Meson Spectroscopy at COMPASS
Methods for Amplitude Analysis

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The COMPASS Experiment

Diffractive Pion Dissociation

Amplitude Analysis Formalism

PWA Model Selection
  Mass Independent Fit
  Bayesian Model Evaluation
  Waveset Exploration

Mass Dependent Fit

ROOTPWA Analysis Toolkit
The COMPASS Experiment
Searching for Gluonic Contributions to the Meson Spectrum

Overview

- **C**o**m**mon **M**uon and **P**roton **A**pparatus for **S**tructure and **S**pectroscopy
- Located at **CERN's SPS**
- M2-beamline: high intensity $\pi/p$ beam up to 280GeV/c
The COMPASS Experiment
Searching for Gluonic Contributions to the Meson Spectrum

Overview
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Hadron Program
- Light Meson Spectroscopy
- Diffractive Reactions → **Spin Exotic Mesons**
- Central Production → **Glueballs**
- Low \( Q^2 \): Pion/Kaon Polarizabilities
Diffractive Pion Dissociation
Example: 3 Pion Final State

\[ \pi \rightarrow \pi \pi \pi \]

Number of Events

COMPASS 2004
\[ \pi \text{Pb} \rightarrow \pi \pi \pi \text{Pb} \]
Diffractive Pion Dissociation
Example: 3 Pion Final State

\[
\begin{align*}
\pi & \rightarrow X + \pi^- + \pi^- \\
A & \rightarrow \pi^+ + \pi^0 + A
\end{align*}
\]

\[
\text{Mass of } \pi^0 \text{ System (GeV/c)}^2
\]

\[
\begin{array}{c|c|c|c|c}
0 & 0.5 & 1 & 1.5 & 2 \\
\hline
\text{Events} / (5 \text{ MeV/c})^2 & 2 & 2.5 & 3 & 3.5 & 4 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
1260 & 1320 & 1670 \\
\hline
a_1 & a_2 & \pi_2
\end{array}
\]

* event distribution
* background wave
**Diffractive Pion Dissociation**

Example: 3 Pion Final State

\[ X: J^{PC} M^\epsilon \text{ decay amplitude in reflectivity base} \]

Implement parity conservation:

\[ \psi^\epsilon_{JM} = c(M) \left[ \psi_{JM}(\tau) - \epsilon P(-1)^{J-M} \psi_{J(-M)}(\tau) \right] \]

\[ \epsilon = \pm 1 \quad M \in [0..J] \quad c(M > 0) = \frac{1}{\sqrt{2}} \quad c(M = 0) = \frac{1}{2} \]
**PWA Formalism Overview**

2Stage Isobar-Model Fit

**STEP 1: Mass-Independent PWA**

- Fit angular distributions + isobar systems in independent mass bins

\[
\mathcal{I}(\tau, m) = \sum_{\epsilon = \pm 1} \sum_{r=1}^{N_r} \left| \sum_{\alpha} T_{\alpha r}^\epsilon \psi_{\alpha}^\epsilon(\tau, m) \right|^2
\]

- Production amplitude

- Decay amplitude
**PWA Formalism Overview**

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- Production amplitude
- Decay amplitude

**STEP 2: Mass-Dependent \( \chi^2 \) fit → Extract Resonance Parameters**

- Parameterization of spin-density matrix elements \( \sum_r T_{ir}^\epsilon T_{jr}^{\epsilon*}(m_x) \)
- Takes into account interference terms
- Coherent background for some waves
Model n-body decay by a chain of successive 2-body decays:

\[ J^{PCM \epsilon} \]

\( \pi^-(\text{beam}) \) \rightarrow X \rightarrow \pi^-(\text{bachelor}) \rightarrow \pi^- \rightarrow \pi^+ \rightarrow \pi^+(\text{target recoil})

\( \epsilon = +: \text{natural parity exchange} \)

\( \epsilon = -: \text{unnatural parity exchange} \)

\( \pi^- \) 

For fixed n-body mass \( m \) there are \( 3n - 4 \) parameters (angles, intermediate state masses)

Parameterization of isobar subsystems

Example angular distributions:

\( X(2^{-+}) \rightarrow f_2(1275)\pi \)

\( f_2(1275) \rightarrow \pi\pi \)
Model n-body decay by a chain of successive 2-body decays:

For fixed n-body mass $m$ there are $3^n - 4$ parameters (angles, intermediate state masses).

Parameterization of isobar subsystems:

- Unitarity violation
- Rescattering effects
- Potential for improvement
- Input from theory needed (see e.g. talk by B. Kubis)

Example angular distributions:

$X(2^{-}) \rightarrow f_2(1275)\pi$

$f_2(1275) \rightarrow \pi\pi$
Intensity distribution $I$ as a function of decay-kinematic variables $\tau$:

$$I(\tau) = \sum_{\epsilon=\pm 1} \sum_r \sum_{\alpha \in M} |T_{\epsilon r}^{\epsilon} \bar{\psi}_{\alpha}^{\epsilon}(\tau)|^2$$

- Finite waveset $M$
- Production amplitude
- Decay amplitude
Intensity distribution $\mathcal{I}$ as a function of decay-kinematic variables $\tau$:

\[
\mathcal{I}(\tau) = \sum_{\epsilon = \pm 1} \sum_{r} \sum_{\alpha \in M} T^\epsilon_{\alpha r} \psi^\epsilon_{\alpha}(\tau)
\]

- Finite waveset $M$
- Production amplitude
- Decay amplitude

The likelihood $\mathcal{L}$ to observe (a specific set of) $N$ events in a bin with finite acceptance $\eta(\tau)$ (assuming a model $M$, parameters $T^\epsilon_{ir}$) is:

\[
P(\text{Data}|T_{ir}, M) = \mathcal{L} = \left[ \frac{\bar{N}^N}{N!} e^{-\bar{N}} \right] \prod_{i}^{N} \frac{\mathcal{I}(\tau_i) \eta(\tau_i) f(\tau_i)}{\int \mathcal{I}(\tau) \eta(\tau) d\rho(\tau)}
\]

with $d\rho(\tau) = f(\tau) d\tau$
Mass Independent Amplitude Fit

Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\bar{N}^N}{N!} e^{-\bar{N}} \right] \prod_i \frac{\mathcal{I}(\tau_i)}{\bar{N}} \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_i \mathcal{I}(\tau_i) \cdot \prod_i \eta(\tau_i) f(\tau_i) \cdot e^{-\bar{N}} \]
Mass Independent Amplitude Fit

Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\tilde{N}^N}{N!} e^{-\tilde{N}} \right] \prod_i \frac{\mathcal{I}(\tau_i)}{\tilde{N}} \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_i \mathcal{I}(\tau_i) \cdot \prod_i \eta(\tau_i) f(\tau_i) \cdot e^{-\tilde{N}} \]

Taking the logarithm leads to and inserting for \( \tilde{N} \)

\[ \ln \mathcal{L} = -N \ln N + \sum_i \eta(\tau_i) f(\tau_i) + \sum_i \ln \mathcal{I}(\tau_i) - \int \mathcal{I}(\tau) \eta(\tau) d\rho(\tau) \]
Mass Independent Amplitude Fit

Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\bar{N}^N}{N!} e^{-\bar{N}} \right] \prod_{i=1}^{N} \frac{\mathcal{I}(\tau_i)}{\bar{N}} \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_{i=1}^{N} \mathcal{I}(\tau_i) \cdot \prod_{i=1}^{N} \eta(\tau_i) f(\tau_i) \cdot e^{-\bar{N}} \]

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drop \((-N \ln N + \sum_{i=1}^{N} \eta(\tau_i) f(\tau_i))\) and insert intensity parameterization

\[ \ln \mathcal{L} = \sum_{n=1}^{\text{Nevents}} \ln \left[ \sum_{\epsilon, \tau} \sum_{\alpha, \beta \in M} T_{\alpha \tau}^\epsilon T_{\beta r}^{\epsilon*} \bar{\psi}_\alpha^{\epsilon}(\tau_n) \bar{\psi}_\beta^\epsilon(\tau_n)^* \right] - \sum_{\epsilon, r} \sum_{\alpha, \beta \in M} T_{\alpha \tau}^\epsilon T_{\beta r}^{\epsilon*} \mathcal{I} \mathcal{A}^\epsilon_{\alpha \beta} \]
Mass Independent Amplitude Fit

Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\bar{N}^N}{N!} e^{-\bar{N}} \right] \frac{1}{N} \prod_i \frac{\mathcal{I}(\tau_i)}{\bar{N}} \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_i \mathcal{I}(\tau_i) \cdot \prod_i \eta(\tau_i) f(\tau_i) \cdot e^{-\bar{N}} \]

Taking the logarithm leads to and inserting for \( \bar{N} \)

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drop \((-N \ln N + \sum_i \eta(\tau_i) f(\tau_i))\) and insert intensity parameterization

\[ \ln \mathcal{L} = \sum_{n=1}^{N_{\text{events}}} \ln \left[ \sum_{\epsilon, r} \sum_{\alpha, \beta \in M} T^\epsilon_{\alpha r} T^{\epsilon*}_{\beta r} \psi^\epsilon_{\alpha}(\tau_n) \bar{\psi}^\epsilon_{\beta}(\tau_n)^* \right] - \sum_{\epsilon, r} \sum_{\alpha, \beta \in M} T^\epsilon_{\alpha r} T^{\epsilon*}_{\beta r} IA^\epsilon_{\alpha \beta} \]

With acceptance-corrected phase space integral

\[ IA^\epsilon_{\alpha \beta} = \int \bar{\psi}^\epsilon_{\alpha}(\tau_n) \bar{\psi}^\epsilon_{\beta}(\tau_n)^* \eta(\tau) d\tau \]
Which waves to include into the waveset?
Which waves to include into the waveset?

Avoid overfitting
Which waves to include into the waveset?

Avoid overfitting

→ Data driven method
Bayes’ Theorem (for the Model Probability after Observation)

\[ P(M_k|\text{Data}) = \frac{P(\text{Data}|M_k)P(M_k)}{\sum_{k'} P(\text{Data}|M_{k'})P(M_{k'})} \]

with model-priors \( P(M_k) \)  
\[ \sum_{k'} P(M_{k'}) = 1 \]
How to Measure the Goodness of a Model

Marginal Likelihood Definition

Bayes’ Theorem (for the Model Probability after Observation)

\[ P(M_k|\text{Data}) = \frac{P(\text{Data}|M_k)P(M_k)}{\sum_{k'} P(\text{Data}|M_{k'})P(M_{k'})} \]

with model-priors \( P(M_k) \) \( \sum_{k'} P(M_{k'}) = 1 \)

Marginal Likelihood or Evidence

\[ P(\text{Data}|M_k) = \int P(\text{Data}|T^k, M_k) \underbrace{P(T^k|M_k)}_{\text{Prior}} \, dT^k \]

\( P(T^k|M_k) \) contains any pre-knowledge on the model-parameters \( T \)

- Marginalization (\( = \int dT \)) is not trivial in high-dimensional spaces
- Numerically stable is only the LogLikelihood
\[
P(\text{Data}|M_k) = \int_{\mathcal{L}} P(\text{Data}|T^k, M_k) P(T^k|M_k) dT^k
\]
The Occam Factor Approximation


\[
P(\text{Data}|M_k) = \int P(\text{Data}|T_k^k, M_k) P(T_k^k| M_k) \, dT_k^k
\]

Approximate with Laplace’s method:

\[
P(\text{Data}|M_k) \approx P(\text{Data}|T_{ML}^k, M_k) \cdot P(T_{ML}^k| M_k) \cdot \sqrt{(2\pi)^d |C_T|_{\text{Data}}}
\]

Occam factor
\[ P(\text{Data}|M_k) = \int \left[ \frac{P(\text{Data}|T_k, M_k)}{\mathcal{L}} \right] P(T_k|M_k) \, dT_k \] 

Approximate with Laplace’s method:

\[ P(\text{Data}|M_k) \approx P(\text{Data}|T_{\text{ML}}^k, M_k) \cdot P(T_{\text{ML}}^k|M_k) \cdot \sqrt{(2\pi)^d |C_T|_{\text{Data}}} \]

- \( P(\text{Data}|T_{\text{ML}}^k, M_k) \): LogLikelihood at maximum likelihood solution \( T_{\text{ML}} \)
- \( |C_T|_{\text{Data}} \): determinant of covariance matrix
- Dimension of parameter space: \( d \)
The Occam Factor Approximation


\[
P(\text{Data}|M_k) = \int \frac{P(\text{Data}|T^k, M_k) P(T^k|M_k)}{\mathcal{L}} dT^k
\]

Approximate with Laplace’s method:

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P(\text{Data}|M_k) \approx P(\text{Data}|T_{\text{ML}}^k, M_k) \cdot P(T_{\text{ML}}^k|M_k) \cdot \sqrt{(2\pi)^d |C_T|_{\text{Data}}}
\]

- \(P(\text{Data}|T_{\text{ML}}^k, M_k)\) LogLikelihood at maximum likelihood solution \(T_{\text{ML}}\)
- \(|C_T|_{\text{Data}}\) determinant of covariance matrix
- Dimension of parameter space: \(d\)

Logarithmic evidence:

\[
\ln P(\text{Data}|M_k) \approx \ln P(\text{Data}|T_{\text{ML}}^k, M_k) + \ln P(T^k|M_k) + \ln \sqrt{(2\pi)^d |C_T|_D}
\]
Final Definition

Log-Evidence

\[
\ln P(Data|M_k) \approx \ln \mathcal{L}_{ML} + \ln \sqrt{(2\pi)^d|C_T|_{Data}} - \ln V_T^k + \sum_{i \in M} \ln S_i
\]

where \(V_T^k\) is the (prior) volume of parameter space

- Models (=wavesets) compared through the Bayes-Factor
  \[
  B_{12} = \frac{P(Data|M_1)}{P(Data|M_2)}
  \]

- Interpretation according to Kass&Raftery:

<table>
<thead>
<tr>
<th>(2 \ln B_{12})</th>
<th>(B_{12})</th>
<th>Evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 2</td>
<td>1 to 3</td>
<td>Not worth mentioning</td>
</tr>
<tr>
<td>2 to 6</td>
<td>3 to 20</td>
<td>Positive</td>
</tr>
<tr>
<td>6 to 10</td>
<td>20 to 150</td>
<td>Strong</td>
</tr>
<tr>
<td>&gt; 10</td>
<td>&gt; 150</td>
<td>Very strong</td>
</tr>
</tbody>
</table>

Strategies:

- Start with population of small (2-15 waves) wavesets (adding waves)
- Start with diverse population (10 - 80 waves)
- Start with population of large wavesets (not done yet)
Number of waves optimizes at around 35
Automatic Waveset Exploration
Genetic Algorithm – 50 generations, population size 50

- Diverse initial population run $\rightarrow$ better results
- Typical log-Evidence differences: 30-100
Example: Top 20 Fits from Genetic Search
Example Mass Dependent Fit

\[ T^\epsilon_i T^\epsilon_j^* = \rho^\epsilon_{ij}(m) = \left( \sum_k C^\epsilon_{ik} BW_k(m) \sqrt{\int |\psi^\epsilon_i|^2 d\tau} \right) \left( \sum_l C^\epsilon_{lj} BW_l(m) \sqrt{\int |\psi^\epsilon_j|^2 d\tau} \right) \]

(1)

with Breit-Wigner amplitude:

\[ BW_{ik}(m, M_0, \Gamma_0) = \frac{M_0 \Gamma_0}{m^2 - M_0^2 + i \Gamma_{tot}(m) M_0} \]

(2)

and dynamic width:

\[ \Gamma_{tot}(m) = \sum_n \gamma_n \frac{\rho_n(m)}{\rho_n(M_0)} \quad \rho_n(m) \sim \int |\psi^\epsilon_i|^2 dq \quad \sum \gamma_n = \Gamma_0 \]

(3)

and background terms:

\[ bkg(m) = e^{-\alpha q} \quad q \text{ – Breakup momentum} \]

(4)
Fit Results Overview - Spin Density Matrix

7 waves, 8 resonances
- Based on BNL code “pwa2000”
- Largely rewritten
- Workflow for mass-dependent fit:

```
+-----------------+                      +-----------------+
| Phase Space     | Amplitude Calculator   | Integrator (INT) |
| Eventgenerator  | (GAMP)                 | ps-mc integrals  |
|                 |                        | all              |
| ps-mc           |                        | ps-mc amp        |
| 4-vectors       |                        | amp              |
| accepted        |                        | all              |
| events          |                        |
+-----------------+                      +-----------------+
| Amplitude       | ps-mc amplitudes       | TFitBin          |
| specifications  |                        | Observables     |
| (keygen)        |                        | calculator       |
+-----------------+                      +-----------------+
| COMGEANT        | Amplitude Calculator   | TPWALikelihood   |
| Acceptance      | (GAMP)                 | Fitter           |
| Monte Carlo     |                        | data             |
+-----------------+                      +-----------------+
| ps-mc           |                        | Spin density     |
| 4-vectors       |                        | matrix           |
| all events      |                        |                  |
+-----------------+                      +-----------------+
| PHAST Event     | ps-mc 4-vectors        |                  |
| Selection       | accepted final state   |                  |
|                 | particles              |                  |
|                 | data                   |                  |
|                 | 4-vectors              |                  |
|                 |                          |                  |
```

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PWA Model Selection  
Mass Dependent Fit  
ROOTPWA: Open Source Analysis Toolkit  
http://sourceforge.net/projects/rootpwa

Sebastian Neubert — Meson Spectroscopy at COMPASS
Main Features:

- **Amplitude calculator** for diffractive production (Helicity Form.)
- General **Amplitude Framework** upcoming (B. Grube)
- MC generators (diffraction)
- Numerical tools
  - MC integrator
  - Fitters
  - Genetic Optimization
- Resonance parameterizations (under development)
- Visualization & Plotting tools (ROOT-based)
- **CUDA support**
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- ROOTPWA is one of 2 PWA programs used at COMPASS
- 2 step analysis:
  1. Fit angular correlations with Isobar Model decay
  2. Parameterize dynamics → resonance extraction
- Genetic search for waveset exploration
- Open source toolkit http://sourceforge.net/projects/rootpwa
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Outlook

- Improvements in amplitude parameterizations
- Study of non-resonant contributions (Deck effect)
- Theory input needed (Rescattering etc.)
- Status of Analyses and Results $\rightarrow$ Talk by B. Ketzer tomorrow