Dynamical RG approach to O(N) scalar field theory

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- Motivation
- Critical statics on O(N) scalar field theory
- Stochastic equation of motion
- Dynamical renormalization group :
  Order-parameter relaxation,
  Shear viscosity, Energy diffusion
- Summary and Outlook

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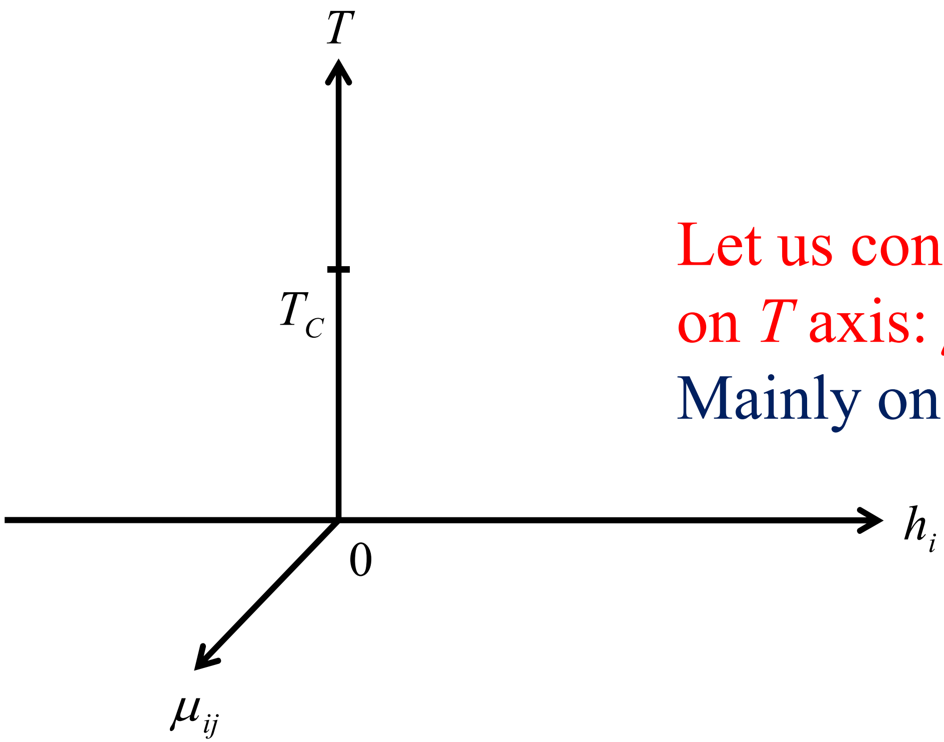
0) Motivation

O(N) scalar field theory :

$$L = \frac{1}{2}(\partial\phi_a)^2 - \frac{1}{2}r(\phi_a)^2 - \frac{1}{4}u(\phi_a)^2(\phi_a)^2 - h_a\phi_a - Q_{ab}\mu_{ab}$$

Define the theory with UV cutoff : Λ $\quad a = 1, \cdots, N$



Let us consider transport properties on *T* axis: $\mu = h = 0$.

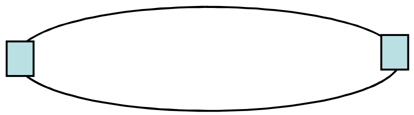
Mainly on shear viscosity.

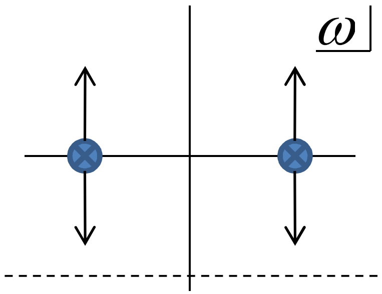
``Shear viscosity in O(N) scalar field theory at high T''

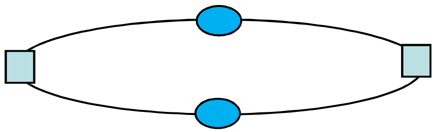
'95 Jeon and Yaffe at N=1	'04 Aarts et.al at Large N
$\eta \sim \dfrac{1}{u^2} T^3$	$\eta \sim \dfrac{N^2}{u^2} T^3$

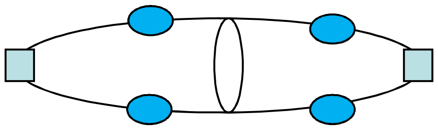
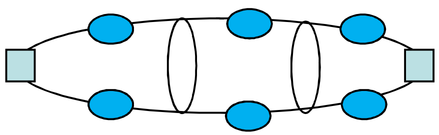
$$\text{Kubo formula}: \eta = \frac{1}{20T} \lim_{p \to 0} \int dx^4 e^{ipx} i \left\langle T^{ij}(x) T_{ij}(0) \right\rangle^R$$

$$\text{with } T^{ij} = \partial_i \phi \partial_j \phi$$

$\eta \sim$  $\sim$ divergent



 $\sim \dfrac{T^3}{\mathrm{Im} \ominus} \sim O(u^{-2})$

$+$  $+$  $+ \cdots$

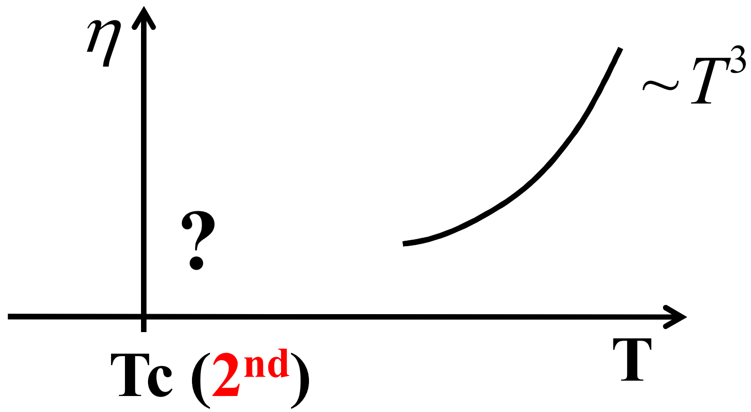
$\sim O(u^{-2})$ $\sim O(u^{-2})$

pinching singularity

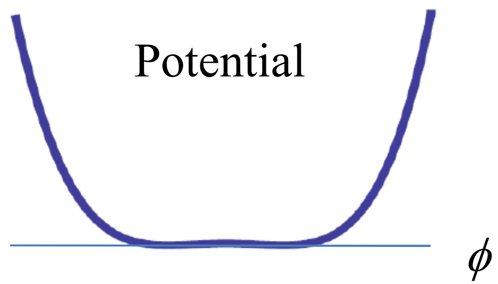
Imaginary part of Self-energy

Resummation to leading order is equivalent to solve linear Boltzmann equation.

What happens to Transport coefficients near phase transition?



- Boltzmann may not work.
- Low energy fluctuations, non-linear interactions, near Tc
  $\Rightarrow$ Non-perturbative treatment such as lattice, ….



- Whether transport coefficients get divergent or remain finite?
  $\Rightarrow$ Dynamical renormalization group (based on Wilson's RG)

1) Critical statics on O(N) scalar field theory

Wilson's renormalization group method (+ ε expansion)

Basic idea : Zoom out the system by scale transformation, and see if there exist non-trivial fixed points (scale invariant theory = critical point/2nd PT)

to see Long wave-length behavior

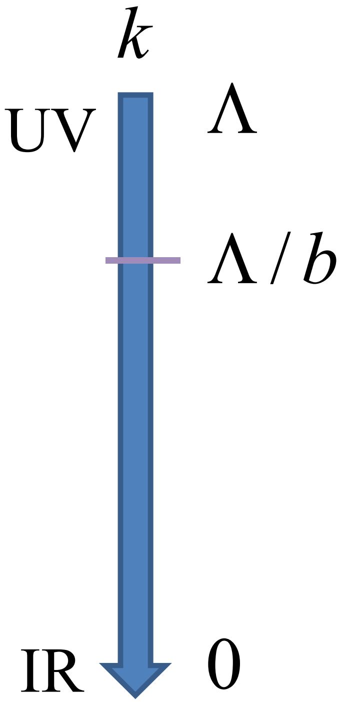
$$L = \frac{1}{2}(\partial\phi_a)^2 - \frac{1}{2}r(\phi_a)^2 - \frac{1}{4}u(\phi_a)^2(\phi_a)^2$$

$$L' = \frac{1}{2}(\partial\phi'_a)^2 - \frac{1}{2}r'(\phi'_a)^2 - \frac{1}{4}u'(\phi'_a)^2(\phi'_a)^2 + \cdots$$

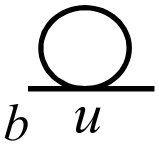
Flow equations for r and u $\Rightarrow$ fixed points

RG transformation = 2 steps

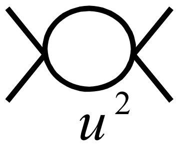
parameter: $b > 1$



1) Integrate the momentum shell $\Lambda < |k| < \Lambda / b$ in loop corrections

$$r' = r + \int_{\Lambda/b}^{\Lambda}$$


$$+ \cdots$$

$$u' = u + \int_{\Lambda/b}^{\Lambda}$$


$$+ \cdots$$

2) Change the length scale for all the variables (zoom out)

$\lambda \sim 1/k$ $\Longrightarrow$ $1/bk$

Repeat 1) and 2) = RG transformation → flows in r, u

Flow equations :（below critical dimension $d = 4 - \varepsilon$ ）

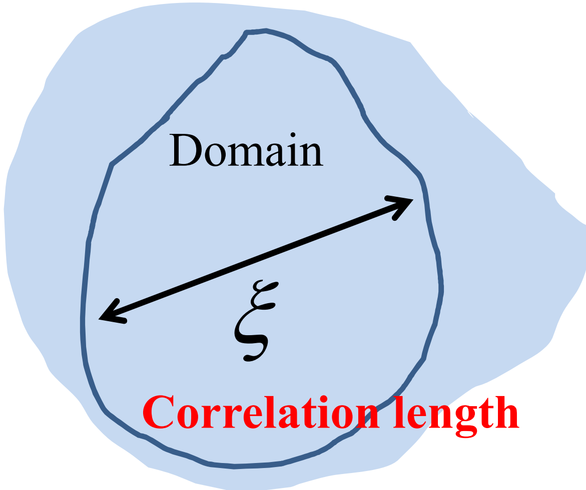
$$\dot{r} = (2 - \eta')r + 4(N+2)\Omega_4 u(\Lambda^2 - r)$$

$$\dot{u} = (\varepsilon - 2\eta')u - 4(N+8)\Omega_4 u^2$$

Non-trivial fixed point:

$$r^* = -\frac{1}{2}\varepsilon\frac{N+2}{N+8}\Lambda^2 + O(\varepsilon^2)$$

$$u^* = \frac{\varepsilon}{4(N+8)\Omega_4} + O(\varepsilon^2)$$



implying a critical point / 2nd order phase transition

From the FP ：
1) Critical Exponents and 2) Scaling Relations

$$r \sim \xi^{-2+\eta'}, \quad \chi_\phi \sim t^{-\gamma} = t^{-\nu(2-\eta')}$$

Unit of length ＝correlation length

$$b^L / \Lambda = \xi$$

2) Stochastic equation of motion (Langevin eq.)

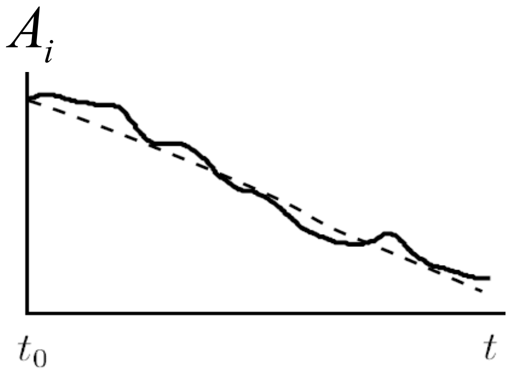
$$\partial_t A_i(k) \simeq L_{ij}(k)\frac{\delta H}{\delta A_j(k)} - [A_i, A_j]_{PB}\frac{\delta H}{\delta A_j(k)} + \zeta_i'(k,t)$$

Mode-mode couplings

$A_i(k)$: Slow modes {Order parameter, conserved variables}

$L_{ij}(k)$: Kinetic (transport) coefficients

$H = H(\{A_i\})$: Ginzburg-Landau Hamiltonian



$$\langle \zeta_i'(k,t)\zeta_j'(k',t')\rangle \simeq 2L_{ij}(k)\delta(t-t')\delta(k-k')$$

White noise (justified for long time scale)

In case of O(N) scalar field theory

Slow modes $A_i(k)$;

ϕ_i : Order parameter (OP)

$\vec{J}$: Transverse momentum

E : Energy density

[Q_{ij} : $O(N)$ charges, $N(N\text{-}1)/2$]

Symmetry + Derivative expansion (up to marginal terms)

$$H = \int \mathrm{d}^d x \left[\frac{1}{2}\left(\vec{\nabla}\phi_i\right)^2 + \frac{r_0}{2}\phi_i^{\,2} + \frac{u_0}{2}\left(\phi_i^{\,2}\right)^2 + \frac{1}{2}\vec{J}^2 \right.$$

$$\left. + \gamma_0 \phi_i^{\,2} E + \frac{1}{2} C_0^{-1} E^2 + \frac{1}{2}\tilde{r}_0 \left(\vec{\nabla} E\right)^2 + H_S \right]$$

$$H_S = -\phi_i h_i - \vec{J}\cdot\vec{H} + \beta E$$

Specific heat

EoM for slow variables:

$$\frac{\partial \phi_i}{\partial t} = -\lambda_0 \frac{\delta \mathcal{H}}{\delta \phi_i} + L_0^{(i)} \vec{\nabla}^2 \frac{\delta \mathcal{H}}{\delta E} - g_0 \left(\vec{\nabla}\phi_i\right) \cdot \frac{\delta \mathcal{H}}{\delta \vec{J}} + \theta_i$$

$$\frac{\partial E}{\partial t} = \Gamma_0 \vec{\nabla}^2 \frac{\delta \mathcal{H}}{\delta E} + \sum_i L_0^{(i)} \vec{\nabla}^2 \frac{\delta \mathcal{H}}{\delta \phi_i} - g_0 \left(\vec{\nabla} E\right) \cdot \frac{\delta \mathcal{H}}{\delta \vec{J}} + \xi$$

$$\frac{\partial \vec{J}}{\partial t} = \mathcal{T} \cdot \left[\eta_0 \vec{\nabla}^2 \frac{\delta \mathcal{H}}{\delta \vec{J}} + g_0 \left(\vec{\nabla}\phi_i\right) \frac{\delta \mathcal{H}}{\delta \phi_i} + g_0 \left(\vec{\nabla} E\right) \frac{\delta \mathcal{H}}{\delta E} + \vec{\zeta}\right]$$

Dynamical response functions:

$$\chi_\phi(k)_{ij} = \frac{\delta \phi_i(k)}{\delta h_j(k)}$$

$$\chi_E(k) = -\frac{\delta E(k)}{\delta \beta(k)}$$

$$\chi_J(k)_{ij} = \chi_J(k)\mathcal{T}_{ij} = \frac{\delta J_i(k)}{\delta H_j(k)}$$

Transport coefficients:

$$\frac{1}{\lambda} = \lim_{k\to 0} \frac{\partial \chi_\phi^{-1}(k)}{-i\partial\omega}$$

$$\frac{1}{\Gamma} = \lim_{k\to 0} \vec{k}^2 \frac{\partial \chi_E^{-1}(k)}{-i\partial\omega}$$

$$\frac{1}{\eta} = \lim_{k\to 0} \vec{k}^2 \frac{\partial \chi_J^{-1}(k)}{-i\partial\omega}$$

3) Dynamical renormalization group (Hohenberg-Halperin)

Basic idea : RG transformation to EoM $\rightarrow$ Fixed points

Static case

r : mass

u : coupling

Dynamical case

r : mass, γ : ϕ-E coupling

u : coupling

Transport coefficients

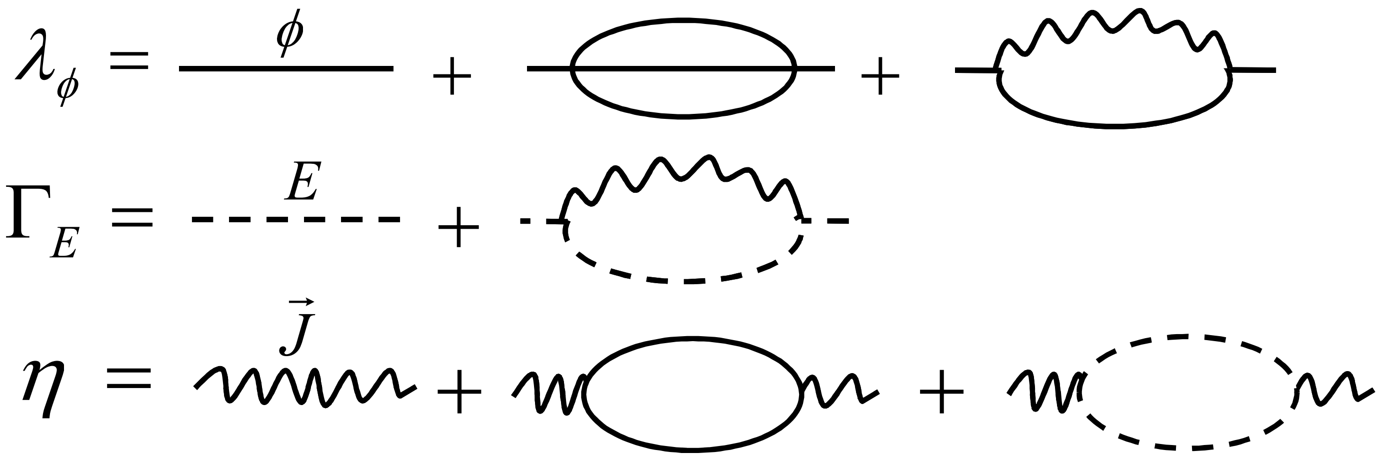
- Γ_E : energy - diffusion const.
- η : shear viscosity const.
- λ_ϕ : OP relaxation rate
- ζ : bulk viscosity const.
- $\vdots$

$\omega \rightarrow k^Z$ Z: Dynamical exp.

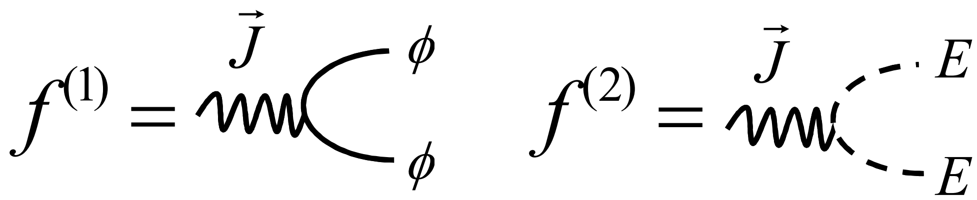
Flow equations for all coefficients!

→ Dynamical universality class = N, d (static universality class)
+ conserved variables + mode-mode couplings
= Model A ,B,… H,…

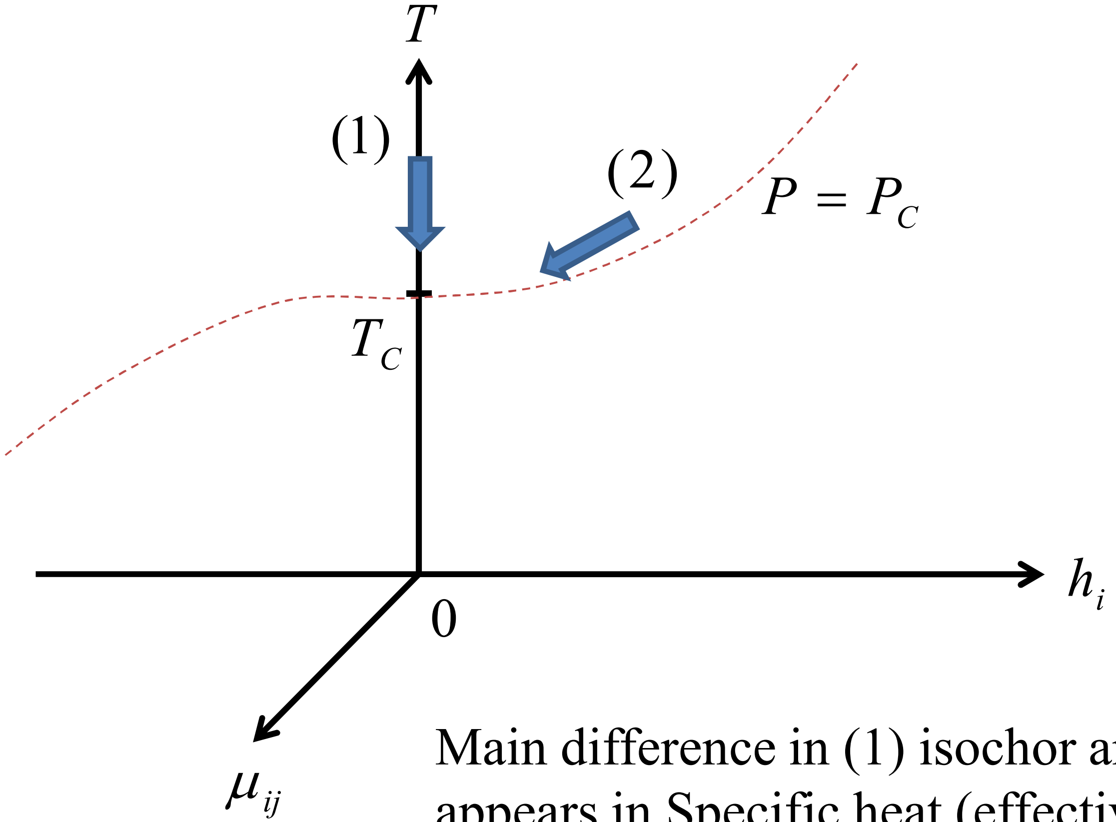
Flow equations for transport coefficients (to leading order of ε):



with 3-point vertices (mode-mode):



Phase diagram and Critical point



Main difference in (1) isochor and (2) isobar
appears in Specific heat (effective mass of Energy fluc)

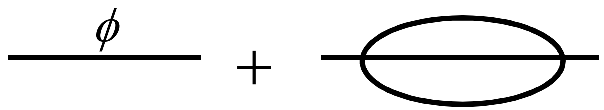
$$\sim \frac{1}{2} C^{-1} E^2$$ Divergent C → Criticality

(1) $C_h \sim t^{-\alpha}$ ($\alpha > 0$ for $N = 1$, $\alpha < 0$ for $N > 1$)

(2) $C_P \sim t^{-1}$

Case 1) isochor (h fixed) $C_h \sim t^{-\alpha}, \ |\alpha| << 1$ Specific heat

$f_1^* = f_2^* = 0$: both of mode-mode couplings vanish
OP and E, J are decoupled at long-length scale!

$\lambda_\phi =$ 

$$Z = 2 + \text{corrections}$$

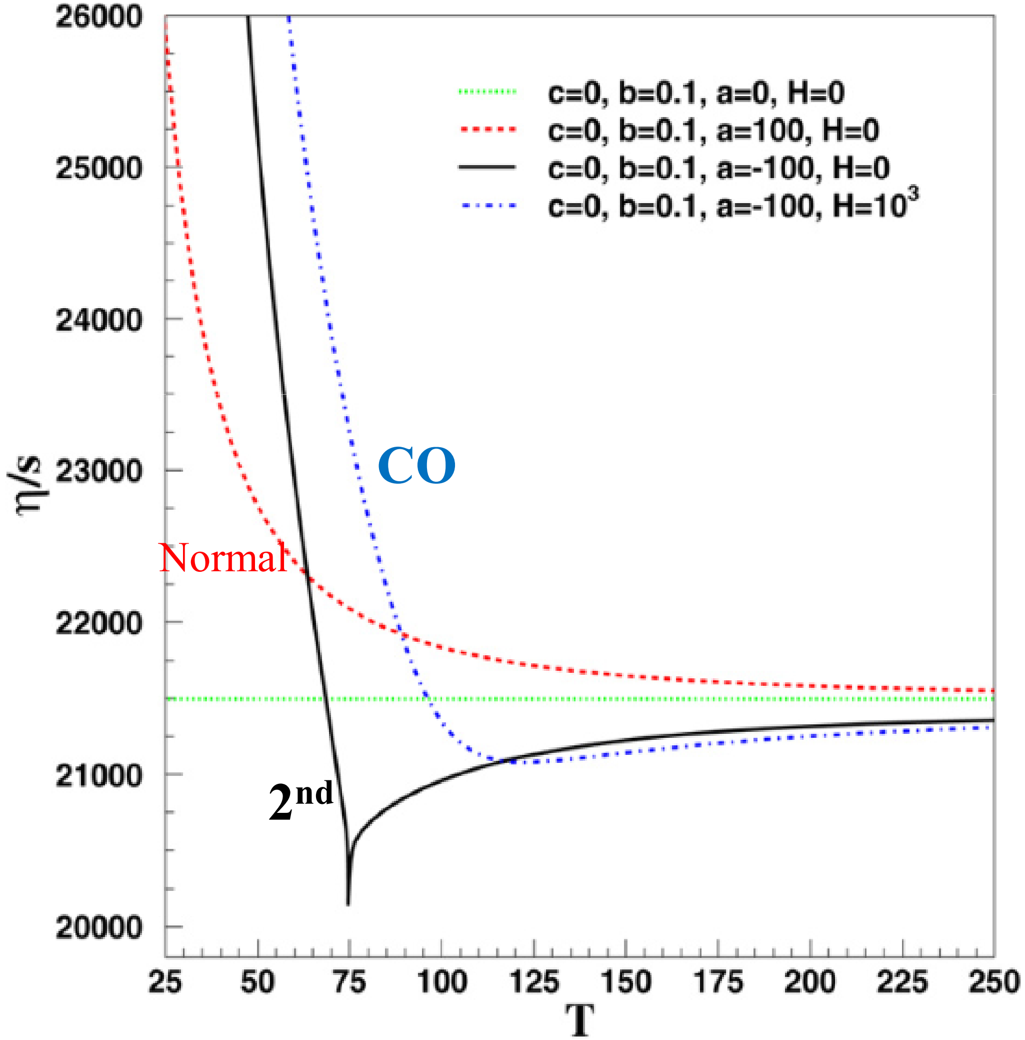
N=1 : Model C $[Z = 2 + O(\varepsilon)]$ ← Correction from Energy fluc.
N>1 : Model A $[Z = 2 + O(\varepsilon^2)]$

No singularity in E and J, so that
shear viscosity and energy diffusion are finite!

The criticality of the OP field does not propagate to E and J !

$\Rightarrow$ only regular part remains.

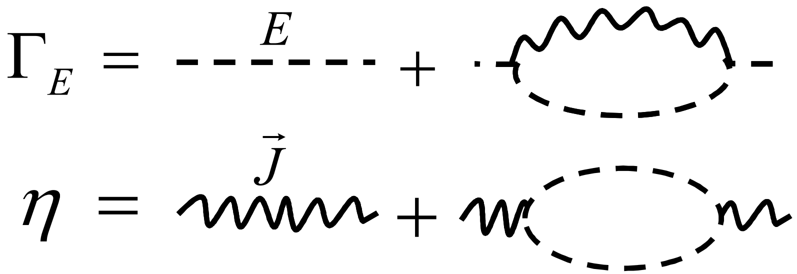
PLB (2008) N=1, Boltzmann + 2PI Potential (Effective Mass)



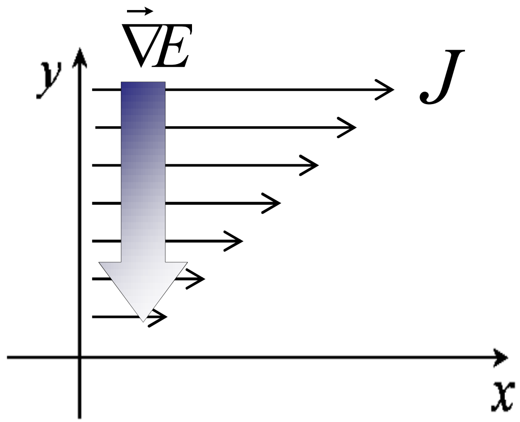
(2) isobar → ? Model H : Liquid- Gas critical point $C_p \sim t^{-1+\nu\eta'}$

$f_1^* = 0,\ f_2^* = O(\varepsilon)$ E-J mode-mode coupling is finite at FP

C_p^{-1} `Mass' of Energy fluctuation vanishes.
Energy has large criticality!



Energy –fluctuation transfers momentum



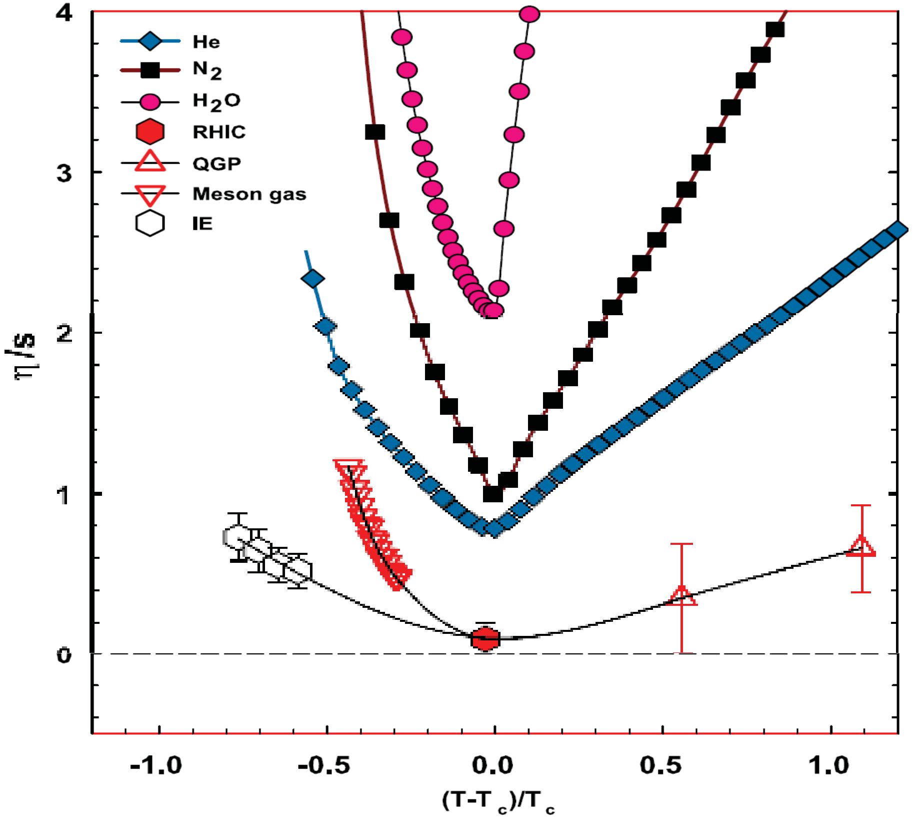
$$\eta \sim \xi^{x}, \quad x = \frac{1}{19}\varepsilon + O(\varepsilon^2)$$

$$\Gamma_E \sim \xi^{y}, \quad y = \frac{18}{19}\varepsilon + O(\varepsilon^2)$$

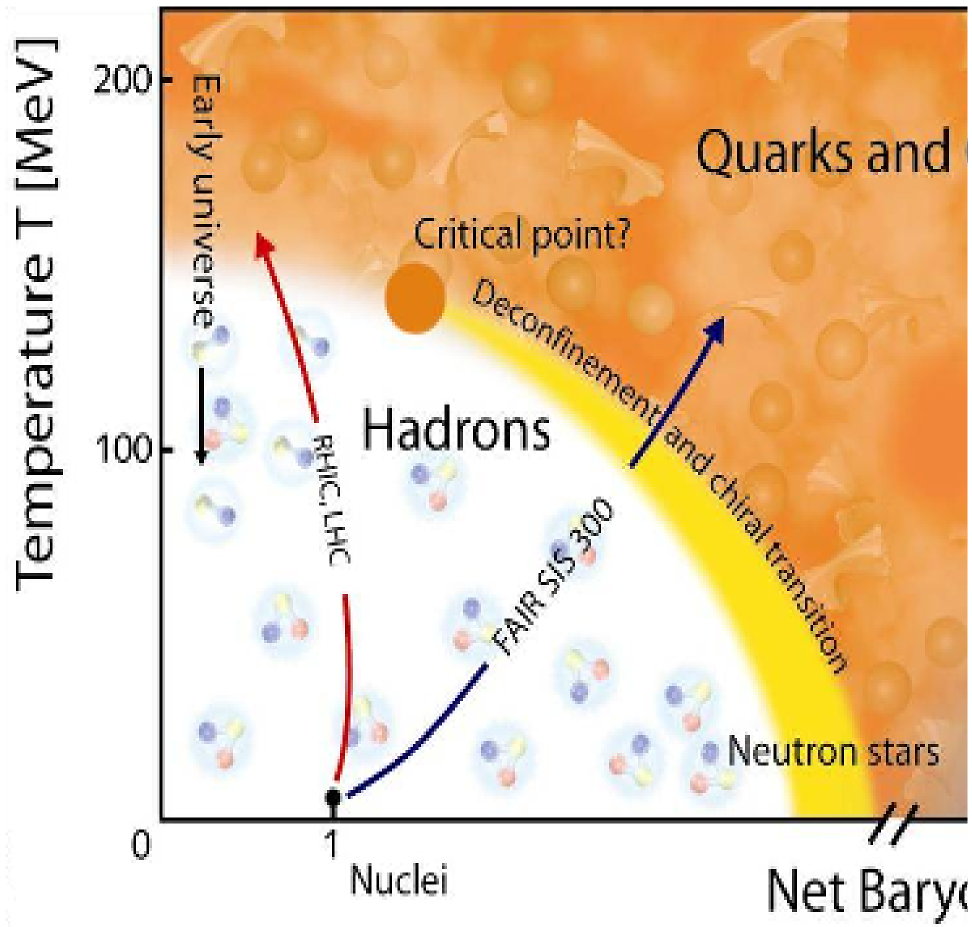
Scaling law: $x + y = \varepsilon + \eta'$

$$Z = 4 - x + \eta' \rightarrow 3 \ \text{at}\ d = 3$$

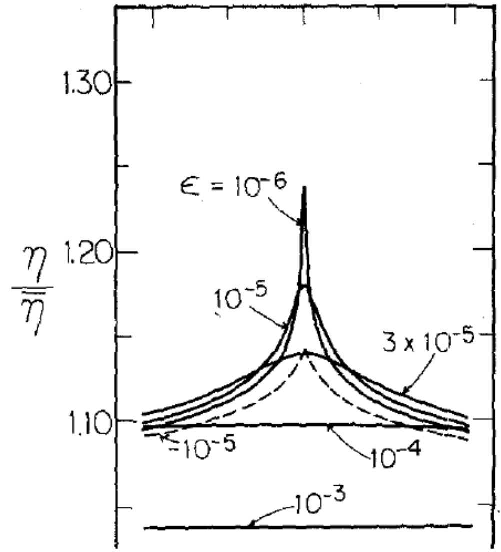
Lacey et al. PRL 98, 092301 (2007)



Model H and QCD CEP



Helium gas-liquid critical point (Model H)



4) Summary and Outlook

- Dynamical RG → EoM (φ, E, J) for O(N) scalar theory
- $h=\mu=0$, approaching Tc from above,

  all transport coefficients finite (except for bulk viscosity)

- Full analysis with O(N) charge fluctuations

  Model A → Model G for $N>1$ still $Z \sim 2$

- Finite μ → BEC (2^{nd} PT)
- Approaching the critical point from different directions

Flow equation (recursion relation) for transport coefficients:

$$\lambda_{l+1} = b^{z+\eta'-2}\lambda_l \left[1 + \left.\frac{\Sigma_\phi(k,\lambda_l,\Gamma_l,\eta_l,g_l,\ldots;\Lambda,b)}{\lambda_l}\right|_{k\to 0}\right]$$

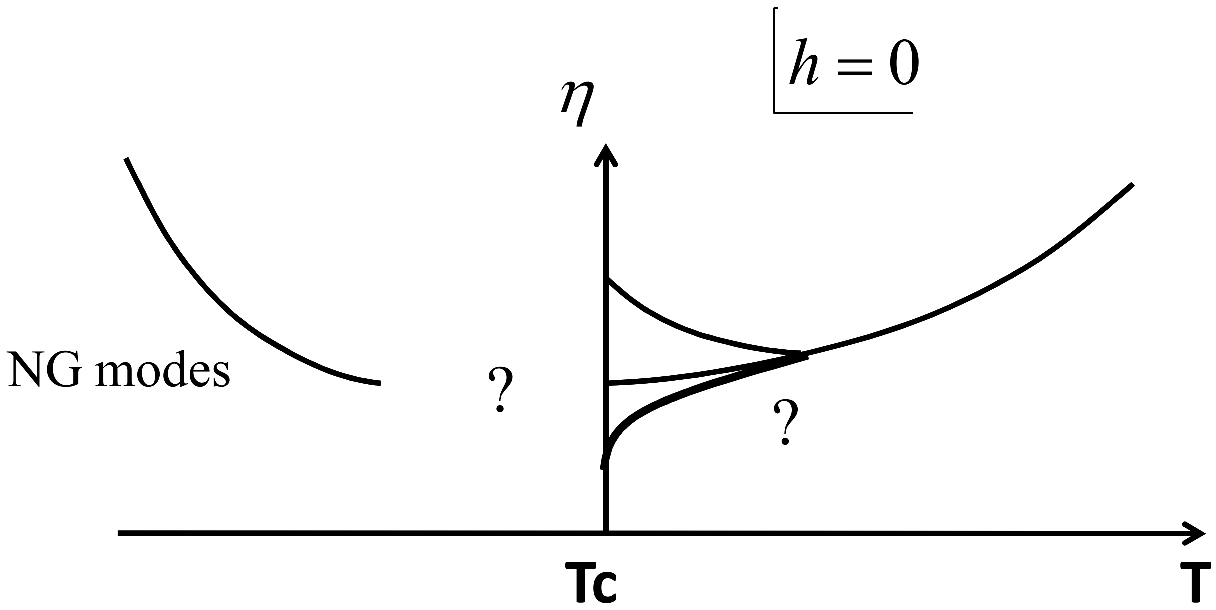
$$\Gamma_{l+1} = b^{z+\eta_E-4}\Gamma_l \left[1 + \left.\frac{\Sigma_E(k,\lambda_l,\Gamma_l,\eta_l,g_l,\ldots;\Lambda,b)}{\Gamma_l\vec{k}^2}\right|_{k\to 0}\right]$$

$$\eta_{l+1} = b^{z-2}\eta_l \left[1 + \left.\frac{\Sigma_J(k,\lambda_l,\Gamma_l,\eta_l,g_l,\ldots;\Lambda,b)}{\eta_l\vec{k}^2}\right|_{k\to 0}\right]$$

$$g_{l+1} = b^{z-3+\epsilon/2}g_l$$

Characteristic frequency scale (relaxation / diffusion) at critical point:

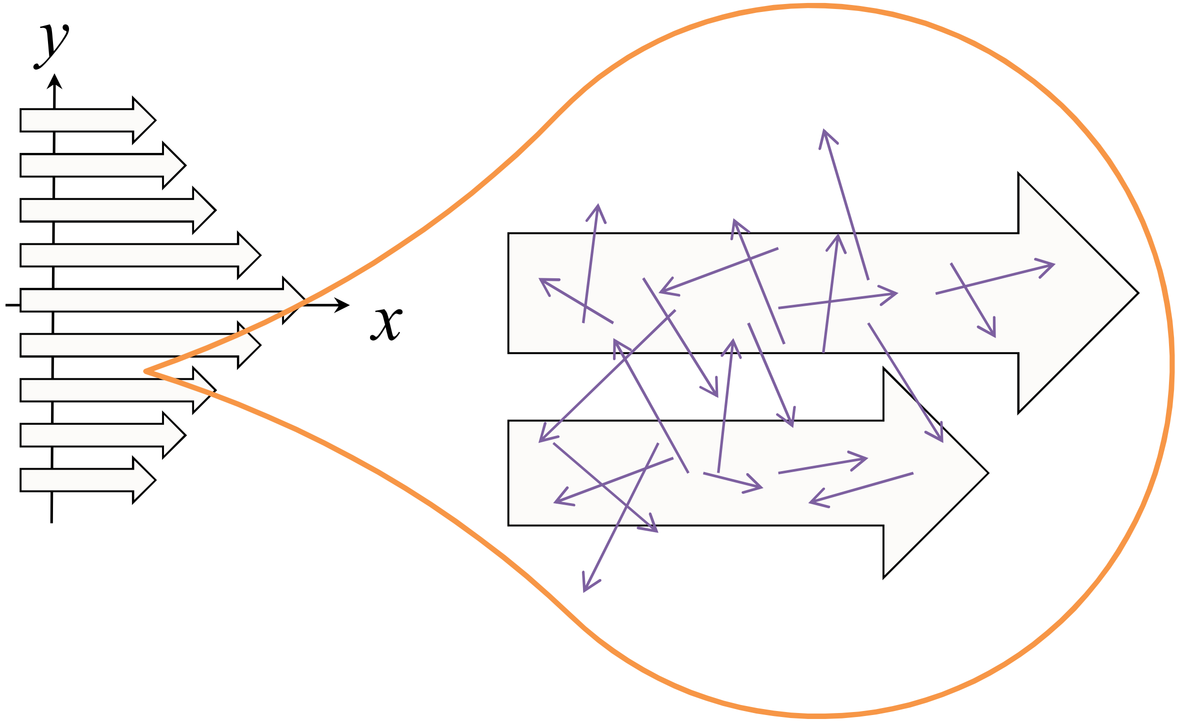
$$\omega \to k^Z \sim \xi^{-Z}$$ Z: Dynamical exponent



In summary, critical dynamics might be different upon how to approach to the critical point. In case of h=0, Shear viscosity and other transport coefficients are all finite. But conventional dynamical RG can not tell us their quantitative critical behaviors.

→ more elaborated methods

Snap shot of neighboring layers



$$\eta \propto \lambda_{mfp} \propto \frac{1}{\sigma_{scs}}$$