mu^\mu(T)/T^4$ (the central quantity)

★ very good agreement between
p4fat3 and asqtad actions



★ except in the peak region, here differences dominantly due to fermionic part Θ_F

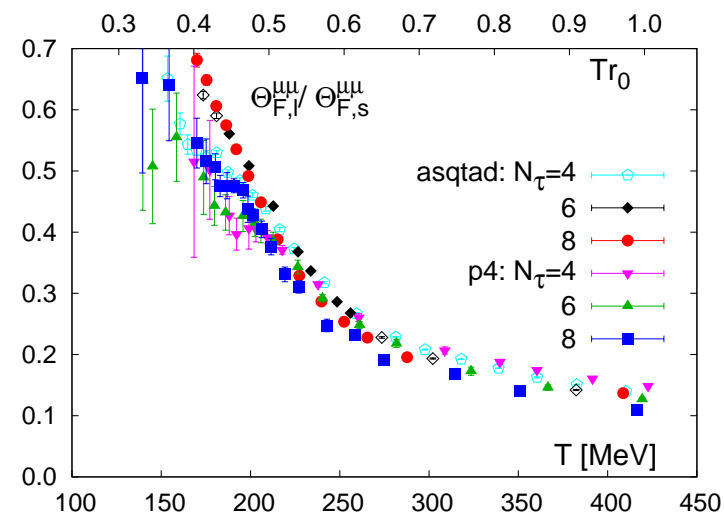


there: 15 % Θ_F contribution
⇒ 5 % overall

affected by R_m at low β

↷ drops out in $\Theta_{F,l}/\Theta_{F,s}$

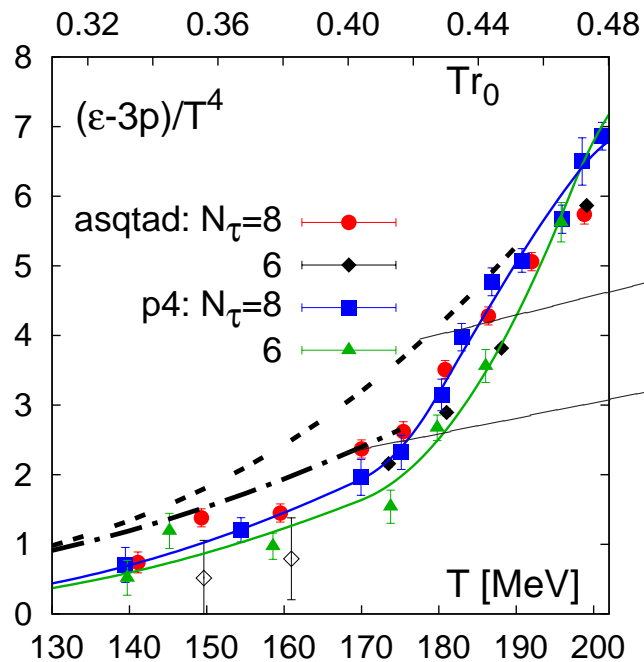
remaining differences due to
slight mismatch in m_s^{phys}



results $m_l = 0.1 m_s$

low T region:

model to compare with:
hadron resonance gas (HRG)



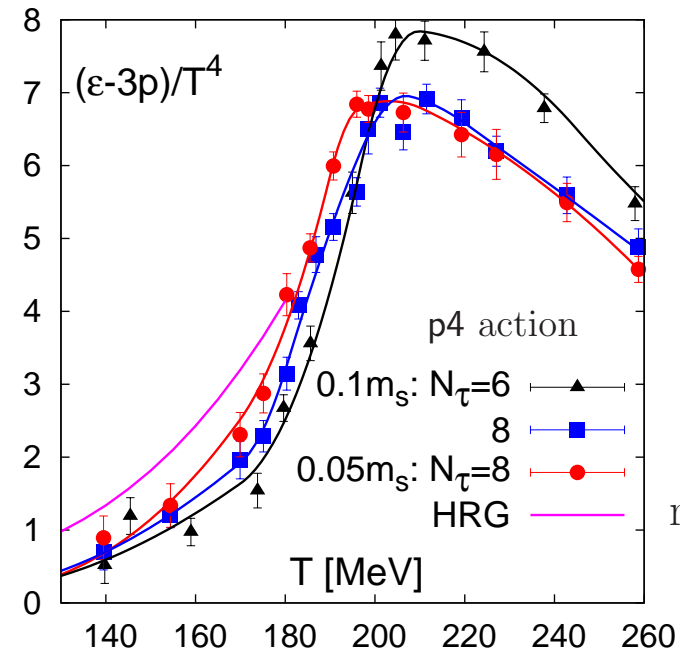
$$\left(\frac{\epsilon - 3p}{T^4}\right)_{low T} = \sum_{m_i \leq m_{max}} \frac{d_i}{2\pi^2} \sum_{k=1}^{\infty} (\pm)^{k+1} \frac{1}{k} \left(\frac{m_i}{T}\right)^3 K_1(km_i/T)$$

- ★ non-negligible contribution of heavy resonances in HRG
- ★ reducing discretization effects lowers the crossover temperature

results $m_l = 0.05 m_s$

low T region:

model to compare with:
hadron resonance gas (HRG)



resonances ≤ 2.5 GeV

$$\left(\frac{\epsilon - 3p}{T^4}\right)_{low T} = \sum_{m_i \leq m_{max}} \frac{d_i}{2\pi^2} \sum_{k=1}^{\infty} (\pm)^{k+1} \frac{1}{k} \left(\frac{m_i}{T}\right)^3 K_1(km_i/T)$$

- ★ reducing discretization effects lowers the crossover temperature
- ★ diminishing quark mass lowers the crossover temperature
- ★ smaller quark masses are expected to raise $\epsilon - 3p$ somewhat at low T

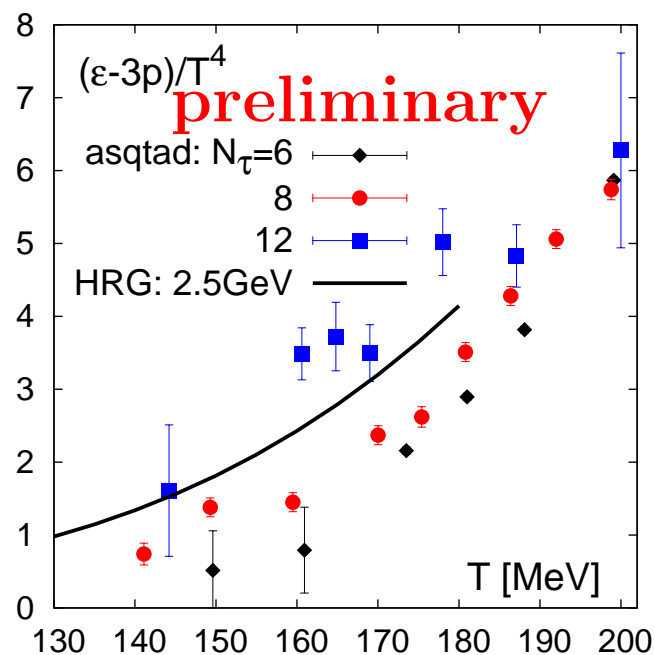
results

low T region:

$$N_\tau = 12, \quad m_l = 0.05 m_s$$

model to compare with:

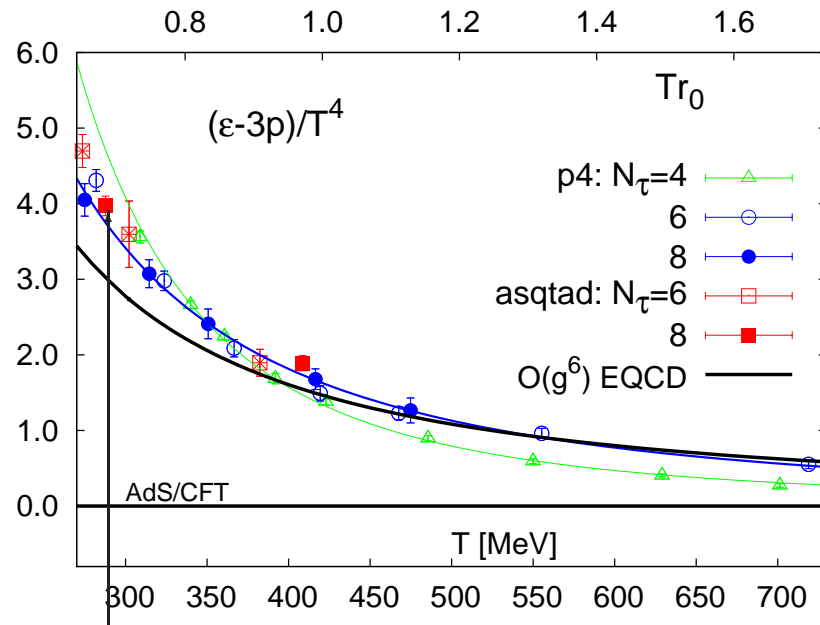
hadron resonance gas (HRG)



$$\left(\frac{\epsilon - 3p}{T^4}\right)_{low T} = \sum_{m_i \leq m_{max}} \frac{d_i}{2\pi^2} \sum_{k=1}^{\infty} (\pm)^{k+1} \frac{1}{k} \left(\frac{m_i}{T}\right)^3 K_1(km_i/T)$$

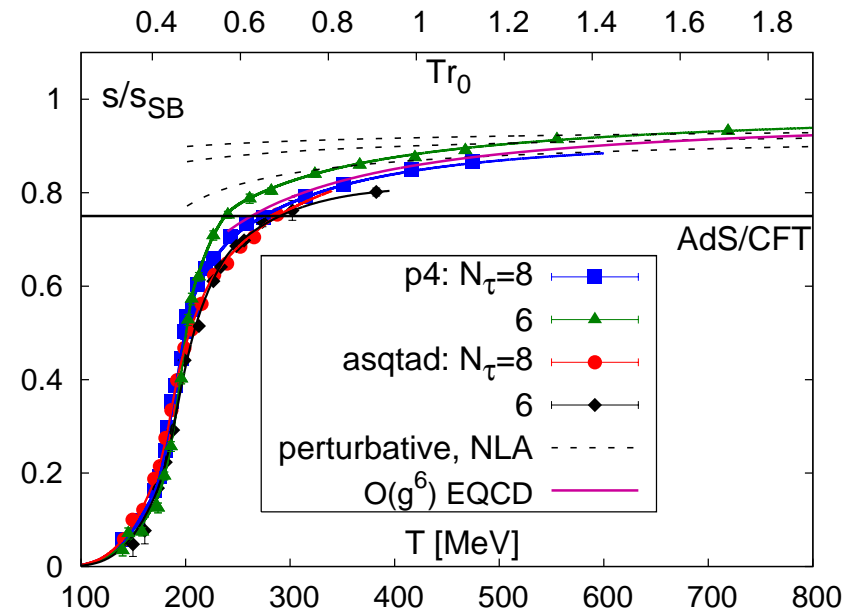
- ★ reducing discretization effects lowers the crossover temperature
- ★ diminishing quark mass lowers the crossover temperature
- ★ smaller quark masses are expected to raise $\epsilon - 3p$ somewhat at low T, but effect is small
- ★ reducing discretization effects from $N_\tau = 8 \rightarrow 12$ seems to raise $\epsilon - 3p$, **preliminary**

results: $m_l = 0.1 m_s$ high T



$$\left(\frac{\epsilon - 3p}{T^4}\right)_{high T} = \frac{3}{4} b_0 g^4(T) + \frac{d_2}{T^2} + \frac{d_4}{T^4}$$

EQCD: 3d effective theory,
with perturbative matching to 4d
Kajantie et al.; Laine, Schröder



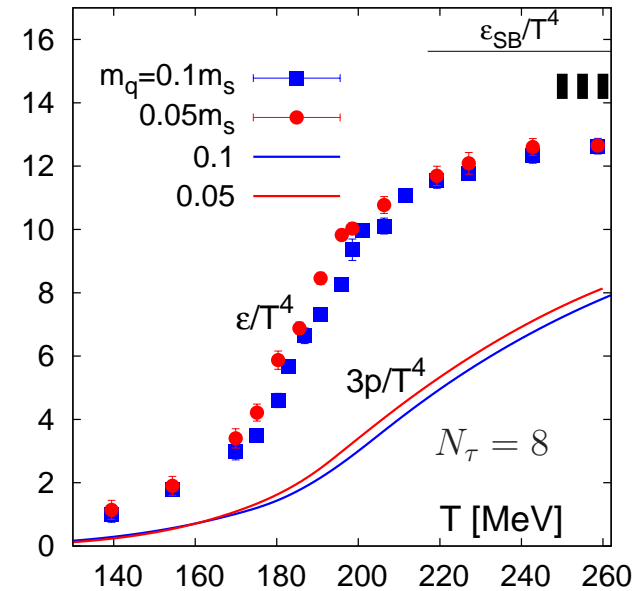
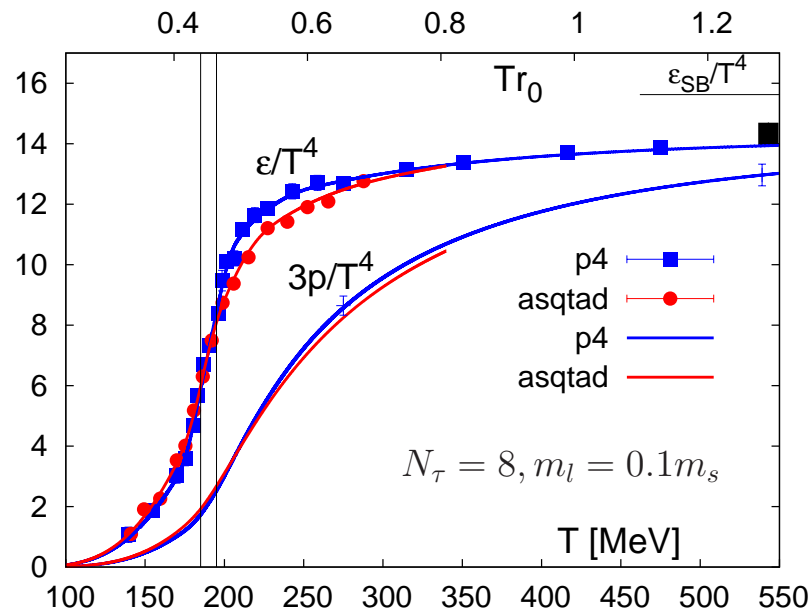
$$s/T^3 = (\epsilon + p)/T^4$$

courtesy P.Petreczky

also agreement with resummed pert.theory
Blaizot et al.

results

pressure and energy density



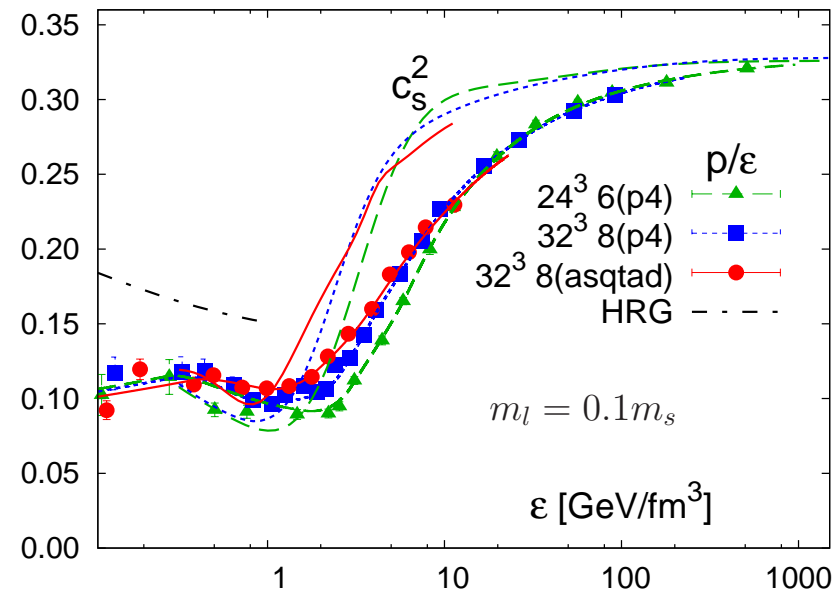
- ★ good agreement between **asqtad** and **p4fat3** actions
- ★ integration error: indicated by little bars to the right
- ★ in comparison with **Stefan-Boltzmann**: 10 % below at 2 - 3 T_c
- ★ quark mass effects negligible at high T

vertical lines denote $N_\tau = 8$ transition region $185 \text{ MeV} < T < 195 \text{ MeV}$ for $m_l = 0.1m_s$

results

speed of sound

$$c_s^2 = \frac{p}{\epsilon} + \epsilon \frac{d(p/\epsilon)}{d\epsilon}$$



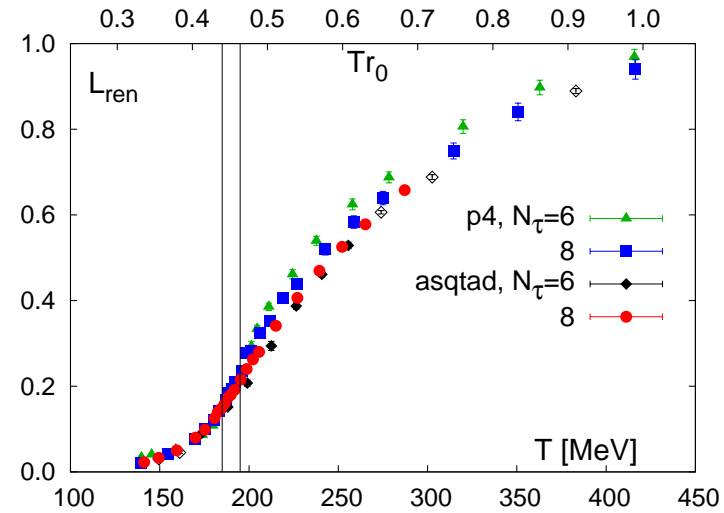
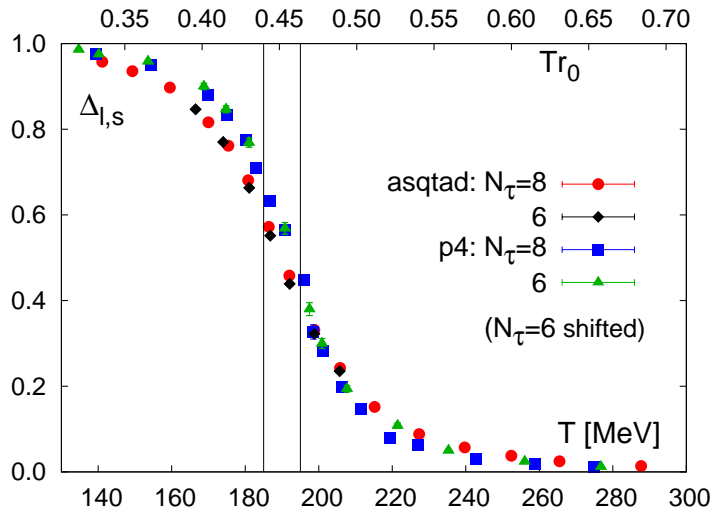
★ good agreement between **asqtad** and **p4fat3** actions

★ softest point corresponds to $\epsilon \simeq 1 \text{ GeV/fm}^3$

II Transition

comparison with chiral condensate and Polyakov loop

- chiral condensate $\langle \bar{\psi}\psi \rangle$ true order parameter in the chiral limit $m_q \rightarrow 0$
- Polyakov loop $L \sim \exp(-F_q/T)$ true order parameter in the static quark limit $m_q \rightarrow \infty$



- both need renormalization

$$\Delta_{l,s}(T) = \frac{\langle \bar{\psi}\psi \rangle_{l,T} - \frac{m_l}{m_s} \langle \bar{\psi}\psi \rangle_{s,T}}{\langle \bar{\psi}\psi \rangle_{l,0} - \frac{m_l}{m_s} \langle \bar{\psi}\psi \rangle_{s,0}}$$

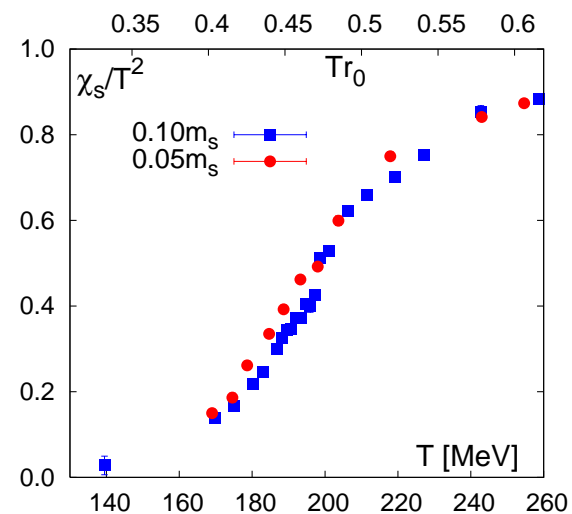
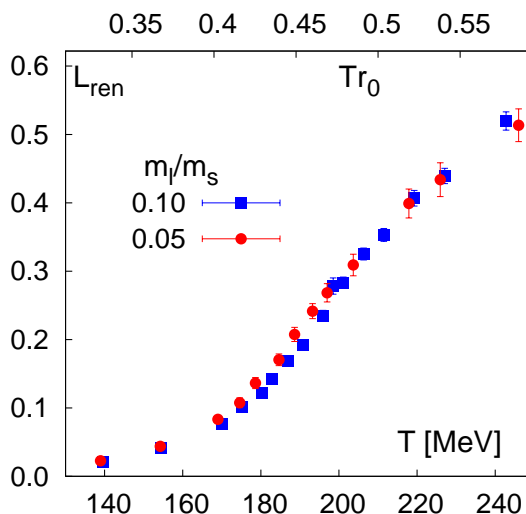
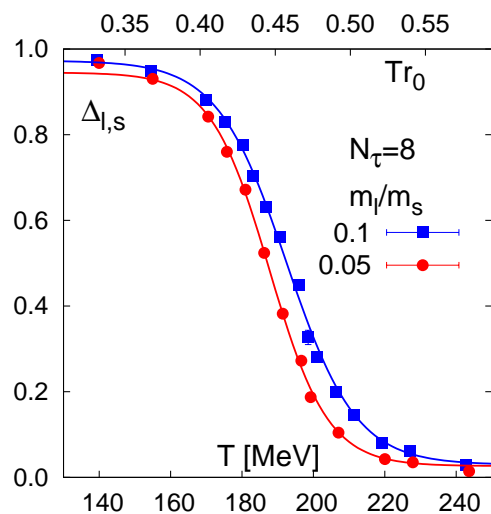
subtracting power-law additive and cancelling multiplicative UV divergencies

UV renormalization via normalization of static quark potential

significant changes in the same temperature range as EoS

going to $m_l = 0.05 m_s \leftrightarrow m_\pi \simeq 150 \text{ MeV}$ ($N_\tau = 8$, p4 action, asqtad $T = 0$ running)

strange quark number susceptibility



$$\chi_s(T) \sim \frac{1}{T^3 V} \left. \frac{\partial^2 \ln Z(T, \mu_s)}{\partial \mu_s^2} \right|_{\mu_s=0}$$

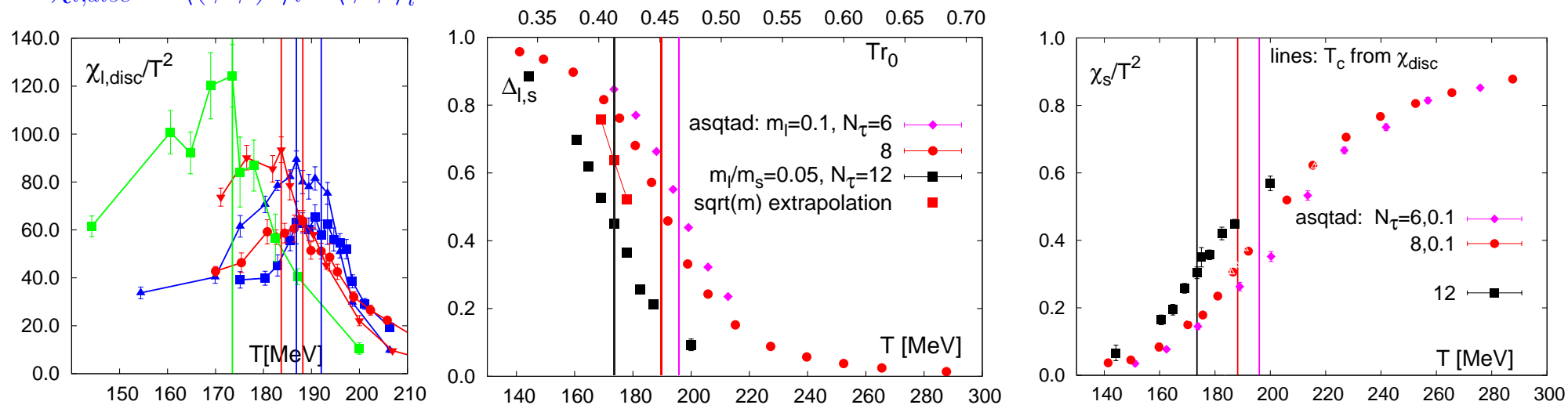
at low T : $\sim \exp(-m_K/T)$
not too sensitive to criticality

– as in EoS: overall shift by about -5 MeV due to smaller quark mass

going to $N_\tau = 12$ ($m_l = 0.05 m_s$, asqtad action)

disc. chiral susceptibility

$$\chi_{l,disc} \sim \langle (\bar{\psi}\psi)^2 \rangle_l - \langle \bar{\psi}\psi \rangle_l^2$$



asqtad: $N_\tau = 8$, $N_\tau = 12$

p4: $N_\tau = 8$

vertical lines mark $\chi_{l,max}$

note the asymmetry around $\chi_{l,max}$

red boxes:

$N_\tau = 12$ at $m_l = 0.1 m_s$

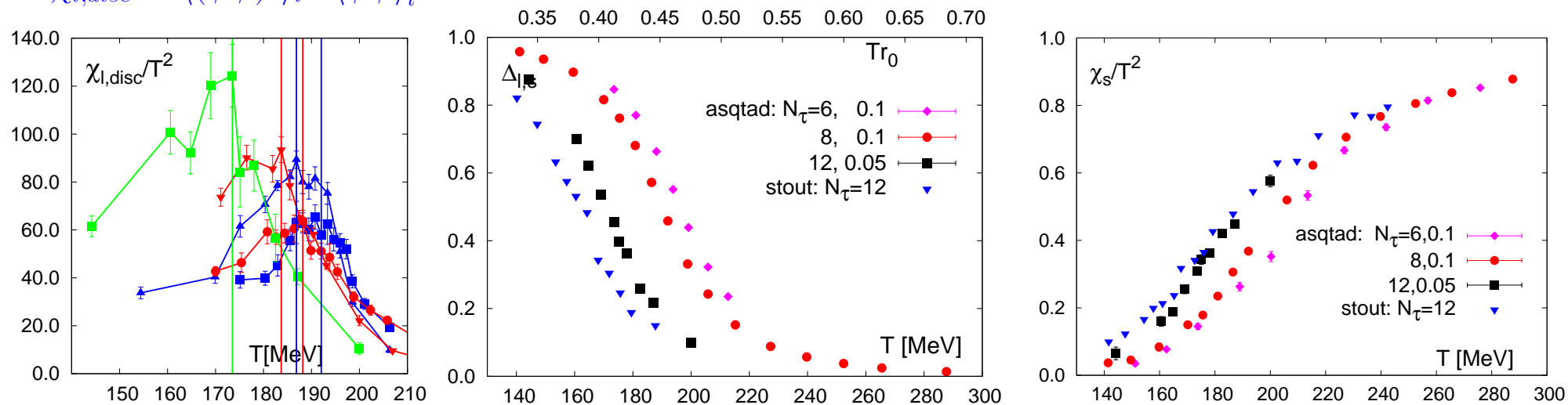
through $\sqrt{m_l}$ extrapolation (see next section)

more work under way, so far: – χ_{disc} , $\Delta_{l,s}$ and χ_s simultaneously exhibit further downward shift in T
 – suggesting a transition temperature $\lesssim 170$ MeV at physical π mass

going to $N_\tau = 12$ ($m_l = 0.05 m_s$, asqtad action)

disc. chiral susceptibility

$$\chi_{l,disc} \sim \langle (\bar{\psi}\psi)^2 \rangle_l - \langle \bar{\psi}\psi \rangle_l^2$$



asqtad: $N_\tau = 8$, $N_\tau = 12$

p4: $N_\tau = 8$

vertical lines mark $\chi_{l,max}$

note the asymmetry around $\chi_{l,max}$

comparison with the stout action (Budapest-Wuppertal):

differences in the $\Delta_{l,s}$, χ_s data decrease with decreasing a

for a comparison with the hisq action see talk by P.Petreczky

- more work under way, so far:
- χ_{disc} , $\Delta_{l,s}$ and χ_s simultaneously exhibit further downward shift in T
 - suggesting a transition temperature $\lesssim 170$ MeV at physical π mass

III Critical behavior ?!

phase diagram of QCD at $\mu_q = 0$

- chiral symmetry of 2 flavor QCD is:

$$SU_L(2) \times SU_R(2) \simeq O(4)$$

- hence, if m_s is large enough in 2+1 QCD

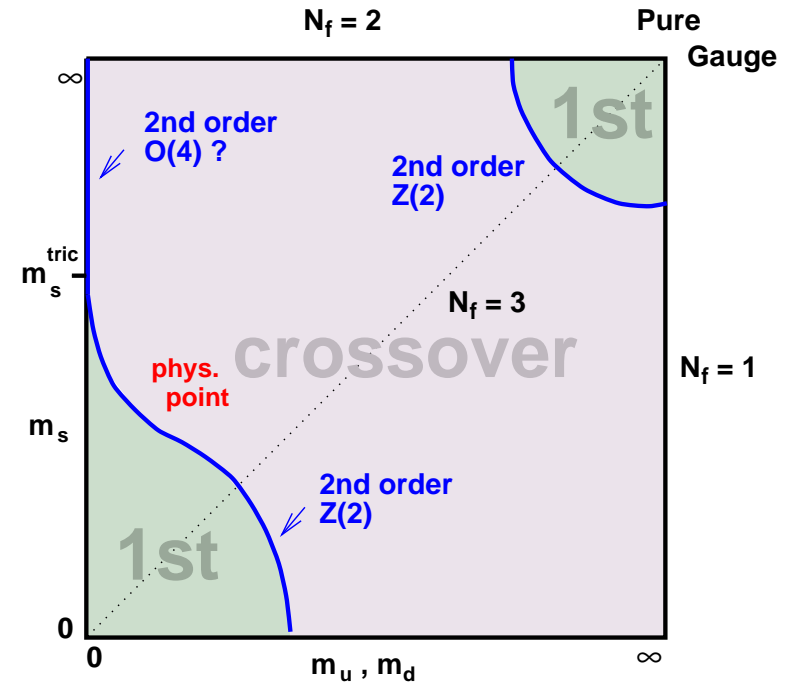
expect universal behavior as of 3d $O(4)$ spins
in the vicinity of finite T chiral limit, $m_l \rightarrow 0$

[Pisarski, Wilczek, ...]

- so far no clear evidence from numerical simulations
and there are caveats due to the fate of the $U_A(1)$

[long list of refs.]

- staggered fermions preserve a flavor non-diagonal $U(1)$ part of chiral symmetry even at $a > 0$
 \Rightarrow search for $O(2)$ critical behavior in the vicinity of chiral limit also at $a > 0$



magnetic equation of state in $O(N)$ spin models

magnetization $M = h^{1/\delta} f_G(z)$

where $z = t/h^{1/\beta\delta}$

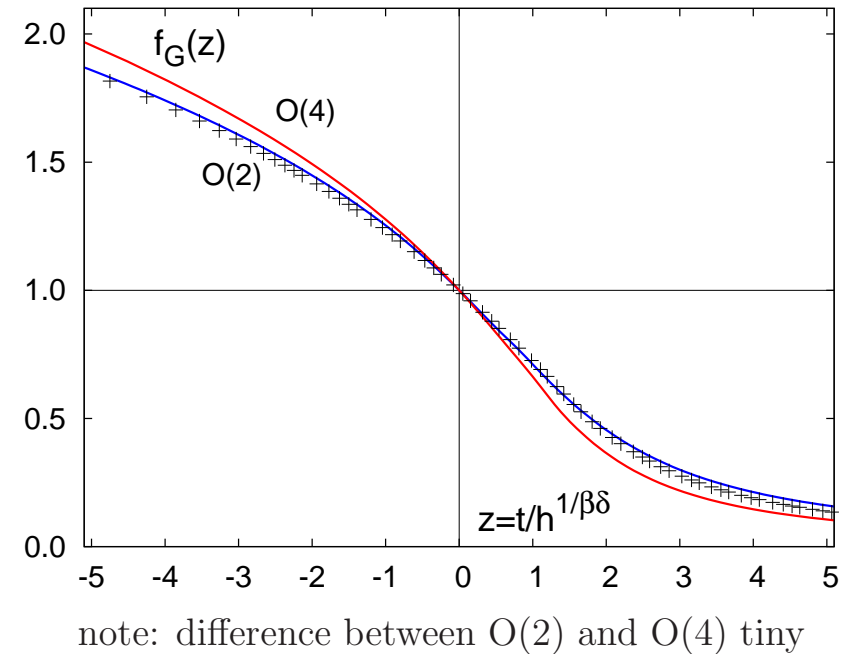
$f_G(z)$ universal scaling function

$t = \frac{1}{t_0} \frac{T - T_c}{T_c}$ reduced temperature

$h = \frac{H}{h_0}$ external field

β, δ universal critical exponents

t_0, h_0 non-universal normalizations



$f_G(z), \beta, \delta$ known to high precision [Engels et al., condensed matter literature](#)

in the limit $z \rightarrow -\infty$, corresponding to $h \rightarrow 0$ and $t < 0$

$f_G(z) \rightarrow (-z)^\beta \{1 + \tilde{c}_2 (-z)^{-\beta\delta/2}\}$ corresponding to $M(t, h) \rightarrow M(t, 0) + c_2(t) \sqrt{h}$

Goldstone effect

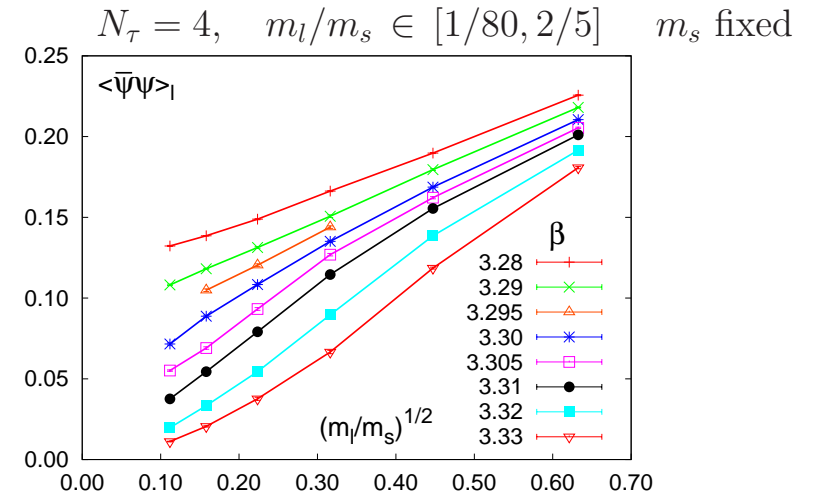
magnetic equation of state in QCD

here $M \sim \langle \bar{\psi}\psi \rangle_l$ $H \sim m_l$

we use $M_b = m_s \langle \bar{\psi}\psi \rangle_l$

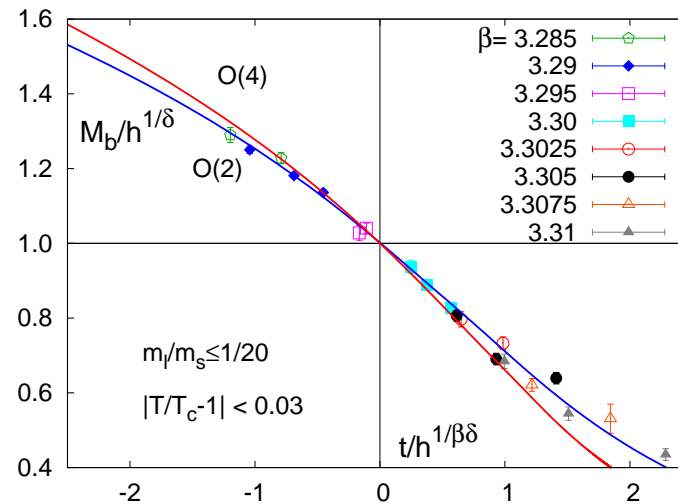
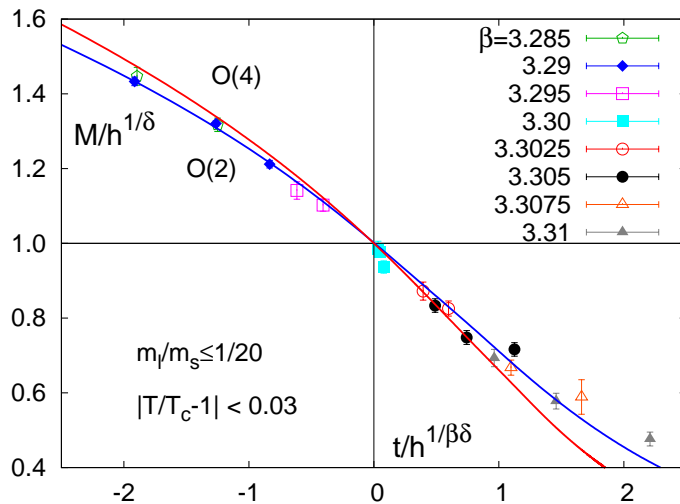
$$M = m_s \left[\langle \bar{\psi}\psi \rangle_l - \frac{m_l}{m_s} \langle \bar{\psi}\psi \rangle_s \right]$$

★ clear evidence for $\sqrt{m_l}$ dependence at low β
i.e. for Goldstone effect



– fit to the $O(2)$ scaling function $f_G(z)$, 3 fit parameters: $t_0, h_0, T_c(m_l = 0)$

m_l/m_s	m_π
1/80	75 MeV
1/40	105 MeV
1/20	150 MeV



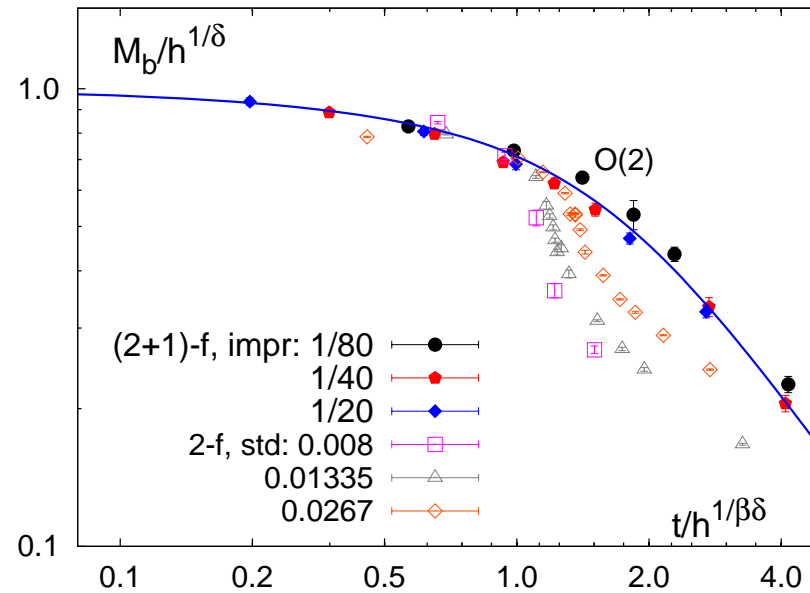
consistency with $O(N)$ scaling behavior at small quark masses

comparison to staggered data in the standard discretization

until now

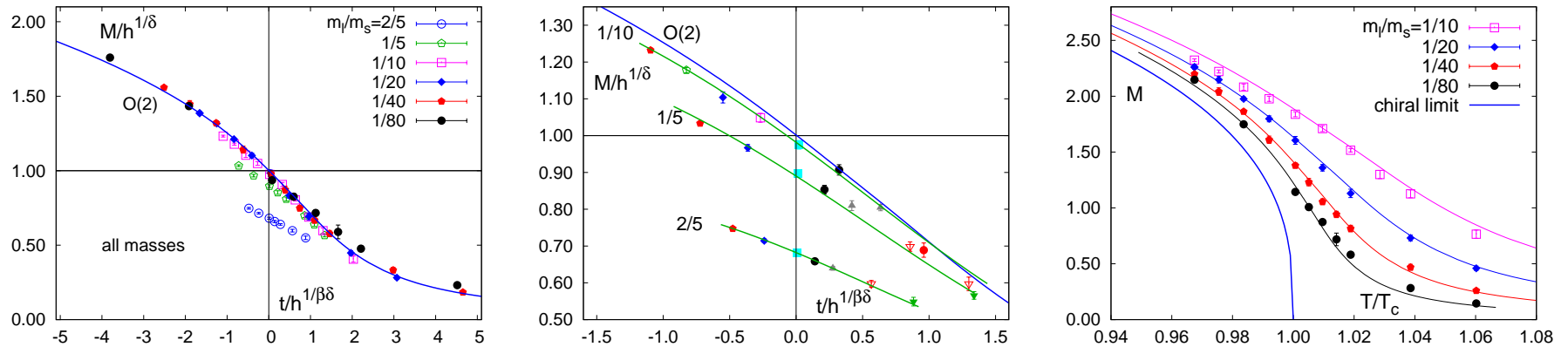
comparison with magn. EoS
for $z > 0$ only

similar difficulties with fits for
critical exponents



MILC, Pisa data

scaling violations



– are clearly visible in the region of small $|z|$ i.e. large m_l and small t , and can be parametrized as

$$M(t, h) = h^{1/\delta} f_G(t/h^{1/\beta\delta}) + a_t t h + b_1 h + b_3 h^3 + b_5 h^5$$

since the scale parameters t_0, h_0 have been determined above

- for the linear term: $b_1 = 0$ for M (subtracted condensate)
 $b_1 \neq 0$ for M_b

–> curves and chiral extrapolation for $M(T)$

★ scaling violations are small for physical values of the quark mass

Summary

- Equation of State
 - extending analysis to $m_\pi \simeq 150$ MeV on $N_\tau = 12$ lattices
 - at low T : reducing lattice spacing and quark mass leads to approaching HRG
 - at high T : making contact with (resummed) perturbation theory and EQCD
- Transition
 - reducing a and m_l affects the studied observables in the same way,
 - suggests a continuum extrapolated value of $T_c \lesssim 170$ MeV preliminary
- critical behavior ?
 - finally, strong indications for $O(N)$ scaling of the magnetic equation of state !
 - although at large lattice spacing, hints at physical point possibly being in the attraction basin of a second order chiral phase transition
 - optimistic signal for further studies of the QCD phase diagram