On causality, unitarity and perturbative expansions

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Introduction and Motivation

Recently a novel scheme for studying hadronic interactions beyond the threshold region was introduced in A. Gasparyan, M.F.M. Lutz, Nucl.Phys. A848 (2010) 126-182. (applied to γN and πN scat.)

This scheme includes:
a) Chiral perturbation theory
b) EM-gauge invariance, causality and unitarity
c) Conformal mapping

• The purpose of the present study is an illustration of this method for a system where the exact solution is known.

• We consider non-relativistic Yukawa interactions of various strengths and ranges.

Lippman-Schwinger equation

The nonrelativistic partial-wave Lippman-Schwinger equation

$$\langle k'|\, t_l(q^2)|k\rangle = \langle k'|V_l\,|k\rangle + \frac{4\,m}{\pi}\int_0^\infty k''^2\, dk''\, \frac{\langle k'|V_l\,|k''\rangle\, \langle k''|\, t_l(q^2)|k\rangle}{q^2 - k''^2 + i\,\epsilon}$$

The Yukawa potential projected onto angular momentum reads

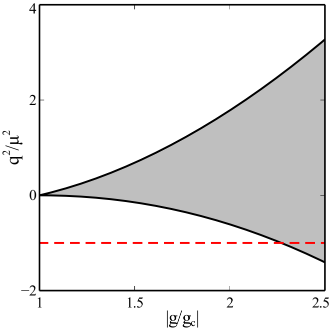
$$\langle k'|V_{l=0}\,|k\rangle = \frac{g}{4\,k'k}\,\log\frac{(k+k')^2+\mu^2}{(k-k')+\mu^2}$$

g - characterizes the strength of interaction
μ - characterizes the range of interaction

Born series

The Lippmann-Schwinger equation can be expanded in powers of $g^{(n)}$:

$$\langle k' |\, t_l(q^2) | k \rangle = t_l^{(1)} + t_l^{(2)} + \ldots$$



The area of convergence (white) for the *s*-wave. The dashed line locates the branch point caused by the 2nd order Born term.

The critical coupling constant is

$$g_c \simeq 1.68 \, \frac{\mu}{2\, m}$$

Born series

The Born series to third order of the scaled on-shell amplitude

$$\bar{t}_l\,(z) = 2\,m\,\mu\,\langle k'|\,t_l(q^2)|k\rangle\Big|_{k'=k=q} \qquad \text{with} \qquad z = \frac{q^2}{\mu^2}$$

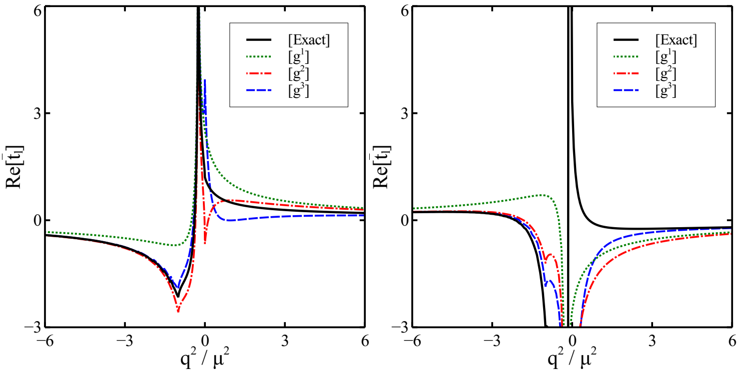


Figure: $g = 3\,g_c/2$ (left), $g = -3\,g_c/2$ (right); $l = 0$

Partial Wave Dispersion relation

Unitarity and Analyticity

$$T_l(q^2) = U_l(q^2) + \int_0^\infty \frac{dq'^2}{\pi} \, \frac{q^2+\mu_M^2}{q'^2+\mu_M^2} \, \frac{\rho(q'^2)}{q'^2-q^2-i\epsilon} \, |T_l(q'^2)|^2$$

- separate left and right-hand cuts
- the generalized potential $U_l(q^2)$ contains all left hand cuts

The unitarity constraint has the simple form

$$\Im \, T_l(q^2) = |T_l(q^2)|^2 \, \rho_l(q^2) \, , \qquad \rho_l(q^2) = \left(\frac{q^2}{\Lambda^2+q^2}\right)^{l+\frac{1}{2}}$$

Λ define the scale at which $\rho \to 1$. $T_{l=0}(q^2) = -\frac{\sqrt{\Lambda^2+q^2}}{\mu} \, \bar{t}_{l=0}(q^2)$

The generalized potential

The perturbation series for $\bar{u}_I\left(q^2\right) = -\frac{\mu}{\sqrt{\Lambda^2+q^2}}\, U_I(q^2)$

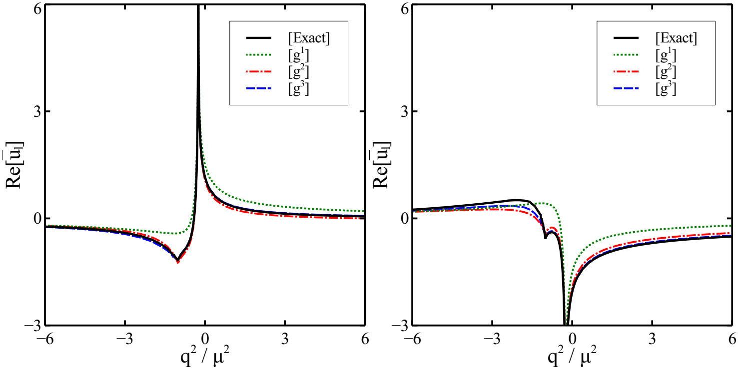


Figure: $g = 3\,g_c/2$ (left), $g = -3\,g_c/2$ (right); $\mu_M^2 = 10\mu^2$; $I = 0$

N/D technique

The non-linear integral equation can be solved by means of N/D technique [G.F.Chew, S.Mandelstam, PR 119 (1960) 467-477.]

$$T_l(q^2) = N_l(q^2)/D_l(q^2)$$

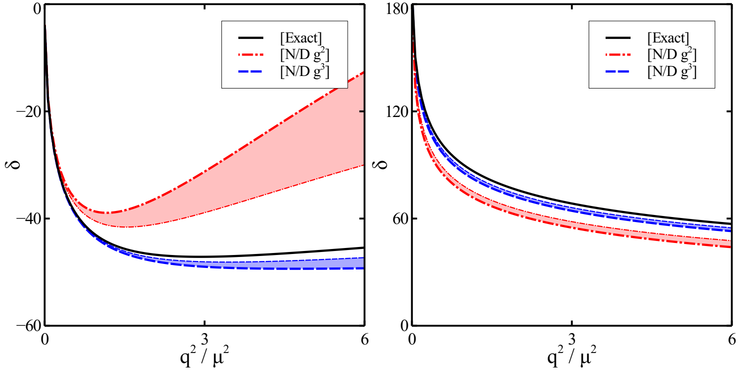


Figure: $g = 3\,g_c/2$ (left), $g = -3\,g_c/2$ (right); $\mu_M = 10\mu^2$; $l = 0$; Λ: $3\,\mu < \Lambda < 9\,\mu$

Approximation for the generalized potential $U_l(q^2)$

- In χPT one can perform a pert. expansion only for small q^2.
- To solve the non-linear integral equation we need $U_l(q^2)$ for all energies above threshold ($q^2 > 0$)

Reliable extrapolation is possible:
We split the contributions from closest and more distant left-hand cuts

$$U(q^2) = U_{\rm inside}(q^2) + U_{\rm outside}(q^2)$$

We calculate exactly $U_{\rm inside}(q^2)$ and $\underline{\text{extrapolate } U_{\rm outside}(q^2)}$

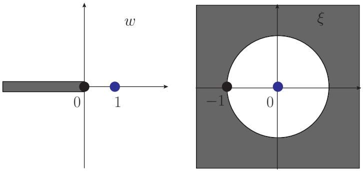
Conformal mapping

Conformal mapping techniques may be used to approximate the generalized potential $U_{\text{outside}}(q^2)$ for higher energies, based on the knowledge of the potential only around threshold.

Typical example: $U_{outside}(w) = ln(w), \quad \mu_E = 1$

$$\hookrightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(\mu_E)}{k!} [w - \mu_E]^k$$

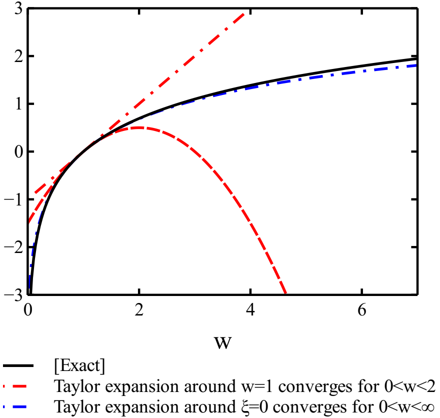
$$\hookrightarrow \sum_{k=0}^{\infty} C_k(f^{(k)}(\mu_E)) \, [\xi(w)]^k$$



$\xi(w) = \frac{1-\sqrt{w}}{1+\sqrt{w}}$ maps the cut onto the unit circle $\xi(0) = 0$, $\xi(1) = 0$

Conformal mapping for $U_{outside}(w) = ln\,(w)$

$$ln\,(w) \hookrightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(\mu_E)}{k!}\,[w - \mu_E]^k \hookrightarrow \sum_{k=0}^{\infty} C_k(f^{(k)}(\mu_E))\,[\xi(w)]^k$$



Results

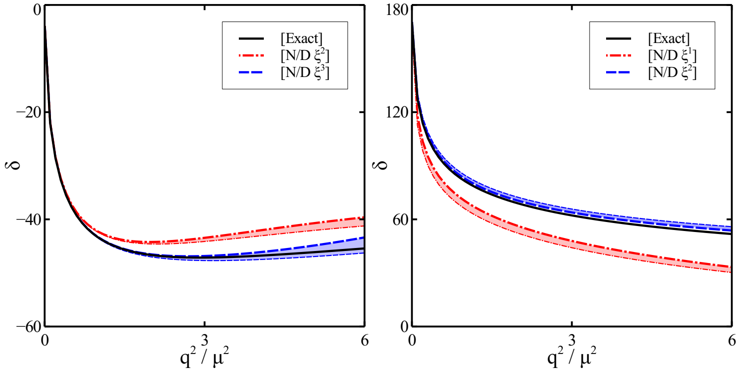


Figure: $g = 3\, g_c/2$ (left), $g = -3\, g_c/2$ (right)

Case of a superposition of strong short-range and weak long-range forces

In contrast to the Yukawa toy model in a realistic system the matching scale μ_M cannot be chosen arbitrarily low (noperturbative effects from u- or t-channel). We set $\mu_M^2 = \mu^2$.

A superposition of two Yukawa potentials

$$\langle k'|V|k\rangle = \langle k'|V_L|k\rangle + \langle k'|V_S|k\rangle$$

where $\mu_S >> \mu_L$.

$$\begin{aligned} t &= \left[1 - (V_L + V_S)\, G\right]^{-1} \left[V_L + V_S\right] \\ &= t_S + (1 + t_S\, G)\, V_L \sum_{n=0}^{\infty} \left[(G + G\, t_S\, G)\, V_L\right]^n (1 + G\, t_S)\,, \end{aligned}$$

Renormalization scheme

Left hand cuts start at

$$q^2 = -\frac{1}{4}\,(n\,\mu_L + k\,\mu_S)^2$$

Thus in the limit $\mu_s \to \infty$ the generalized potential takes the simple form

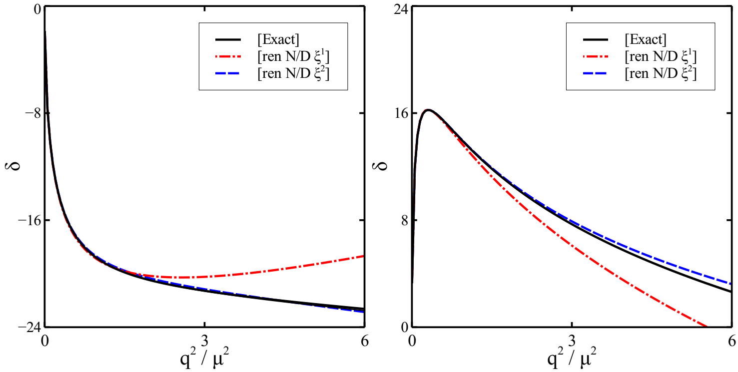
$$U(q^2) = U_L(q^2) + C \qquad \text{with} \qquad C = T(-\mu_M^2) - U_L(-\mu_M^2)\,,$$

For definiteness we consider:

$$\mu_s = 12\,\mu_L,\ |g_S| = 0.95\,g_{c,S},\ |g_L| = 0.5\,g_{c,L},\ \Lambda = \frac{1}{2}\,\mu_S$$

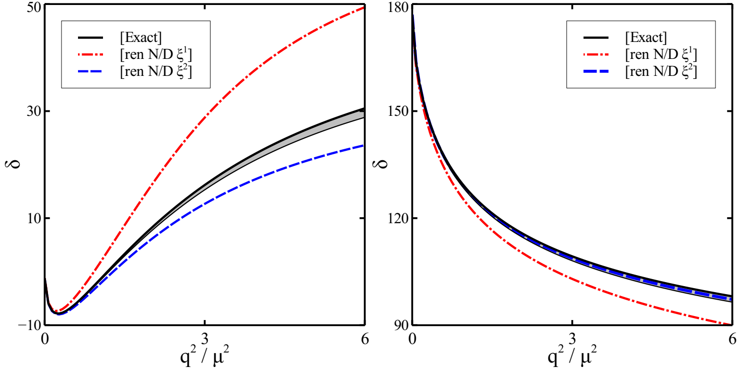
Results

Case of repulsive short-range force. The l.h.p. and r.h.p. correspond to repulsive and attractive long-range potential.

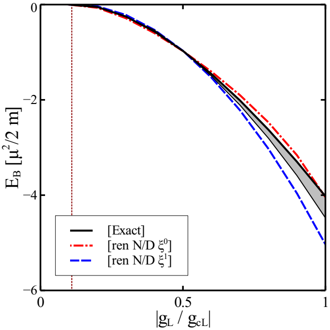


Results

Case of attractive short-range force. The l.h.p. and r.h.p. correspond to repulsive and attractive long-range potential.



Binding energy



Binding energy for the system with two attractive Yukawa interactions as a function of g_L.

Summary

An approach based on

- causality and unitarity
- analytic extrapolation of generalized potential

was successfully applied to a simple system where the exact solution is known.

The constraints set by causality and unitarity can be used to arrive at a quite effective expansion scheme that is suitable for applications in effective field theories.

N/D technique

N/D technique
[G.F.Chew, S.Mandelstam, PR 119 (1960) 467-477.]

$$T_l(q^2) = N_l(q^2)/D_l(q^2)$$

where

$$\begin{aligned} N_l(q^2) &= U_l(q^2) + \int_0^\infty \frac{dq'^2}{\pi} \frac{q^2+\mu_M^2}{q'^2+\mu_M^2} \frac{N_l(q'^2)\,\rho(q'^2)\,(U_l(q'^2) - U_l(q^2))}{q'^2 - q^2} \\ D_l(q^2) &= 1 - \int_0^\infty \frac{dq'^2}{\pi} \frac{q^2+\mu_M^2}{q'^2+\mu_M^2} \frac{N_l(q'^2)\,\rho(q'^2)}{q'^2 - q^2 - i\epsilon} \end{aligned}$$