Combining effective field theories with dispersion relations

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Outline

Introduction

- Chiral perturbation theory and its limitations
- Testing the chiral anomaly

Dispersion relations . . .

- . . . for two pions: pion vector form factor
- . . . for three pions: $\gamma\pi \to \pi\pi,\ \omega/\phi \to 3\pi$
- From hadronic decays to transition form factors: $\omega/\phi \to \pi^0\gamma^*$

Towards the π^0 transition form factor

Summary / Outlook

Light mesons without modeling

Chiral perturbation theory (ChPT) . . .

- Effective field theory: simultaneous expansion in

  quark masses + small momenta

  - ▷ systematically improvable
  - ▷ well-established link to QCD: all symmetry constraints
  - ▷ interrelates many different observables

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. . . and its limitations

- strong final-state interactions render corrections large
- physics of light pseudoscalars (π, K, η) only
  - ▷ (energy) range limited by resonances: $\sigma(500)$, $\rho(770)$ . . .
  - ▷ unitarity is only perturbatively fulfilled
  - ▷ not applicable to decays of (e.g.) vector mesons at all

$\longrightarrow$ find effective ways to resum rescattering / restore unitarity

$\longrightarrow$ dispersion relations

Testing the Wess–Zumino–Witten chiral anomaly

- controls low-energy processes of odd intrinsic parity
- π^0 decay $\pi^0 \to \gamma\gamma$: $F_{\pi^0\gamma\gamma} = \dfrac{e^2}{4\pi^2 F_\pi}$

  F_π: pion decay constant $\longrightarrow$ measured at 1.5% level   PrimEx 2011

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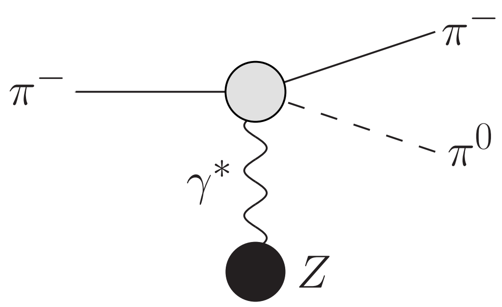
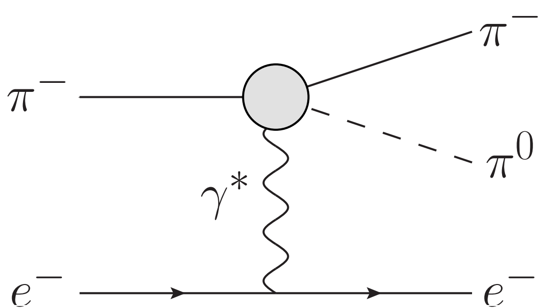
Primakoff reaction



$F_{3\pi} = (10.7 \pm 1.2)$ GeV^{-3}

Serpukhov 1987, Ametller et al. 2001

$\pi^- e^- \to \pi^- e^- \pi^0$



$F_{3\pi} = (9.6 \pm 1.1)$ GeV^{-3}

Giller et al. 2005

$\longrightarrow$ $F_{3\pi}$ tested only at 10% level

Chiral anomaly: Primakoff measurement

- previous analyses based on
  - data in threshold region only — Serpukhov 1987
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- Primakoff measurement of whole spectrum
  COMPASS, work in progress

- idea: use dispersion relations to exploit all data below 1 GeV for anomaly extraction

- effect of ρ resonance included model-independently via $\pi\pi$ P-wave phase shift

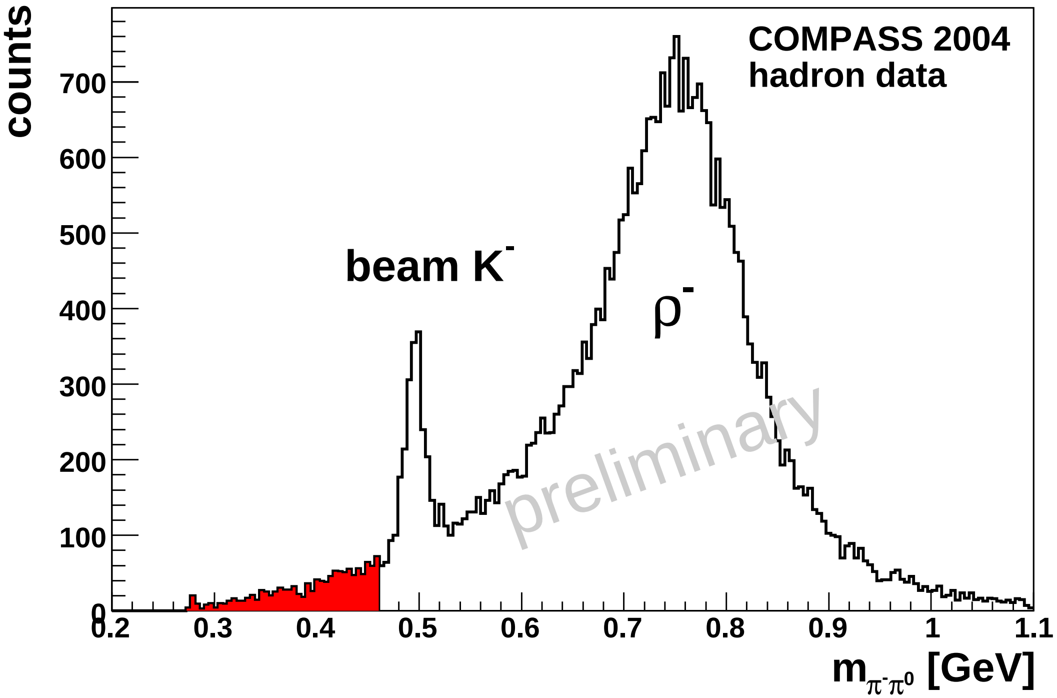
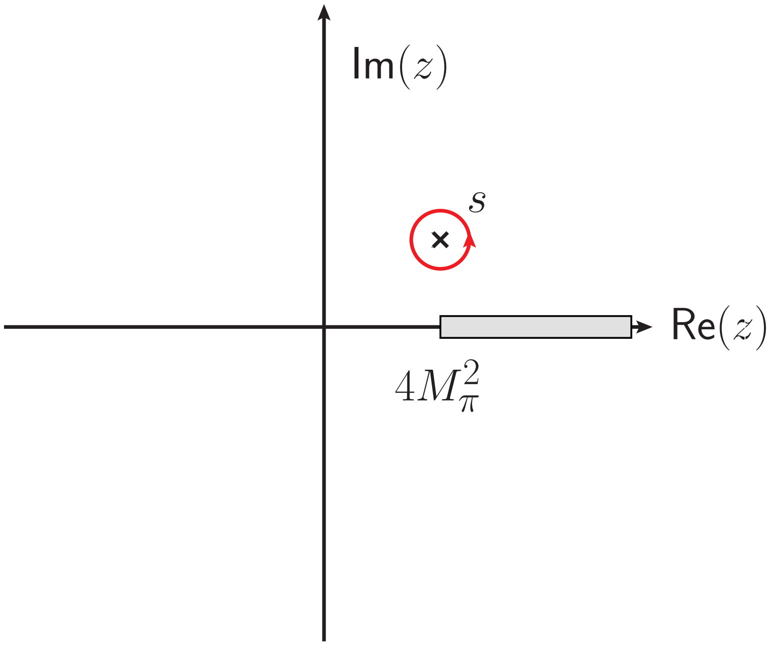


figure courtesy of T. Nagel 2009

Dispersion relations on one page



analyticity & Cauchy's theorem:

$$T(s) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{T(z)dz}{z-s}$$

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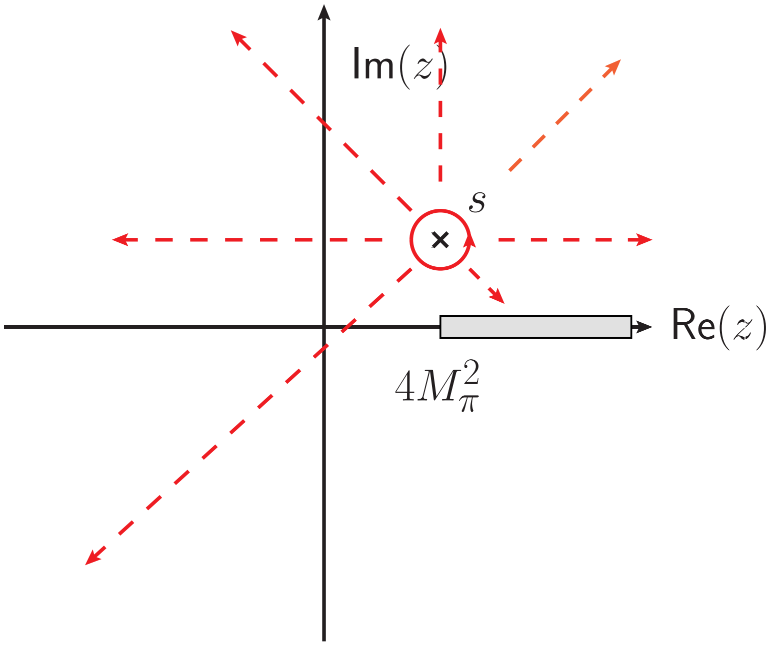

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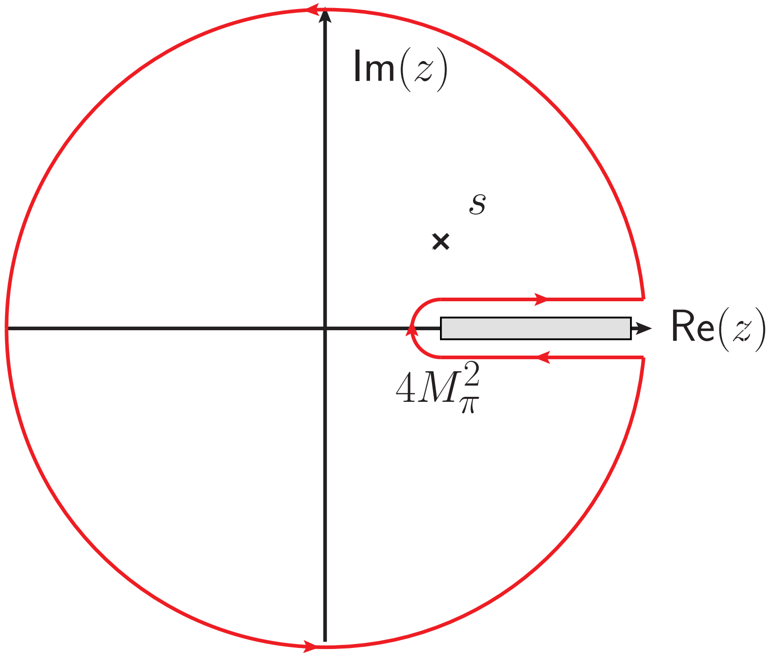

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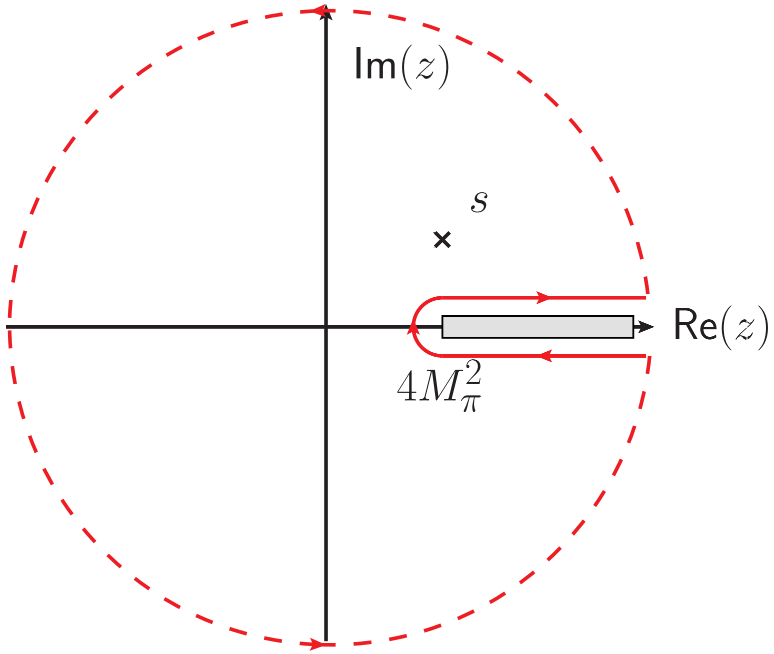

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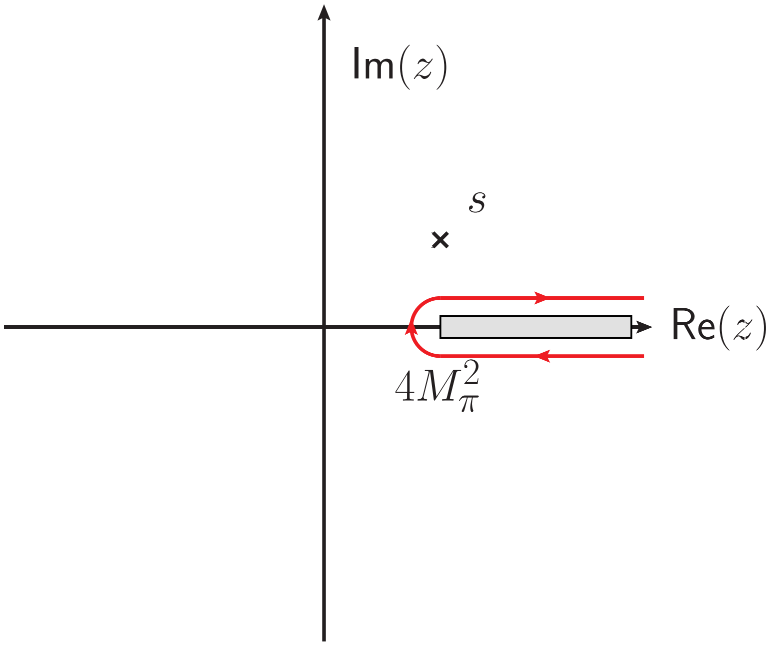

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$$T(s) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{T(z)dz}{z-s}$$

$$\longrightarrow \frac{1}{2\pi i} \int_{4M_\pi^2}^{\infty} \frac{\operatorname{disc} T(z)dz}{z-s}$$

$$= \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\operatorname{Im} T(z)dz}{z-s}$$

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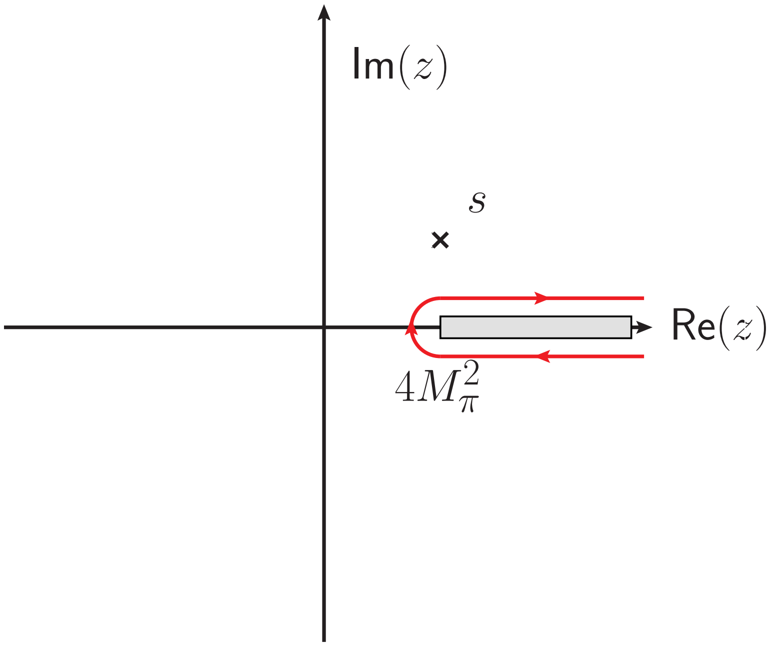

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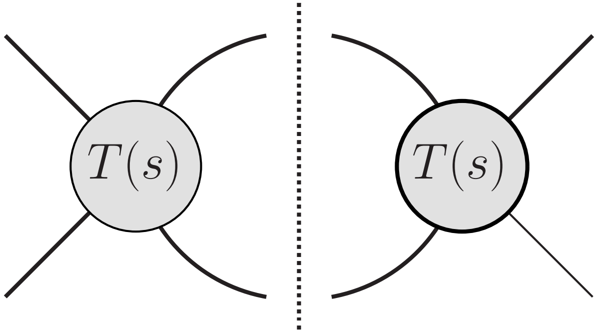
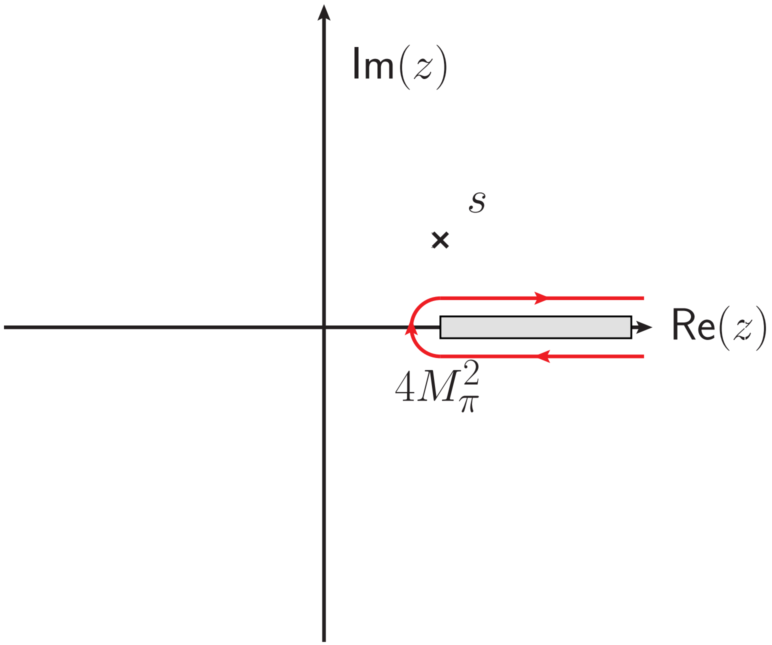


e.g. if $T(s)$ is a $\pi\pi$ partial wave $\longrightarrow$

$$\frac{\operatorname{disc} T(s)}{2i} = \operatorname{Im} T(s) = \frac{2q_\pi}{\sqrt{s}}\theta(s-4M_\pi^2)|T(s)|^2$$

Dispersion relations on one page



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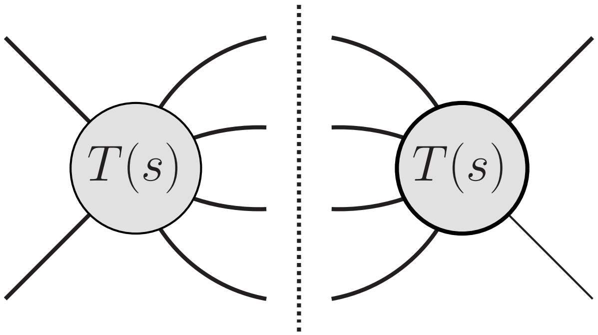
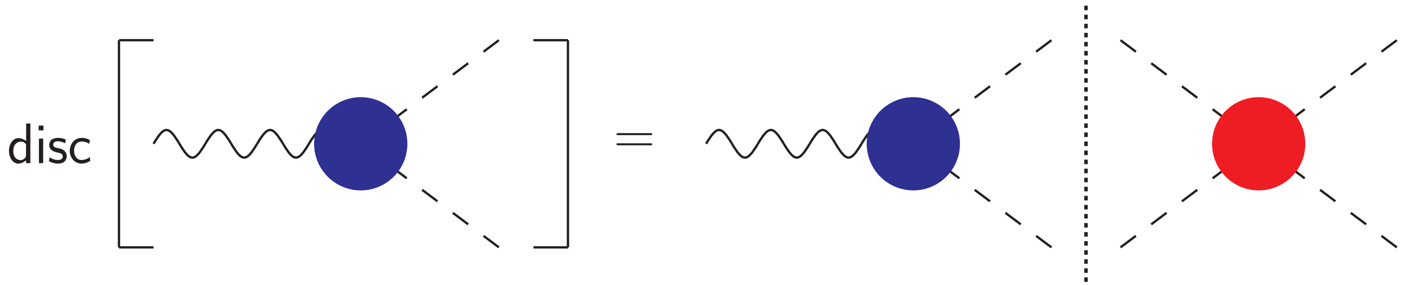


inelastic intermediate states ($K\bar{K}$, 4π)
suppressed at low energies
$\longrightarrow$ will be neglected in the following

Warm-up: pion form factor from dispersion relations

- just two particles in final state: form factor; from unitarity:



$$\frac{1}{2i}\,\text{disc}\,F_I(s) \;=\; \text{Im}\,F_I(s) \;=\; F_I(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_I(s)\,e^{-i\delta_I(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_I(s)$ is just $\delta_I(s)$ Watson 1954

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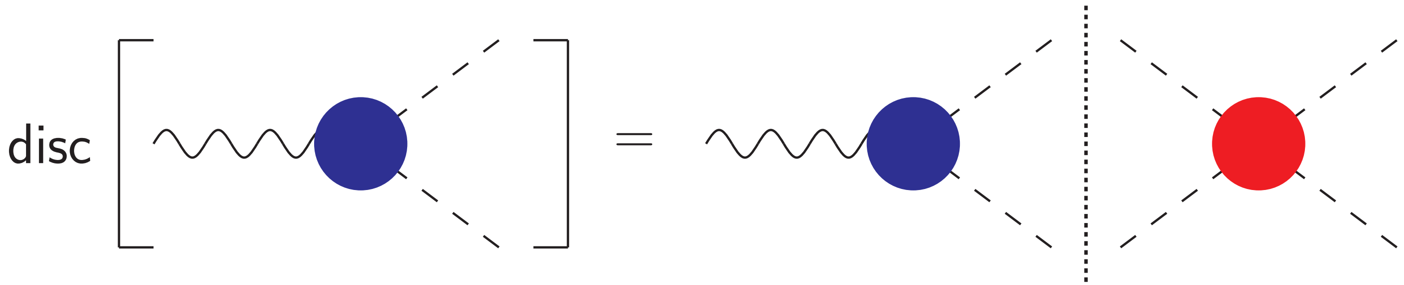

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- solution to this homogeneous integral equation known:

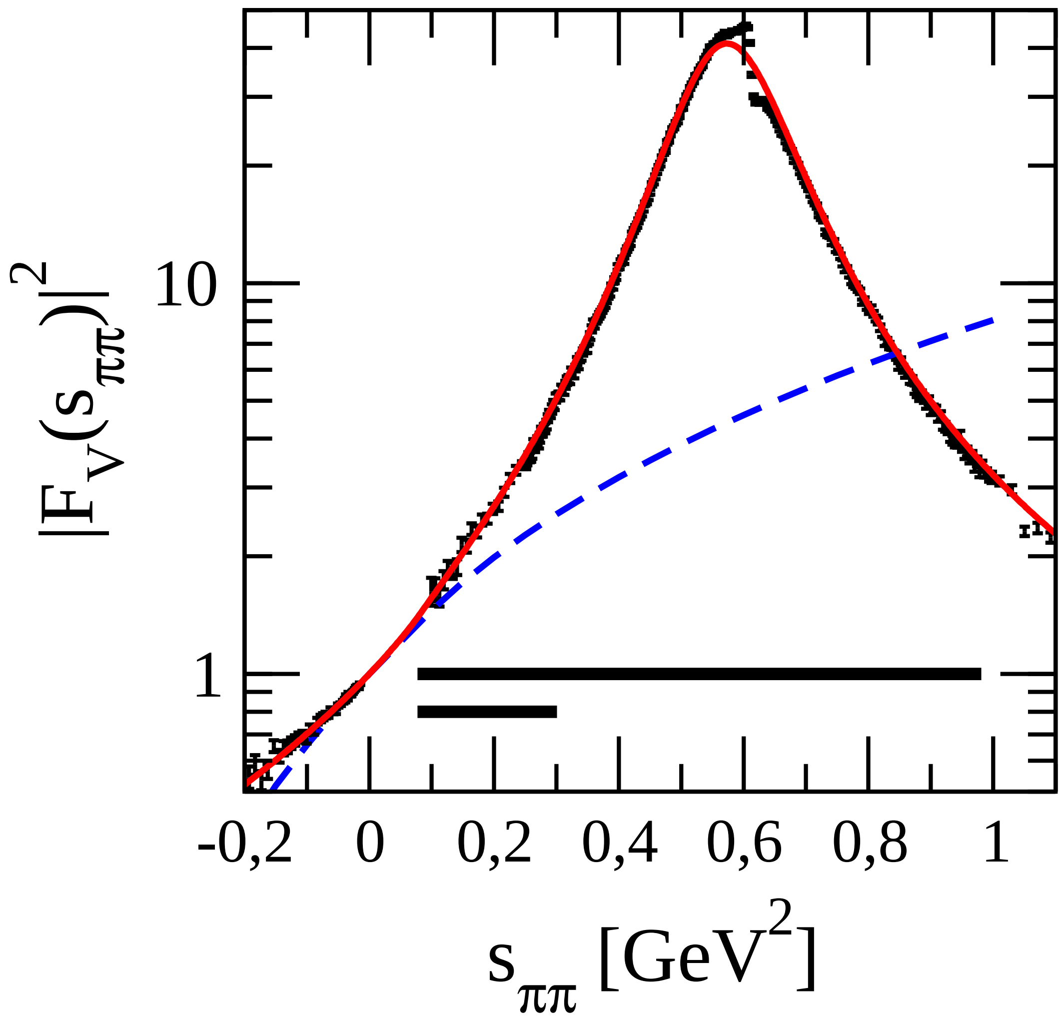
$$F_I(s) = P_I(s)\Omega_I(s)\,, \quad \Omega_I(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds'\,\frac{\delta_I(s')}{s'(s'-s)}\right\}$$

$P_I(s)$ polynomial, $\Omega_I(s)$ Omnès function Omnès 1958

- today: high-accuracy $\pi\pi$ phase shifts available
  Ananthanarayan et al. 2001, García-Martín et al. 2011
- constrain $P_I(s)$ using symmetries (normalisation at $s=0$ etc.)

Pion vector form factor from dispersion relations

- pion vector form factor clearly non-perturbative: ρ resonance



ChPT at one loop

data on $e^+e^- \to \pi^+\pi^-$

Omnès representation

Stollenwerk et al. 2012

$\longrightarrow$ Omnès representation vastly extends range of applicability

Dispersion relations for 3 pions

- $\gamma\pi \to \pi\pi$ particularly simple system: odd partial waves
  $\longrightarrow$ P-wave interactions only (neglecting F- and higher)
- decay amplitude decomposed into single-variable functions

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta} n^\mu p^\nu_{\pi^+} p^\alpha_{\pi^-} p^\beta_{\pi^0}\, \mathcal{F}(s,t,u)$$

$$\mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$$

Unitarity relation for $\mathcal{F}(s)$:

$$\text{disc}\,\mathcal{F}(s) = 2i\{\underbrace{\mathcal{F}(s)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{left-hand cut}}\} \times \theta(s - 4M_\pi^2) \times \sin\delta_1^1(s)\, e^{-i\delta_1^1(s)}$$

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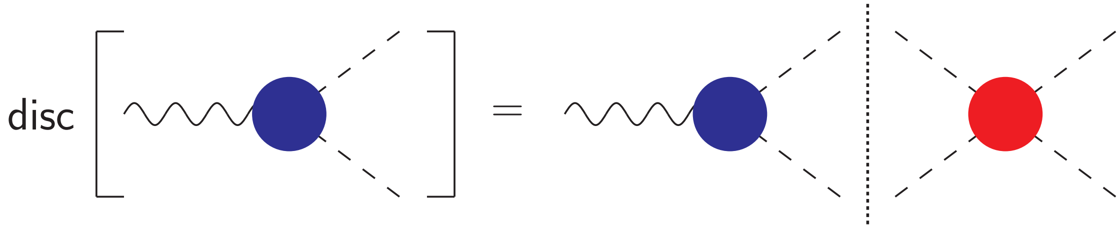
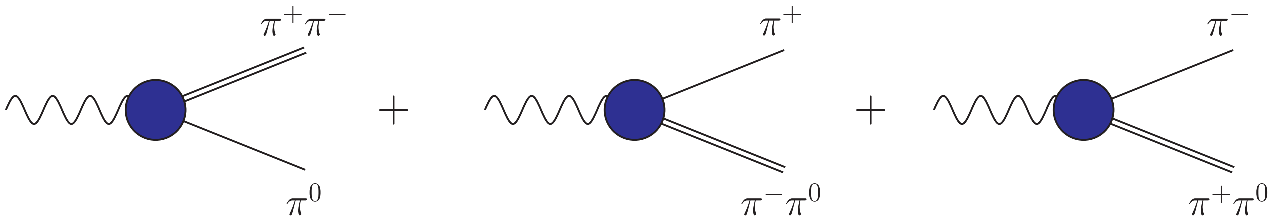


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s) = P(s)\,\Omega(s)\;, \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

$\mathcal{F}(s,t,u) =$



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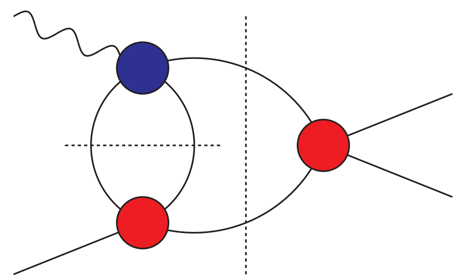

- inhomogeneities $\hat{\mathcal{F}}(s)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s) = \Omega(s)\left\{\frac{C_2^{(1)}}{3}\big(1 - \dot{\Omega}(0)s\big) + \frac{C_2^{(2)}}{3}s + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right\}$$

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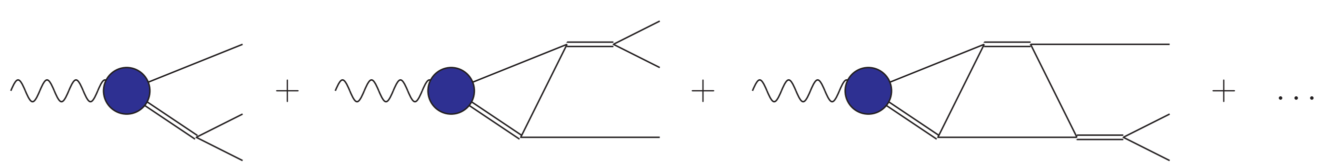

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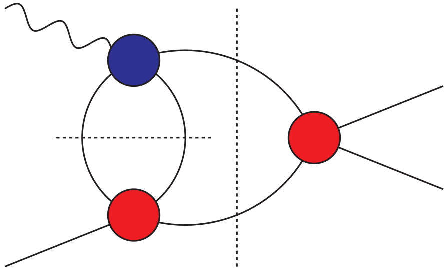

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$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z)\big)$$

- admits crossed-channel scattering between s-, t-, and u-channel (left-hand cuts)

Omnès solution for $\gamma\pi \to \pi\pi$

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- important observation: $\mathcal{F}(s)$ linear in $C_2^{(i)}$

$$\mathcal{F}(s) = C_2^{(1)}\mathcal{F}^{(1)}(s) + C_2^{(2)}\mathcal{F}^{(2)}(s)$$

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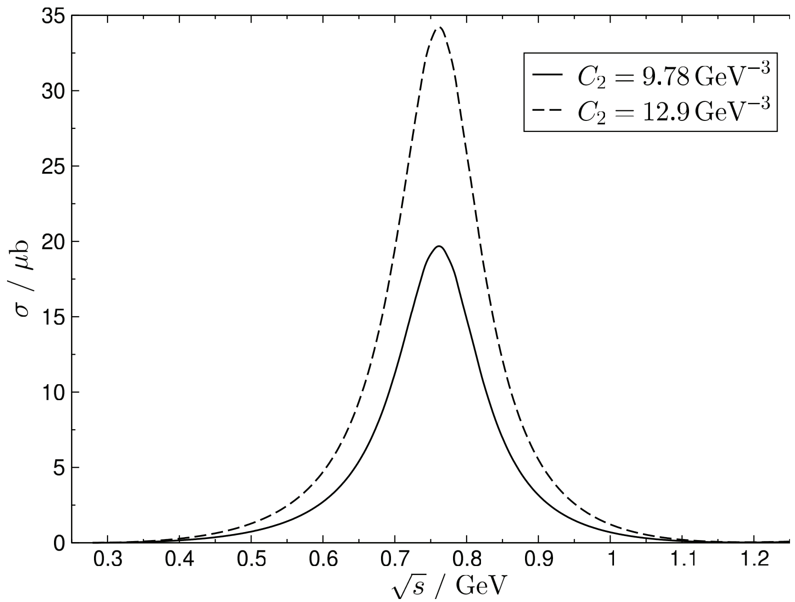
- representation of cross section in terms of two parameters

  $\longrightarrow$ fit to data, extract

$$F_{3\pi} \simeq C_2 = C_2^{(1)} + C_2^{(2)}M_\pi^2$$

$\longrightarrow \sigma \propto (C_2)^2$ also in ρ region



Hoferichter, BK, Sakkas 2012

Extension to decays: $\omega/\phi \to 3\pi$

- identical quantum numbers to $\gamma\pi \to \pi\pi$
- beyond ChPT: copious efforts to develop EFT for vector mesons

  Bijnens et al.; Bruns, Meißner; Lutz, Leupold; Gegelia et al.; Kampf et al.. . .
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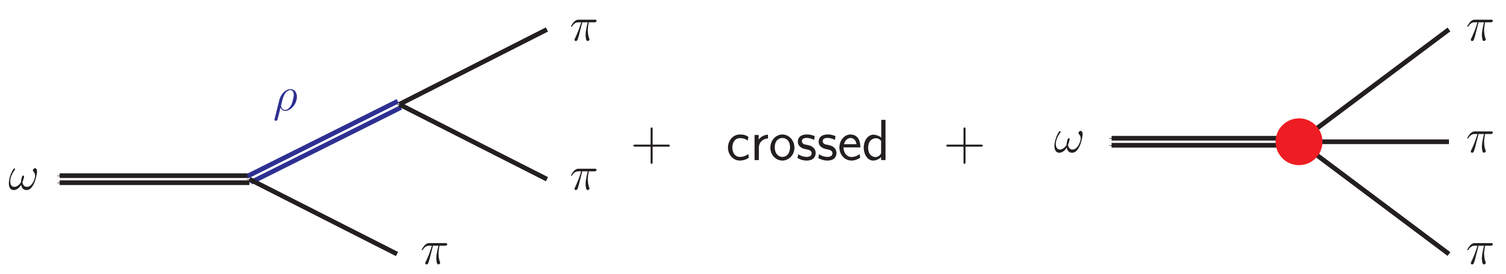
  sum of 3 Breit–Wigners (ρ^+, ρ^-, ρ^0)

  \+ constant background term



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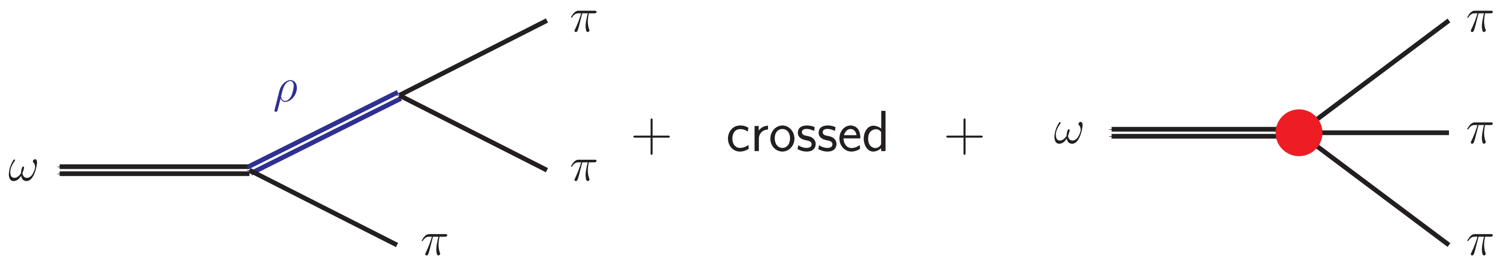


Problem:

$\longrightarrow$ unitarity fixes Im/Re parts

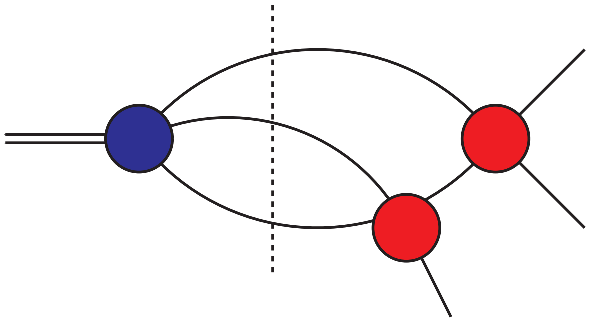
$\longrightarrow$ adding a contact term destroys this relation

$\longrightarrow$ reconcile data with dispersion relations?

$\omega/\phi \to 3\pi$: dispersive solution, Dalitz plots

$$\mathcal{F}(s) = a\,\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s-i\epsilon)}\right\}$$

- fix a to $\omega/\phi \to 3\pi$ partial width(s) $\longrightarrow$ Dalitz plots predicted
- analytic structure of $\hat{\mathcal{F}}(s)$ complicated by 3-particle cuts



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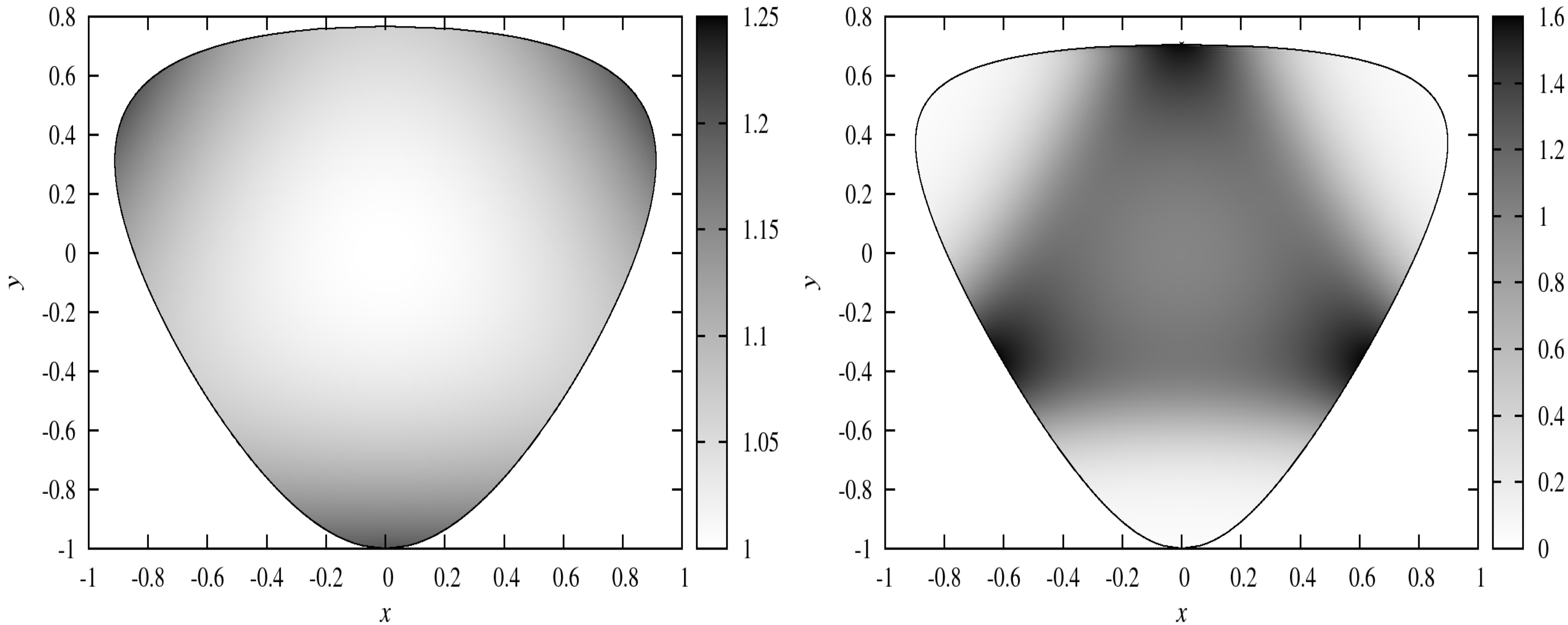
$\omega \to 3\pi$: $\phi \to 3\pi$:



Niecknig, BK, Schneider 2012

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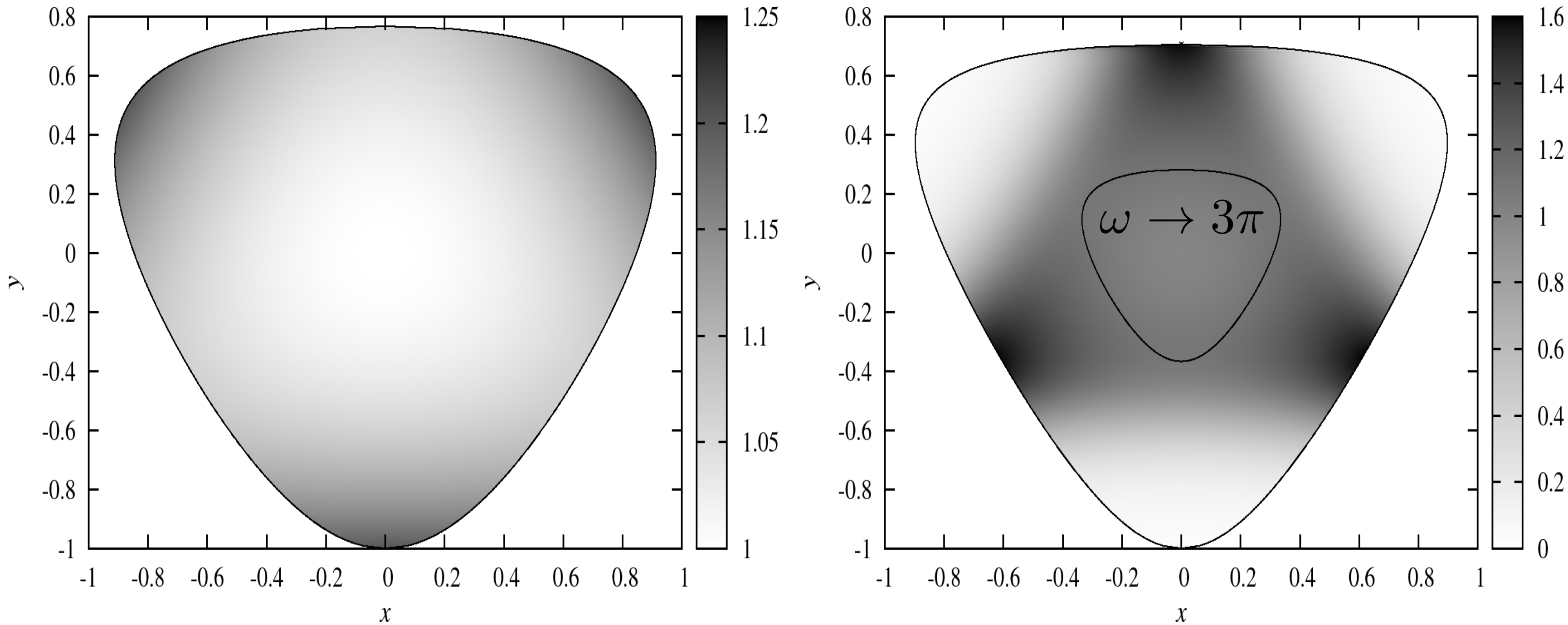
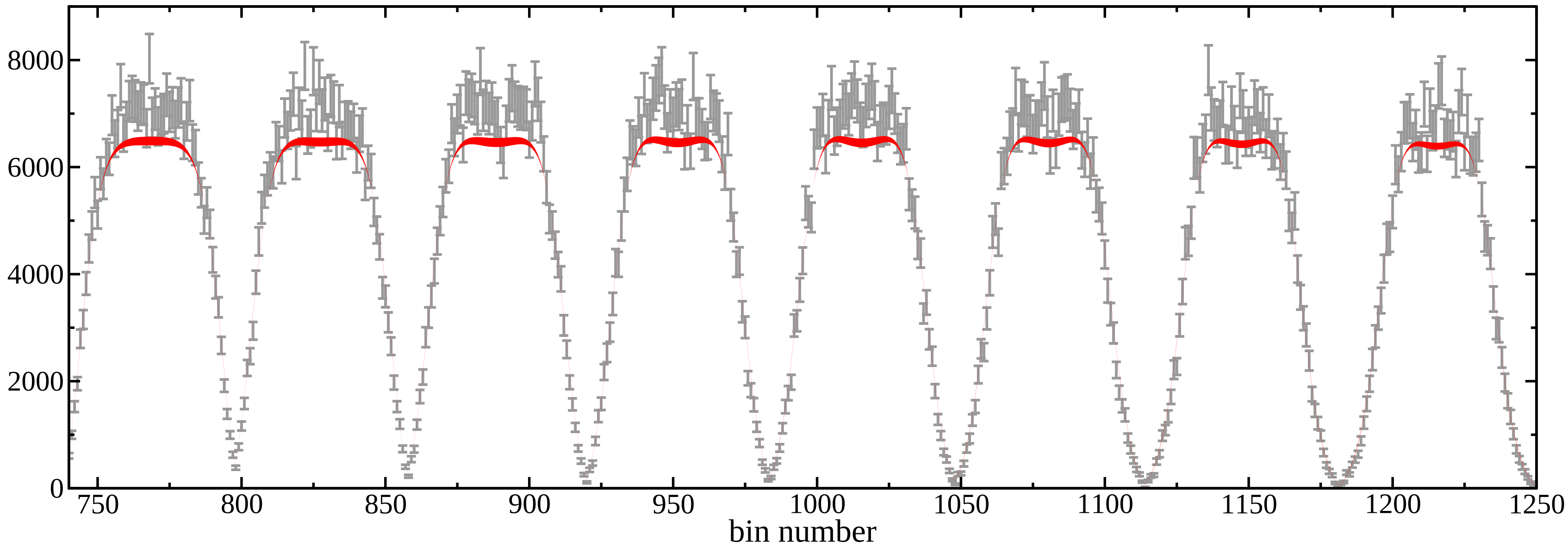
$\omega \to 3\pi$: $\qquad\qquad$ $\phi \to 3\pi$:



Niecknig, BK, Schneider 2012

Experimental comparison to $\phi \to 3\pi$

KLOE Dalitz plot: $2 \cdot 10^6$ events, 1834 bins   Niecknig, BK, Schneider 2012

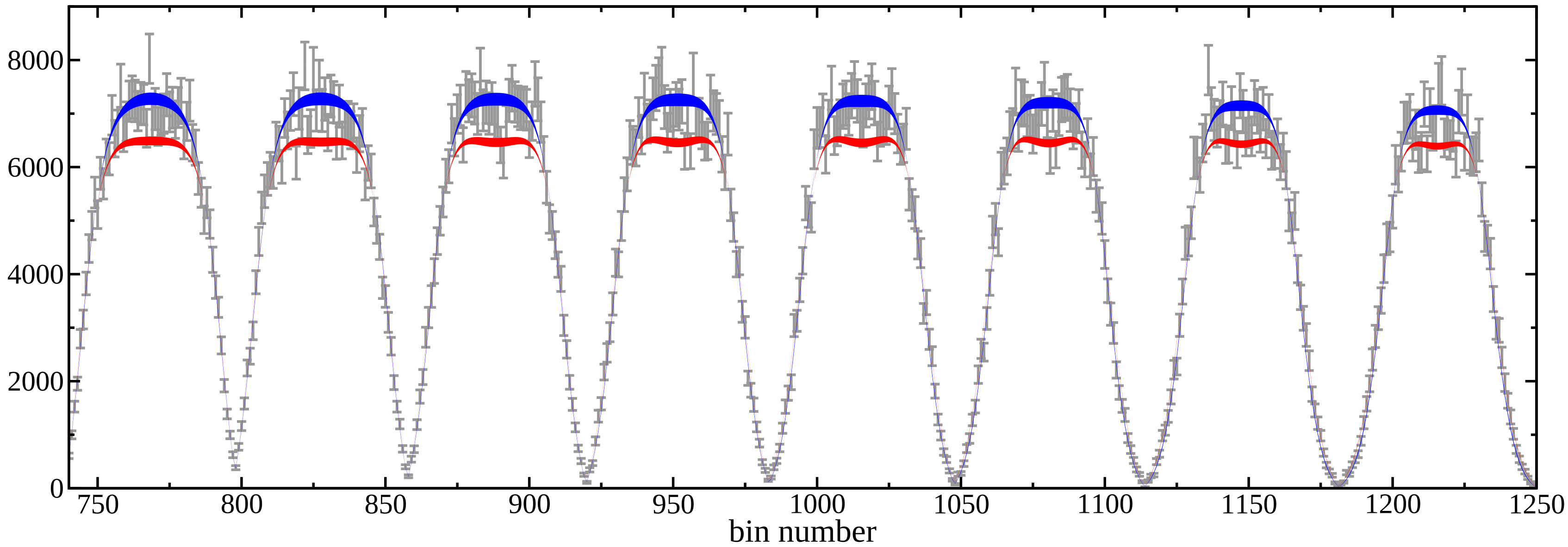


	$\hat{\mathcal{F}} = 0$
χ^2/ndof	$1.71 \ldots 2.06$

$$\mathcal{F}(s) = a\,\Omega(s) = a\,\exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right]$$

Experimental comparison to $\phi \to 3\pi$

KLOE Dalitz plot: $2 \cdot 10^6$ events, 1834 bins   Niecknig, BK, Schneider 2012



	$\hat{\mathcal{F}} = 0$	once-subtracted
χ^2/ndof	$1.71 \ldots 2.06$	$1.17 \ldots 1.50$

$$\mathcal{F}(s) = a\,\Omega(s)\left[1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\hat{\mathcal{F}}(s')\sin\delta_1^1(s')}{|\Omega(s')|(s'-s)}\right]$$

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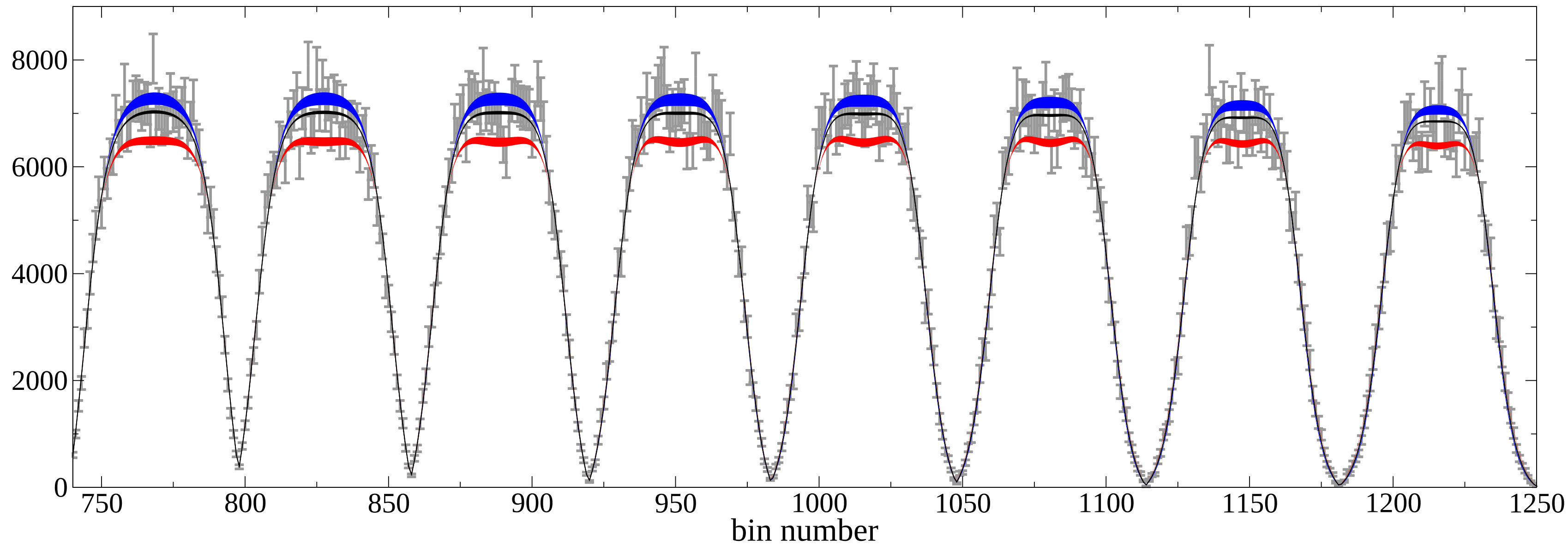

	$\hat{\mathcal{F}} = 0$	once-subtracted	twice-subtracted
χ^2/ndof	$1.71 \ldots 2.06$	$1.17 \ldots 1.50$	$1.02 \ldots 1.03$

$$\mathcal{F}(s) = a\,\Omega(s)\left[1 + b\,s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\hat{\mathcal{F}}(s')\sin\delta_1^1(s')}{|\Omega(s')|(s'-s)}\right]$$

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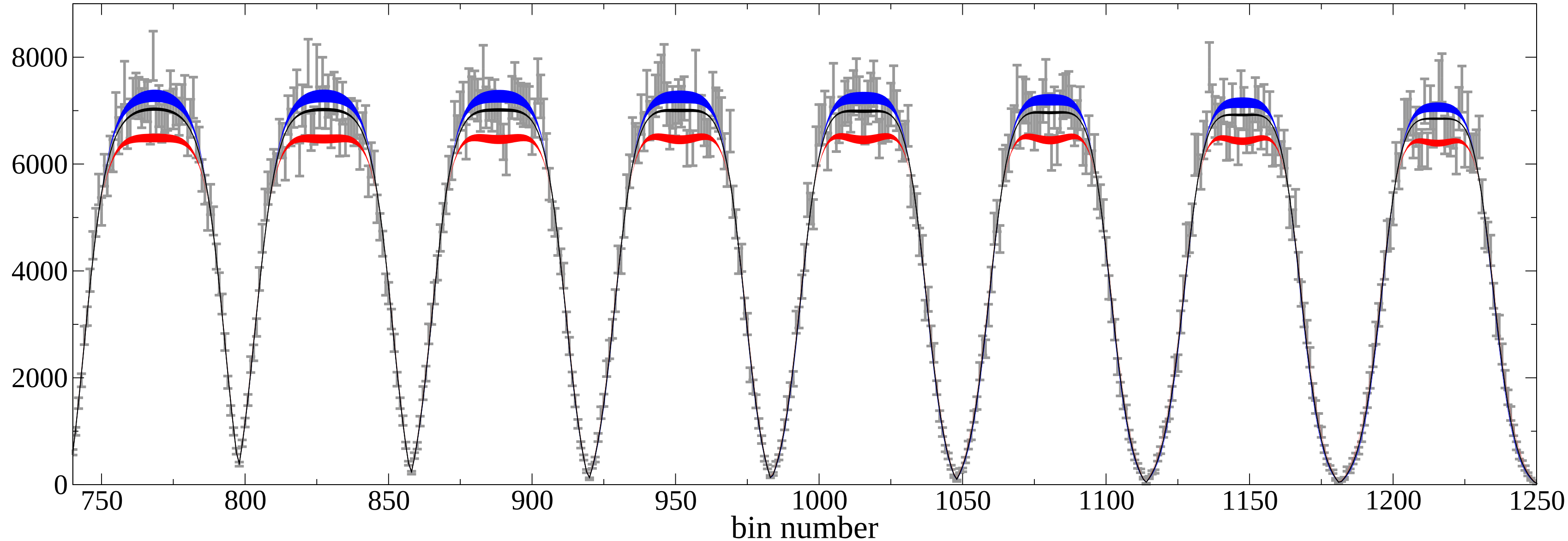


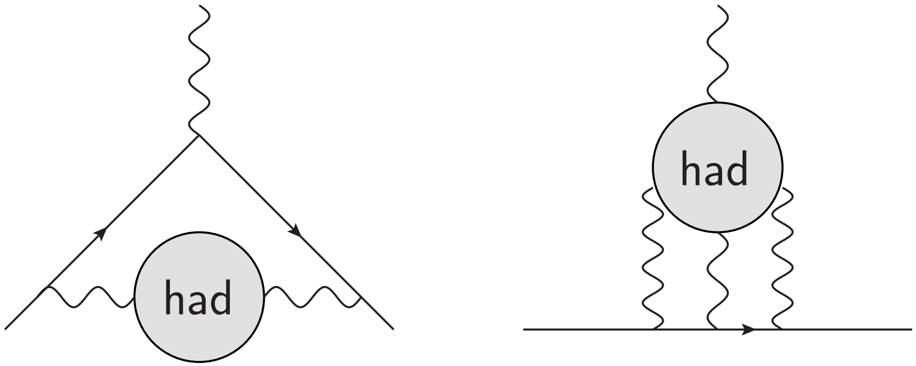
	$\hat{\mathcal{F}} = 0$	once-subtracted	twice-subtracted
χ^2/ndof	$1.71 \ldots 2.06$	$1.17 \ldots 1.50$	$1.02 \ldots 1.03$

- perfect fit respecting analyticity and unitarity possible
- contact term emulates neglected rescattering effects
- no need for "background" — inseparable from "resonance"

Meson transition form factors and $(g-2)_\mu$

Czerwiński et al., arXiv:1207.6556 [hep-ph]

- leading and next-to-leading hadronic effects in $(g-2)_\mu$:



$\longrightarrow$ hadronic light-by-light soon dominant uncertainty

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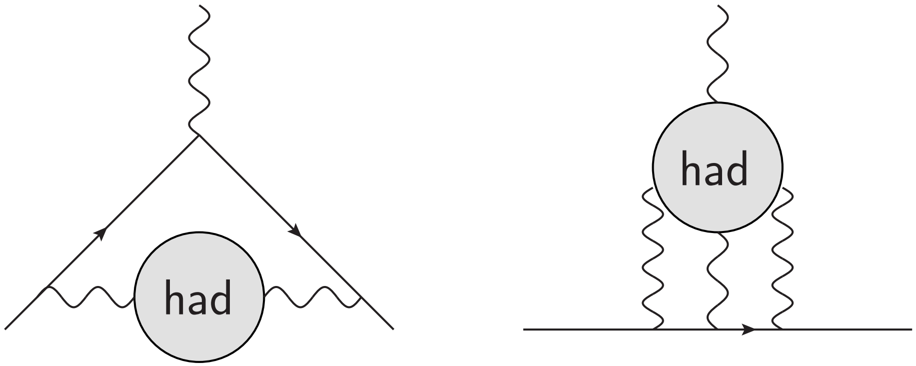
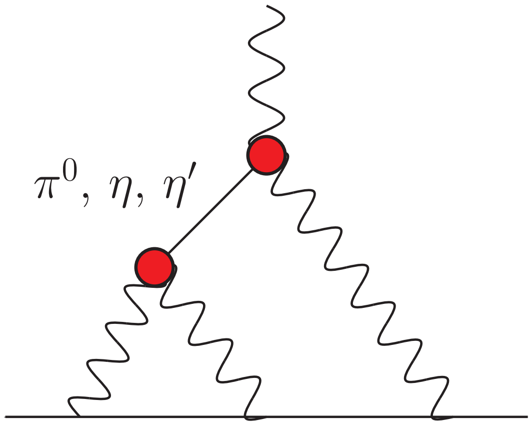


$\longrightarrow$ hadronic light-by-light soon dominant uncertainty

- important contribution: pseudoscalar pole terms

  singly / doubly virtual form factors

  $F_{P\gamma\gamma^*}(q^2,0)$ and $F_{P\gamma^*\gamma^*}(q_1^2,q_2^2)$



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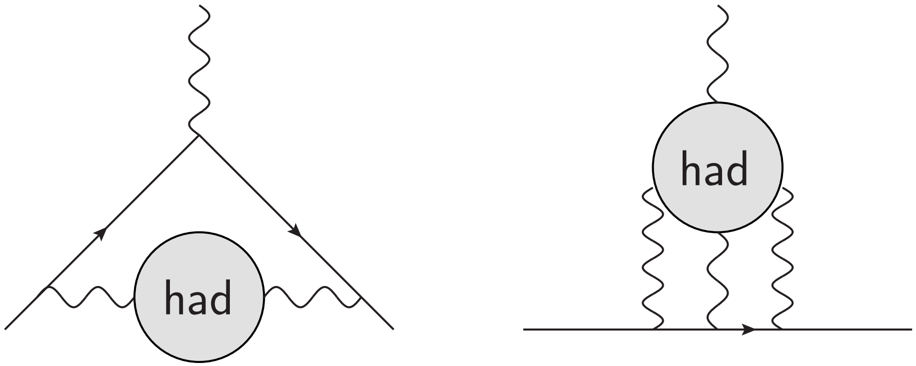

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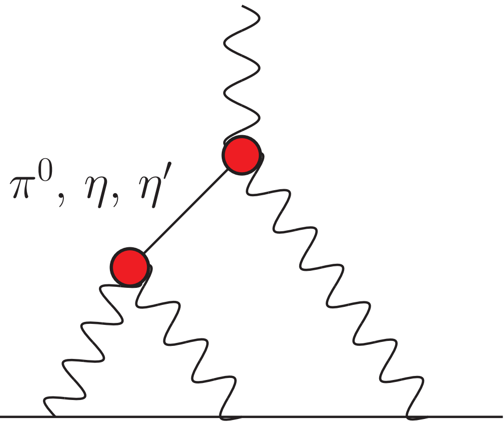
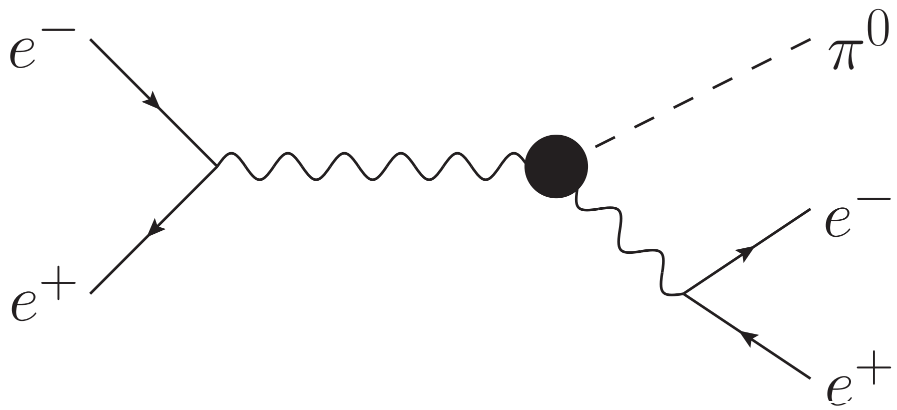


- for specific virtualities: linked to
  vector-meson conversion decays

$\longrightarrow$ e.g. $F_{\pi^0\gamma^*\gamma^*}(q_1^2, M_\omega^2)$ measurable in $\omega \to \pi^0\ell^+\ell^-$ etc.

Transition form factor $\omega \to \pi^0 \ell^+ \ell^-$

- $\pi^0 \to \gamma^* \gamma^*$ form factor linked to $\omega(\phi) \to \pi^0 \gamma^*$ transition:



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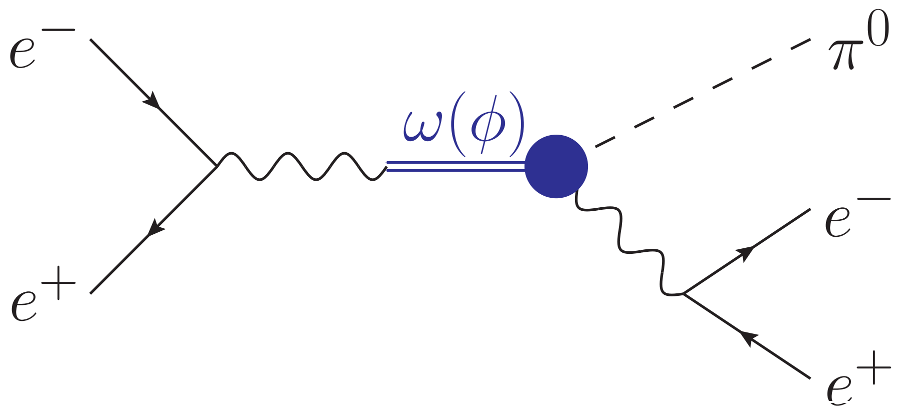

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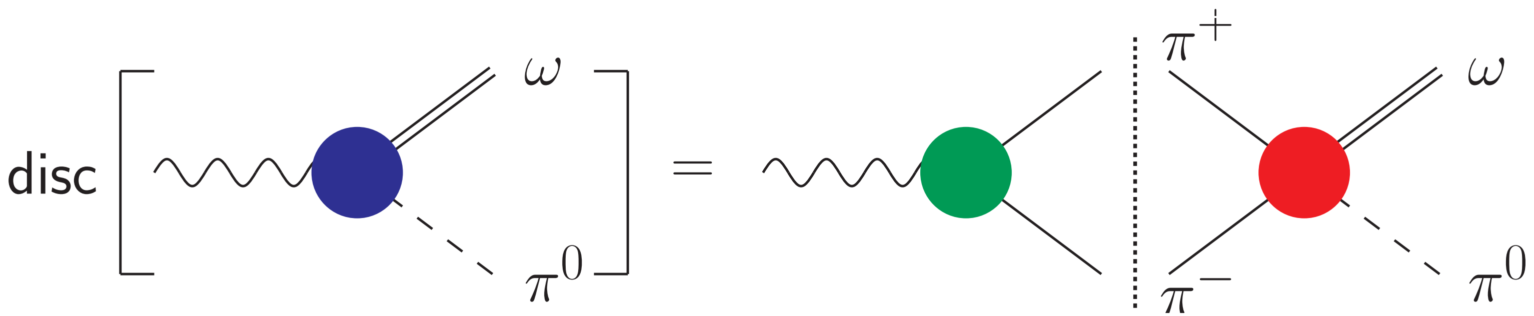


- ω transition form factor related to

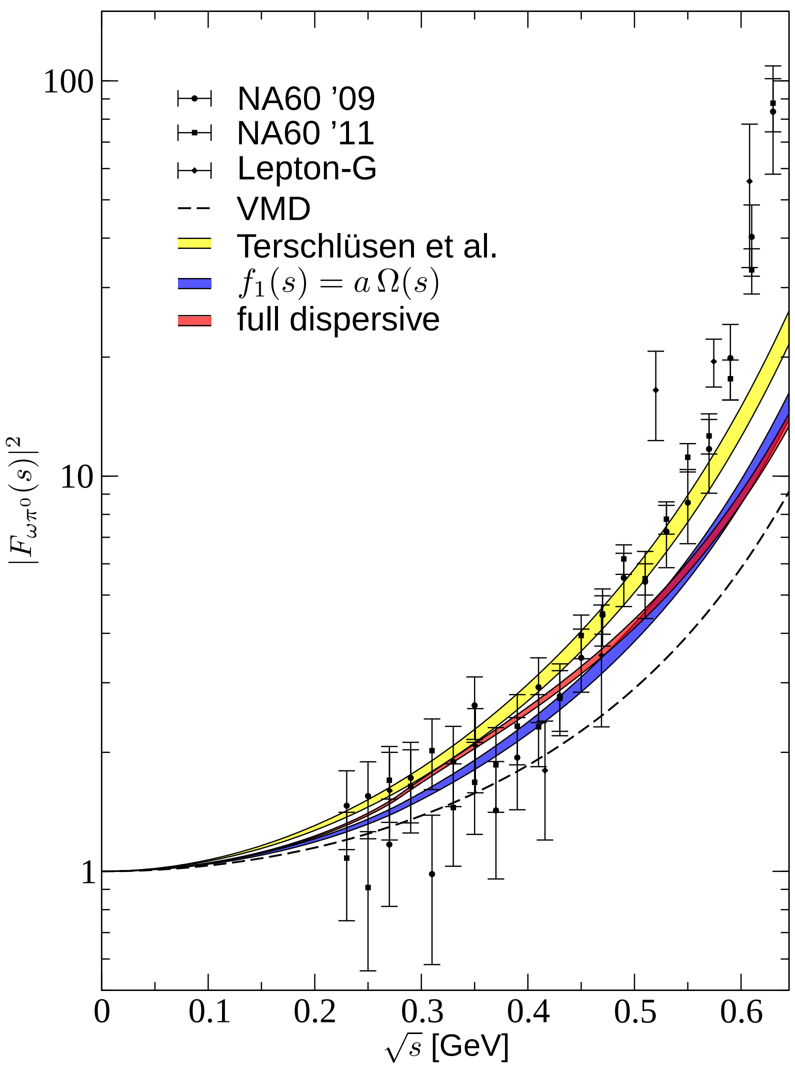
  pion vector form factor $\times$ $\underbrace{\omega \to 3\pi \text{ decay amplitude}}_{\text{calculate like } \phi \to 3\pi}$

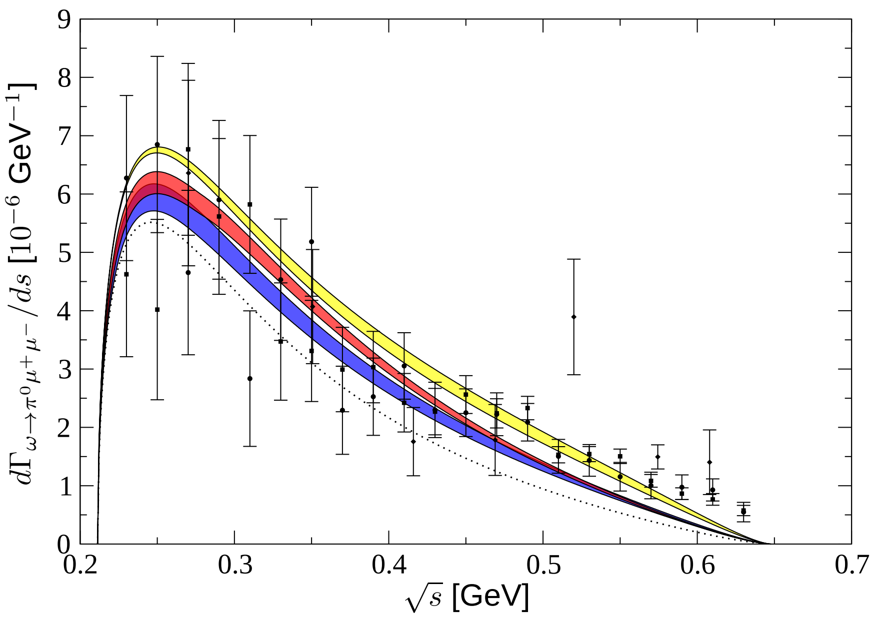
- form factor normalization yields rate $\Gamma(\omega \to \pi^0 \gamma)$

  (2nd most important ω decay channel)

  $\longrightarrow$ works at 95% accuracy   Schneider, BK, Niecknig 2012

Numerical results: $\omega \to \pi^0\mu^+\mu^-$

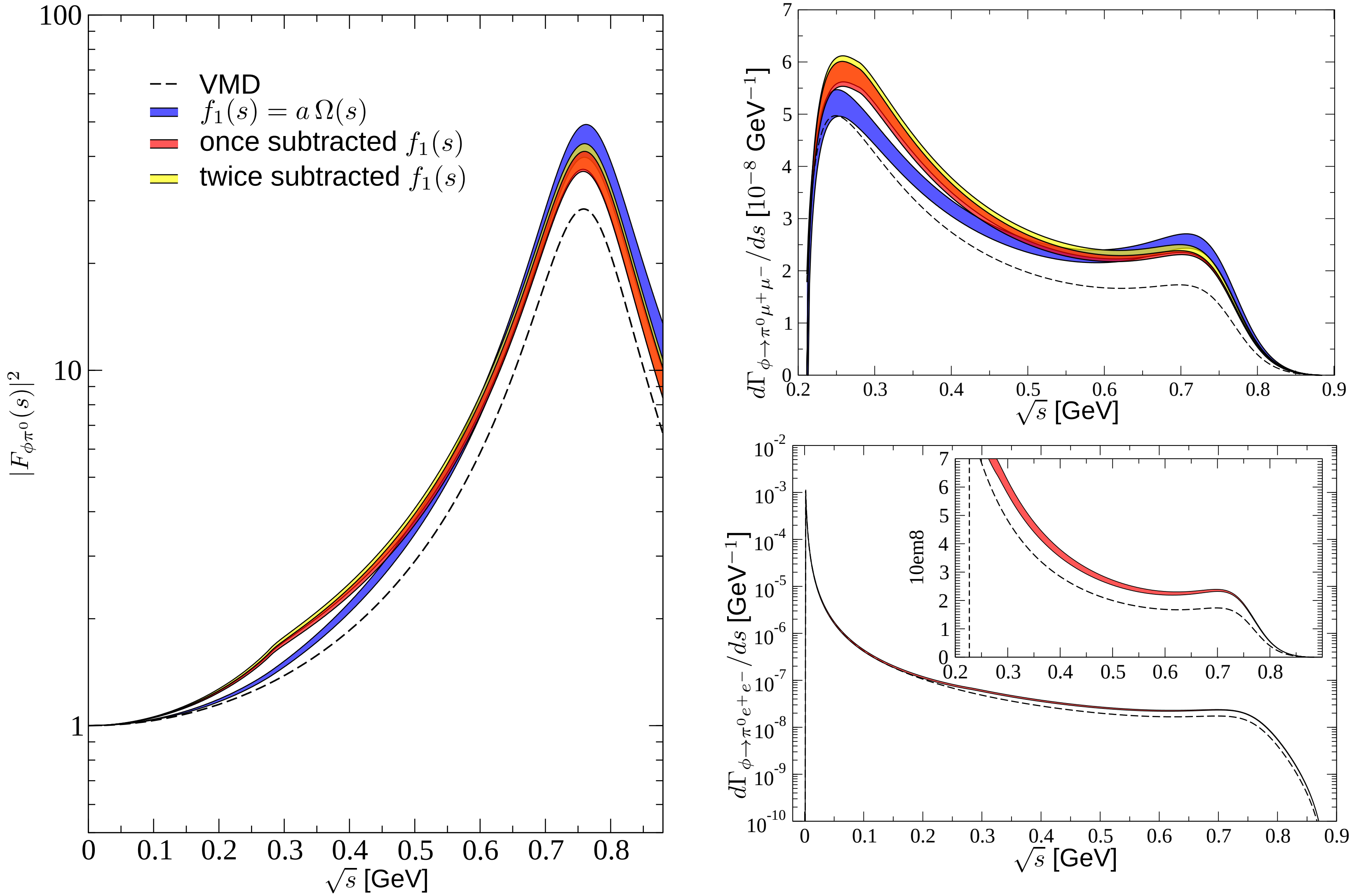




- unable to account for steep rise in data (from heavy-ion collisions) NA60 2009, 2011
- more "exclusive" data?! CLAS?
- $\omega \to 3\pi$ Dalitz plot?

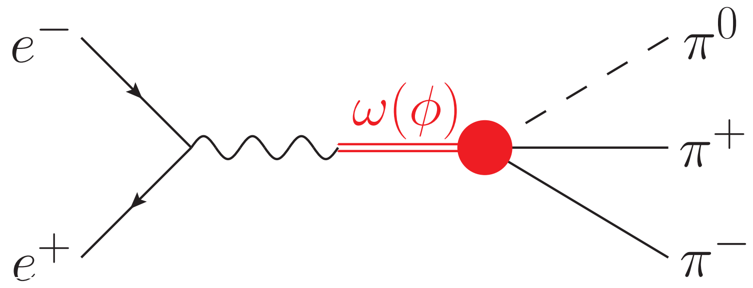
KLOE, WASA-at-COSY, CLAS?

Numerical results: $\phi \to \pi^0 \ell^+ \ell^-$



- measurement would be extremely helpful: ρ in physical region!
- partial-wave amplitude backed up by experiment

One step further: $e^+e^- \to 3\pi$, $e^+e^- \to \pi^0\gamma$



- decay amplitude for $\omega/\phi \to 3\pi$: $\mathcal{M}_{\omega/\phi} \propto \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$

$$\mathcal{F}(s) = a_{\omega/\phi}\,\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right\}$$

$a_{\omega/\phi}$ adjusted to reproduce total width $\omega/\phi \to 3\pi$

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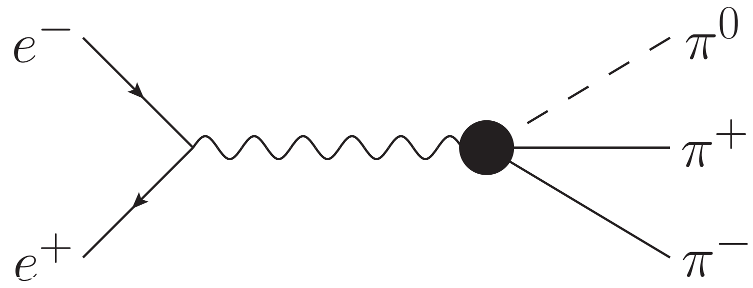

- decay amplitude for $e^+e^- \to 3\pi$: $\mathcal{M}_{e^+e^-} \propto \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$

$$\mathcal{F}(s,q^2) = a_{e^+e^-}(q^2)\,\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)}\right\}$$

$a_{e^+e^-}(q^2)$ adjusted to reproduce spectrum $e^+e^- \to 3\pi$

contains 3π resonances $\longrightarrow$ no dispersive prediction

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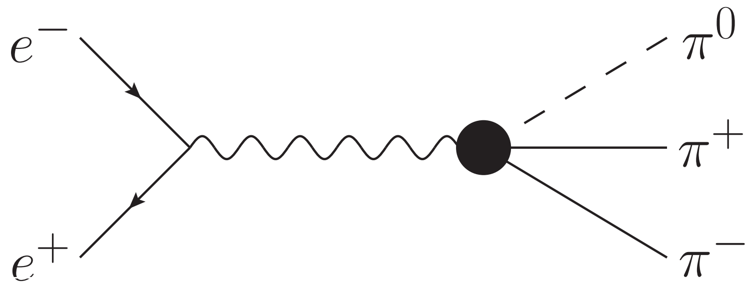

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Lomon, Pacetti 2012; Moussallam 2013

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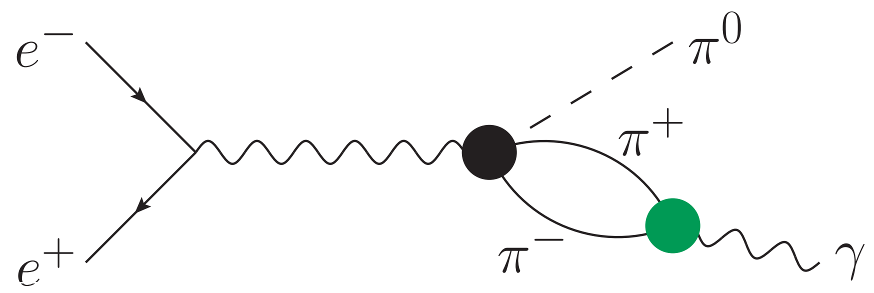

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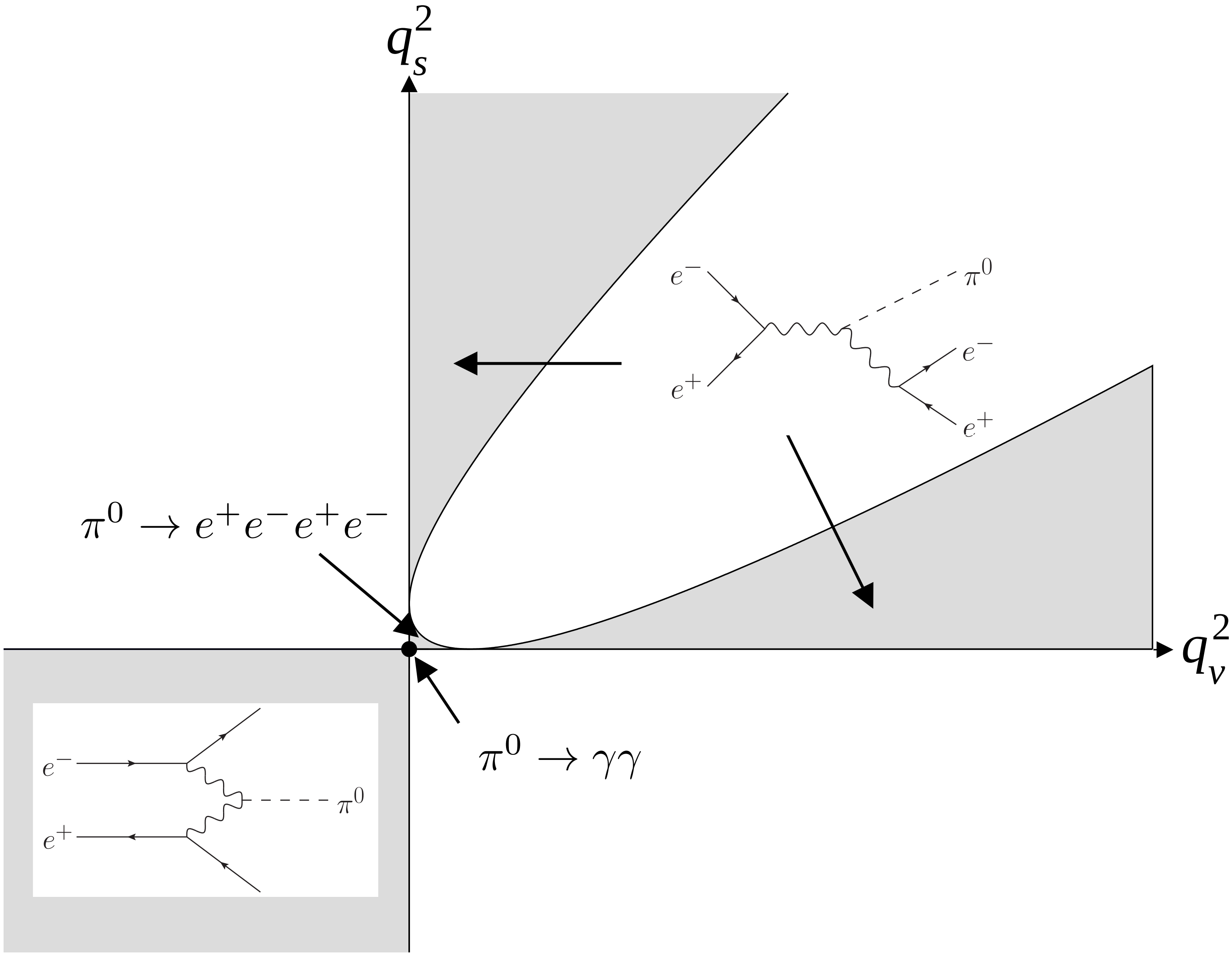
Lomon, Pacetti 2012; Moussallam 2013

- fit to $e^+e^- \to 3\pi$ data $\longrightarrow$ prediction for isoscalar $e^+e^- \to \pi^0\gamma$:

$$F_{\pi\gamma^*\gamma}(q^2,0) = F_{vs}(q^2,0) + F_{vs}(0,q^2)$$

$\pi^0 \to \gamma^*(q_v^2)\gamma^*(q_s^2)$ transition form factor



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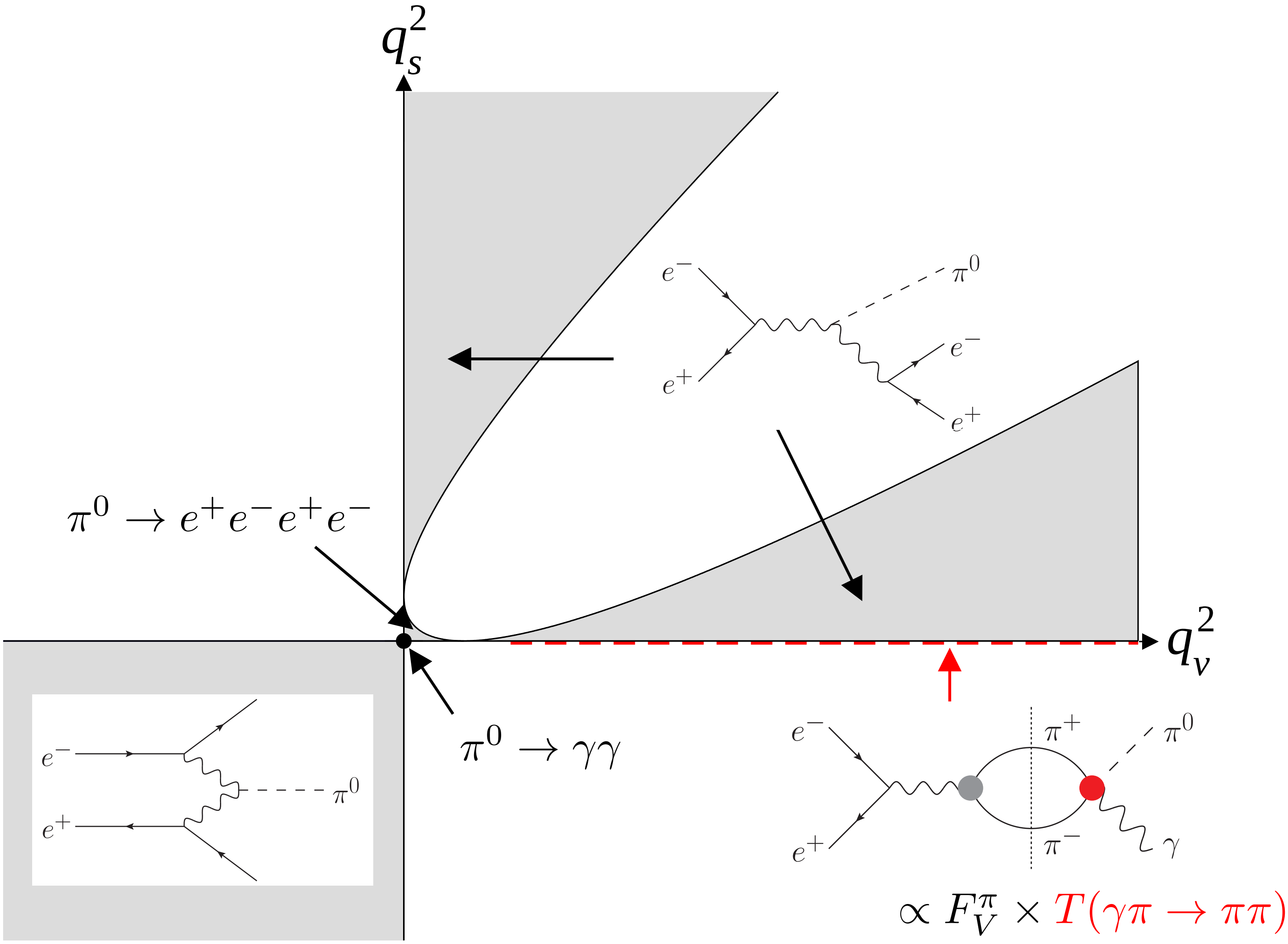

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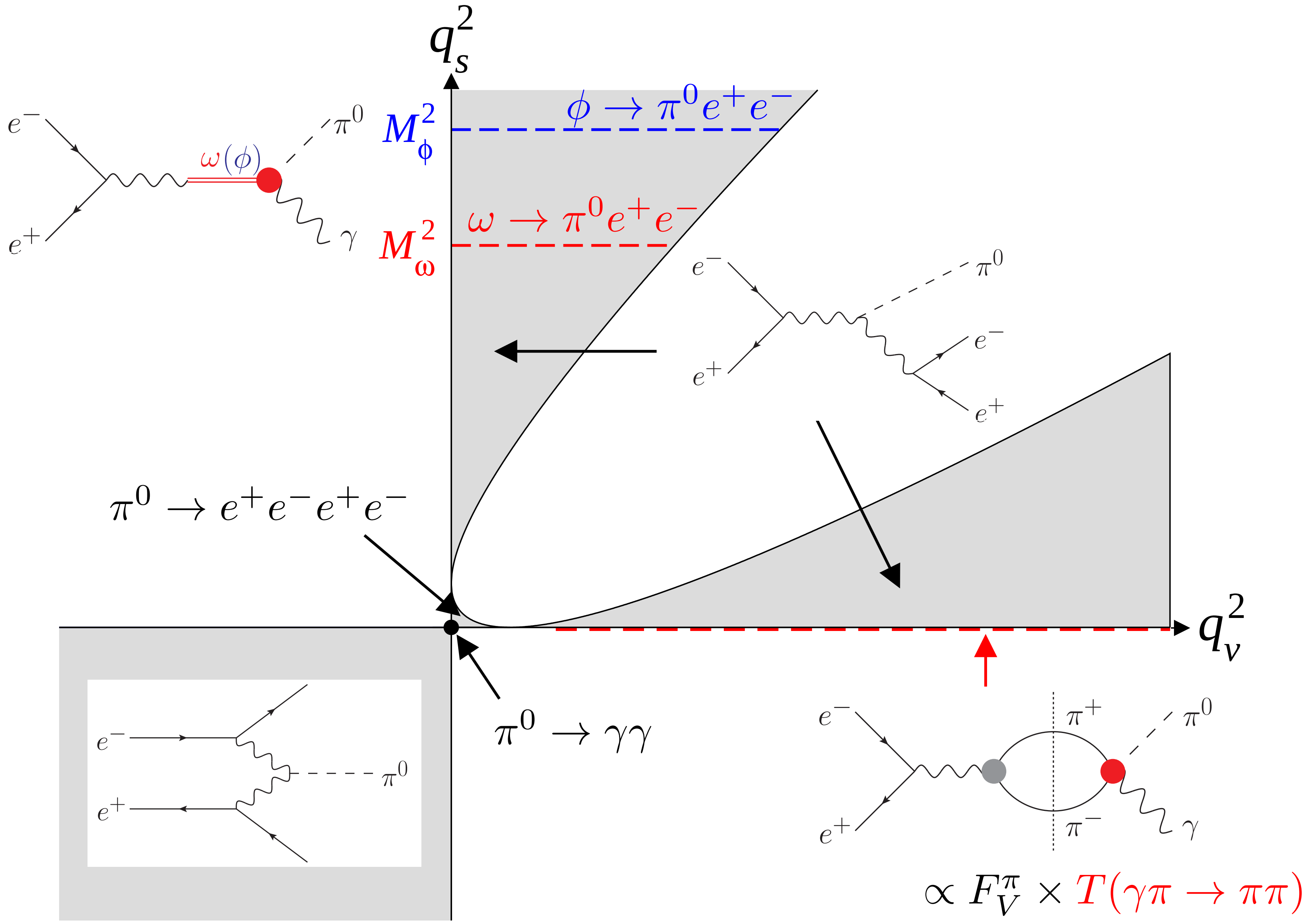
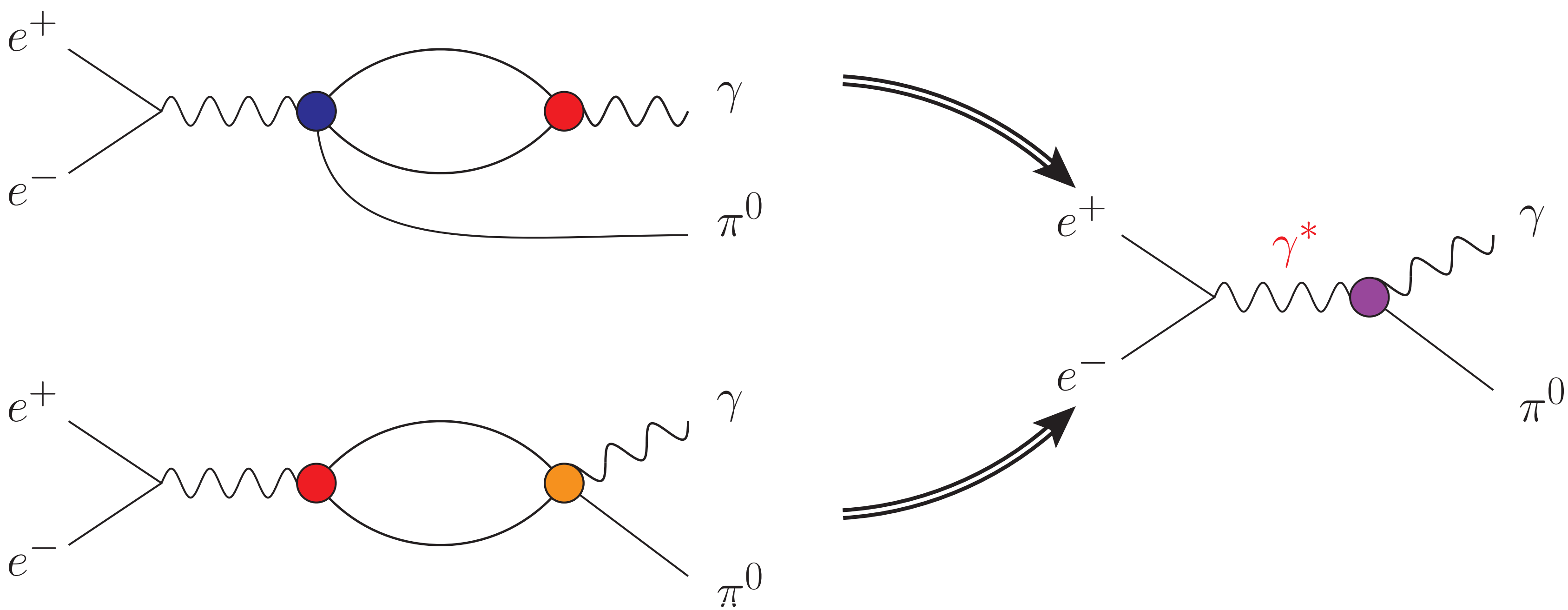


Towards a dispersive analysis of $e^+e^- \to \pi^0\gamma$

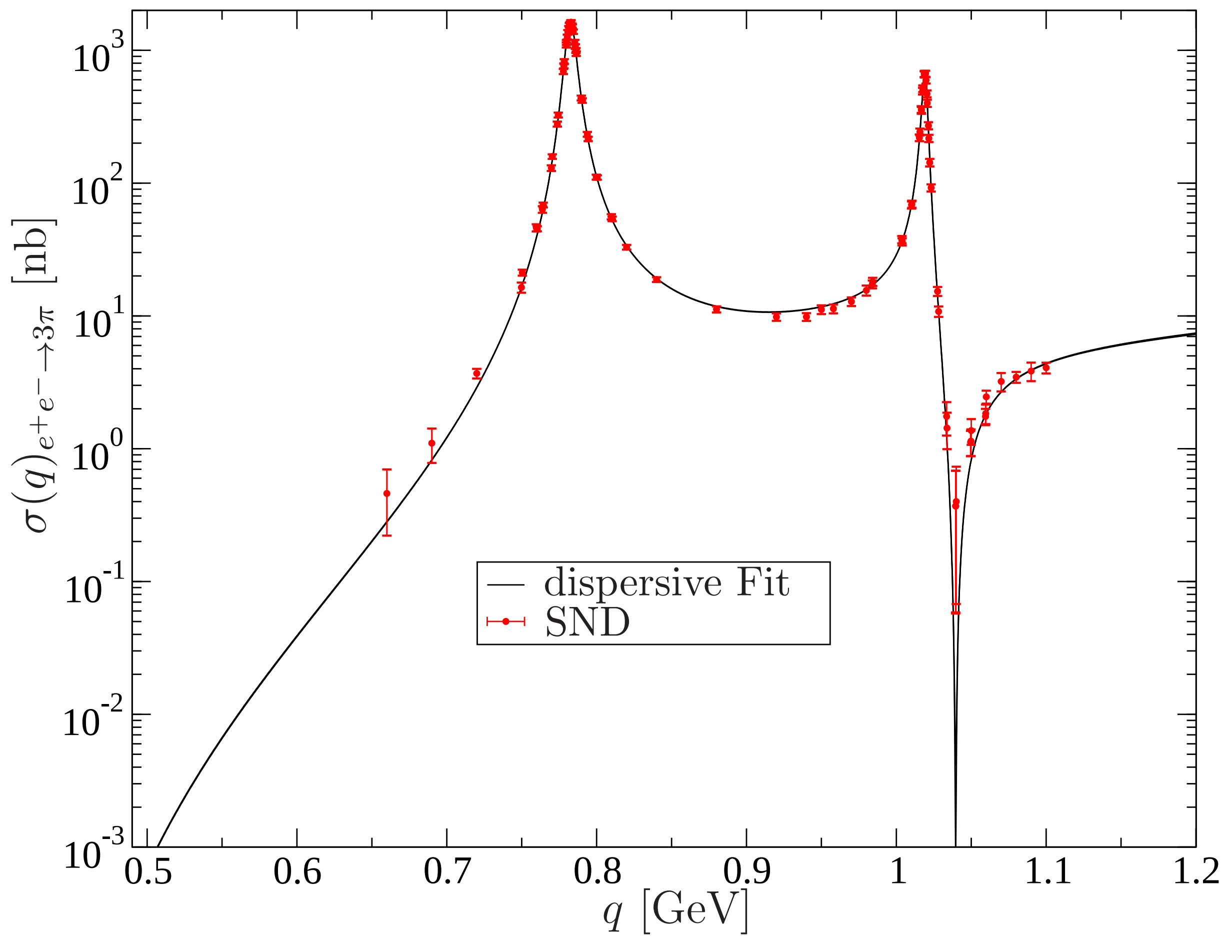


- combine isoscalar and isovector contribution to $e^+e^- \to \pi^0\gamma$

$$F_{\pi\gamma^*\gamma}(q^2,0) = F_{vs}(0,q^2) + F_{vs}(q^2,0)$$

$$= \frac{1}{12\pi^2}\int_{4M_\pi^2}^{\infty} ds' \, \frac{q_\pi^3(s')}{\sqrt{s'}} \left\{ \frac{f_1^{\gamma^*\to 3\pi}(q^2,s')}{s'} + \frac{f_1^{\gamma\pi\to\pi\pi}(s')}{s'-q^2} \right\} F_\pi^{V*}(s')$$

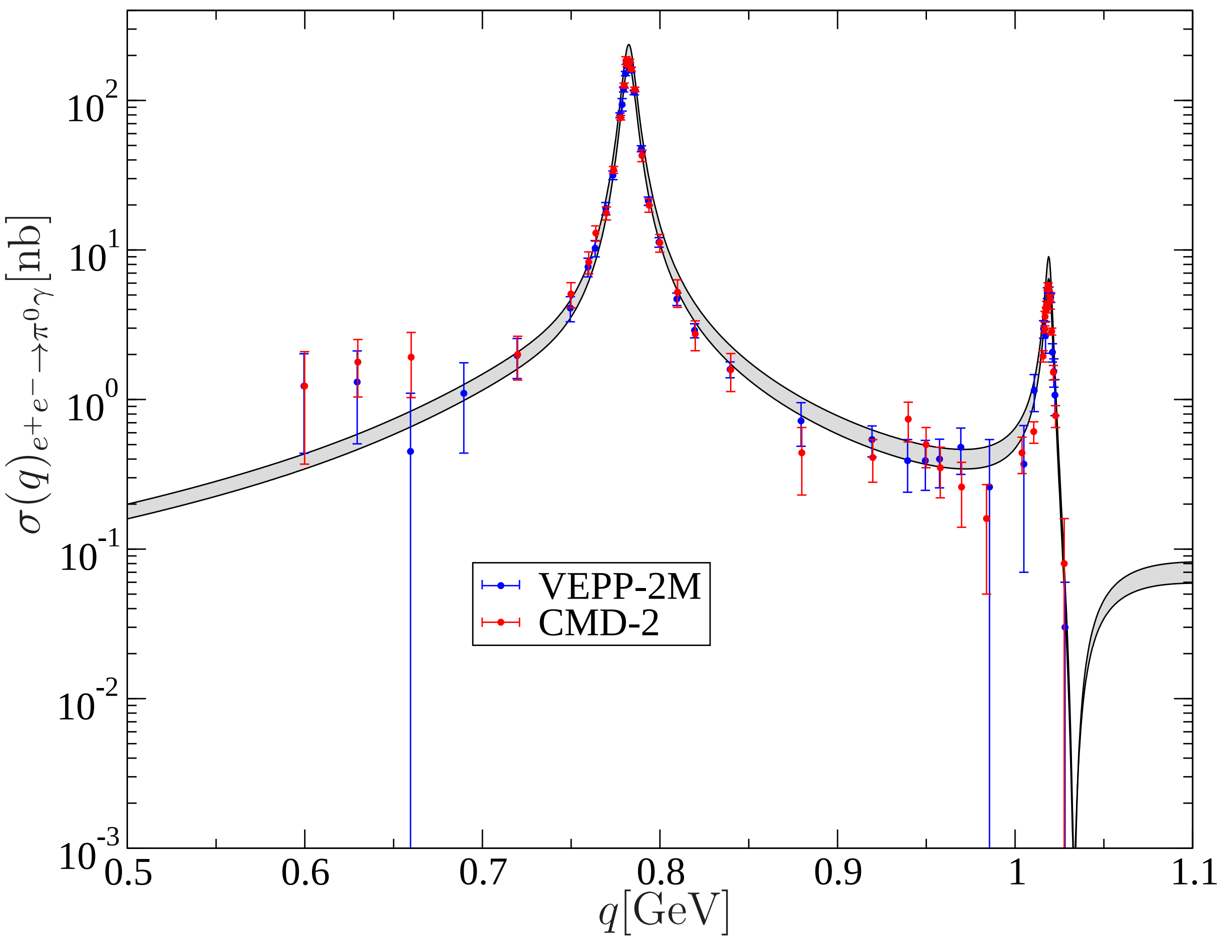
Fit to $e^+e^- \to 3\pi$ data



Hoferichter, BK, Leupold, Niecknig, Schneider preliminary

- one subtraction/normalisation at $q^2 = 0$ fixed by $\gamma \to 3\pi$
- fitted: ω, ϕ residues, one additional (linear) subtraction

Comparison to $e^+e^- \to \pi^0\gamma$ data



Hoferichter, BK, Leupold, Niecknig, Schneider preliminary

- "prediction"—no further parameters adjusted
- data well reproduced

Summary / Outlook

Dispersion relations for light-meson processes

- based on unitarity, analyticity, crossing symmetry
- extends range of applicability (at least) to full elastic regime
- matching to ChPT where it works best

Primakoff reaction $\gamma\pi \to \pi\pi$

- enable improved extraction of $F_{3\pi}$ from data up to 1 GeV

Vector meson decays $\omega/\phi \to 3\pi,\ \pi^0\gamma^*$

- perfect analytic-unitary description of $\phi \to 3\pi$ Dalitz plot

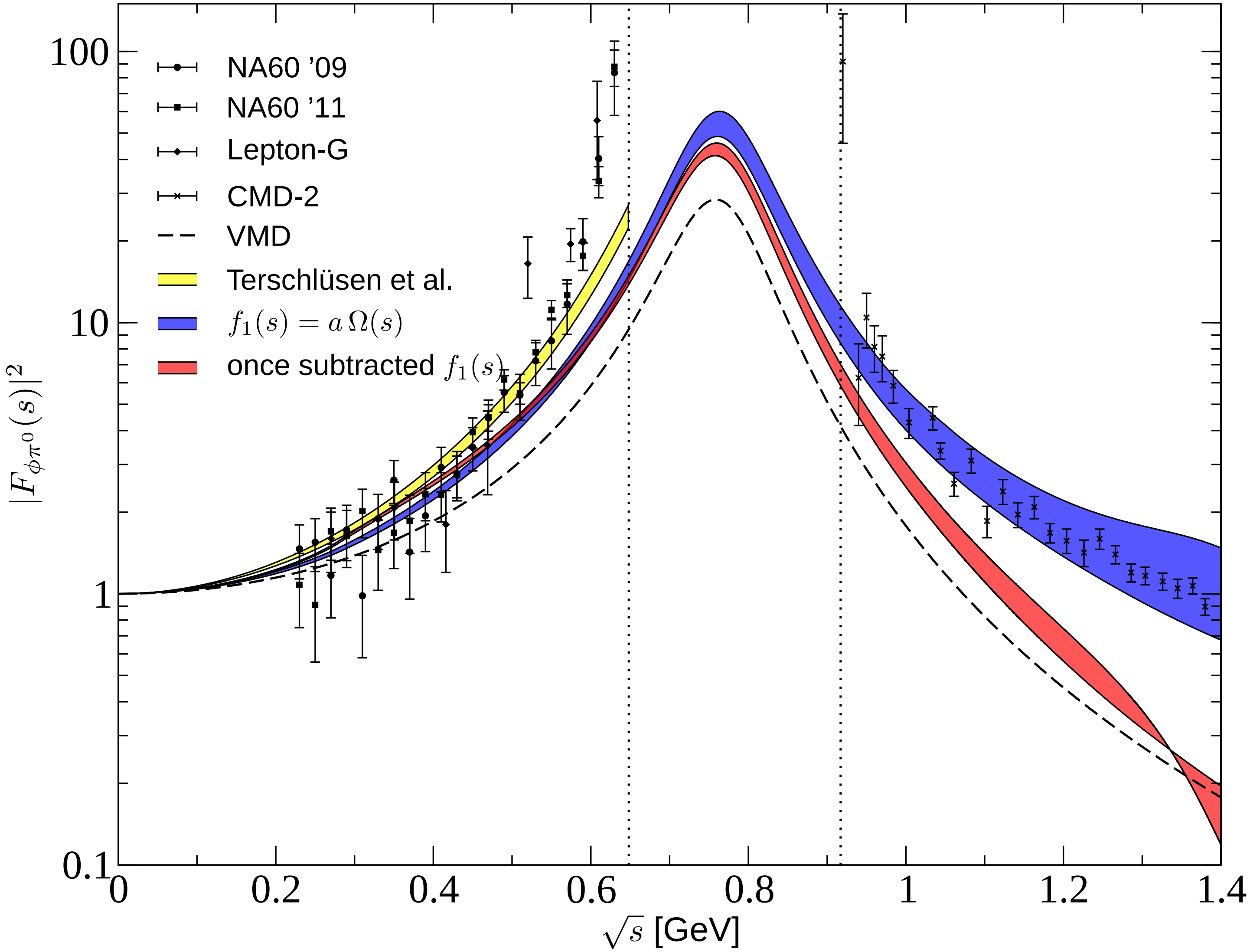
π^0 transition form factor

- successful description of $e^+e^- \to \pi^0\gamma$
- goal: doubly-virtual π^0 transition form factor

$\longrightarrow$ interrelate as much experimental information as possible to constrain hadron physics in $(g-2)_\mu$

Spares

Transition form factor beyond the $\pi\omega$ threshold



- full solution above naive VMD, but still too low
- higher intermediate states (4π / $\pi\omega$) more important?

Improved Breit–Wigner resonances

Lomon, Pacetti 2012; Moussallam 2013

- "standard" Breit–Wigner function with energy-dependent width

$$B^\ell(q^2) = \frac{1}{M_{\rm res}^2 - q^2 - iM_{\rm res}\Gamma_{\rm res}^\ell(q^2)}$$

$$\Gamma_{\rm res}^\ell(q^2) = \theta(q^2 - 4M_\pi^2)\frac{M_{\rm res}}{\sqrt{q^2}}\left(\frac{q^2 - 4M_\pi^2}{M_{\rm res}^2 - 4M_\pi^2}\right)^\ell \Gamma_{\rm res}(M_{\rm res}^2)$$

  - ▷ no correct analytic continuation below threshold $q^2 < 4M_\pi^2$
  - ▷ wrong phase behaviour for $\ell \geq 1$:

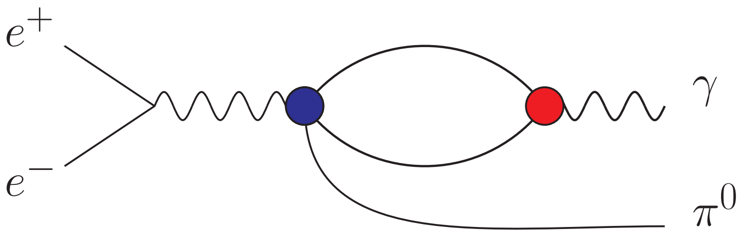
$$\lim_{q^2\to\infty} \arg B^1(q^2) \approx \pi - \arctan\frac{\Gamma_{\rm res}}{M_{\rm res}} \qquad \lim_{q^2\to\infty} \arg B^{\ell\geq 2}(q^2) = \frac{\pi}{2}\ (!)$$

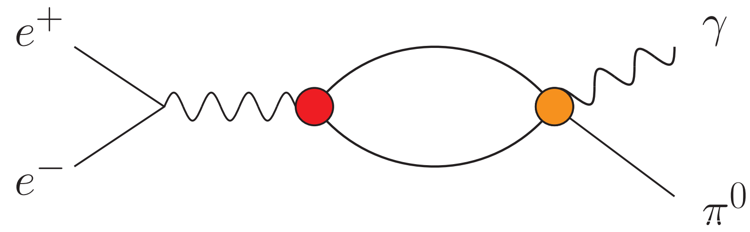
- remedy: reconstruct via dispersion integral

$$\tilde{B}^\ell(q^2) = \frac{1}{\pi}\int_{4M_\pi^2}^\infty \frac{{\rm Im}\, B^\ell(s')ds'}{s' - q^2} \quad \longrightarrow \quad \lim_{s\to\infty} \arg B^\ell(q^2) = \pi$$

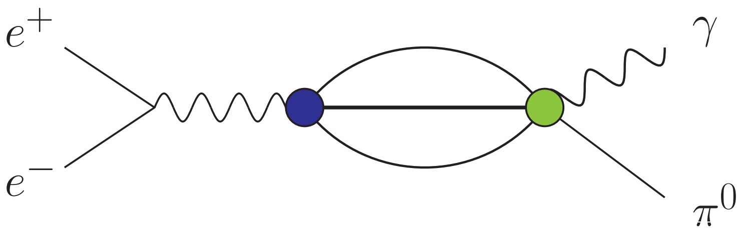
On the approximation for the 3-pion cut

Compare:





$\longrightarrow$ isoscalar contribution looks simplistic; why not instead



$\longrightarrow$ contains amplitude $3\pi \to \gamma\pi$

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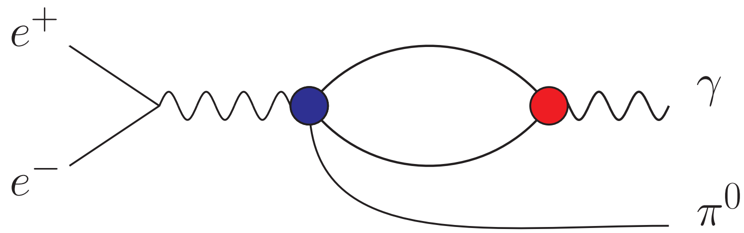

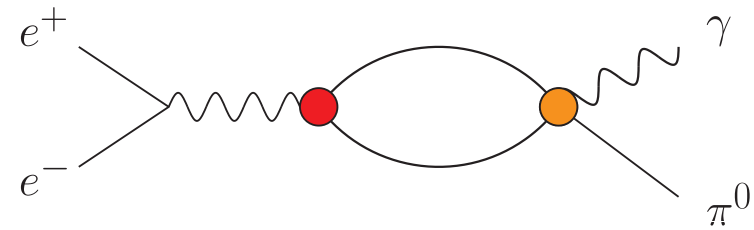

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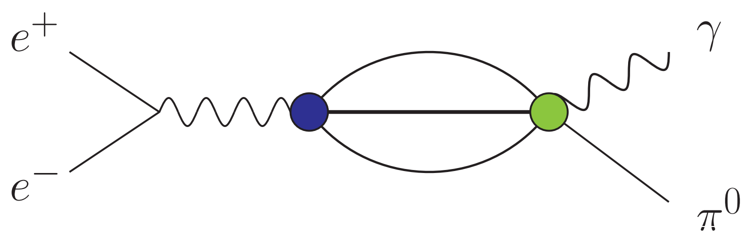

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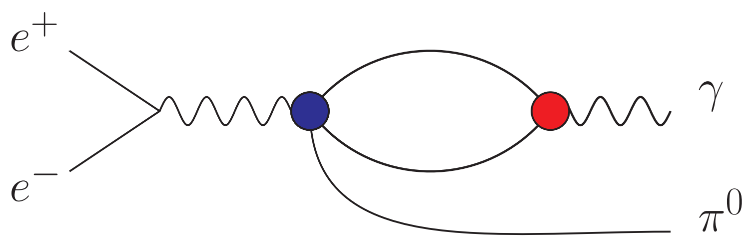

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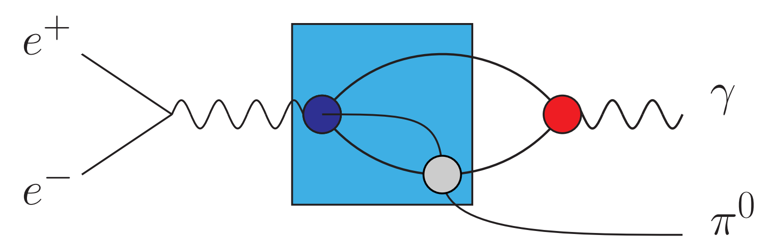

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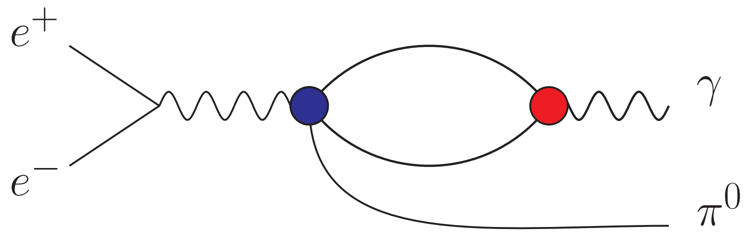

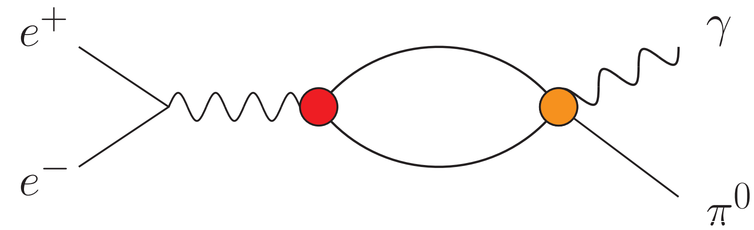

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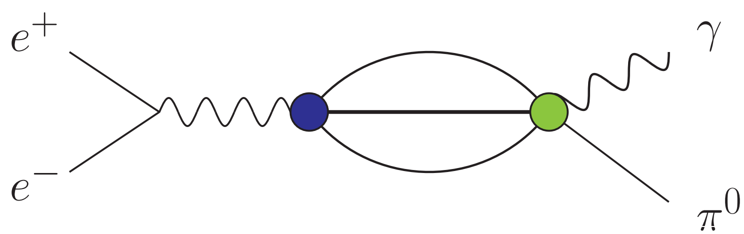

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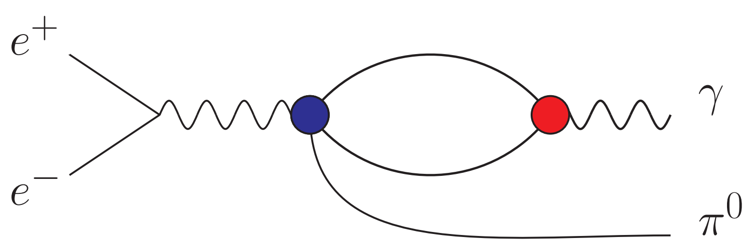

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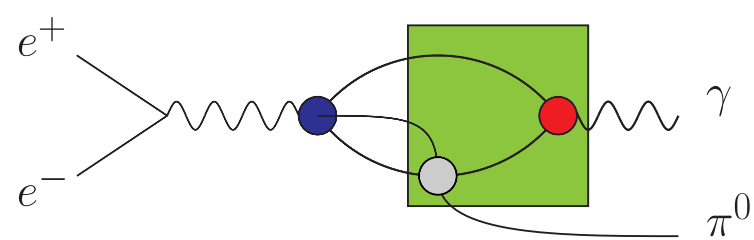

$\longrightarrow$ simplifies left-hand-cut structure in $3\pi \to \gamma\pi$ to pion pole terms

Application: η, $\eta' \to \pi^+\pi^-\gamma$

- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{A}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(s_{\pi\pi}) \times F^V_\pi(s_{\pi\pi})$, $P(s_{\pi\pi}) = 1 + \alpha^{(\prime)} s_{\pi\pi}$

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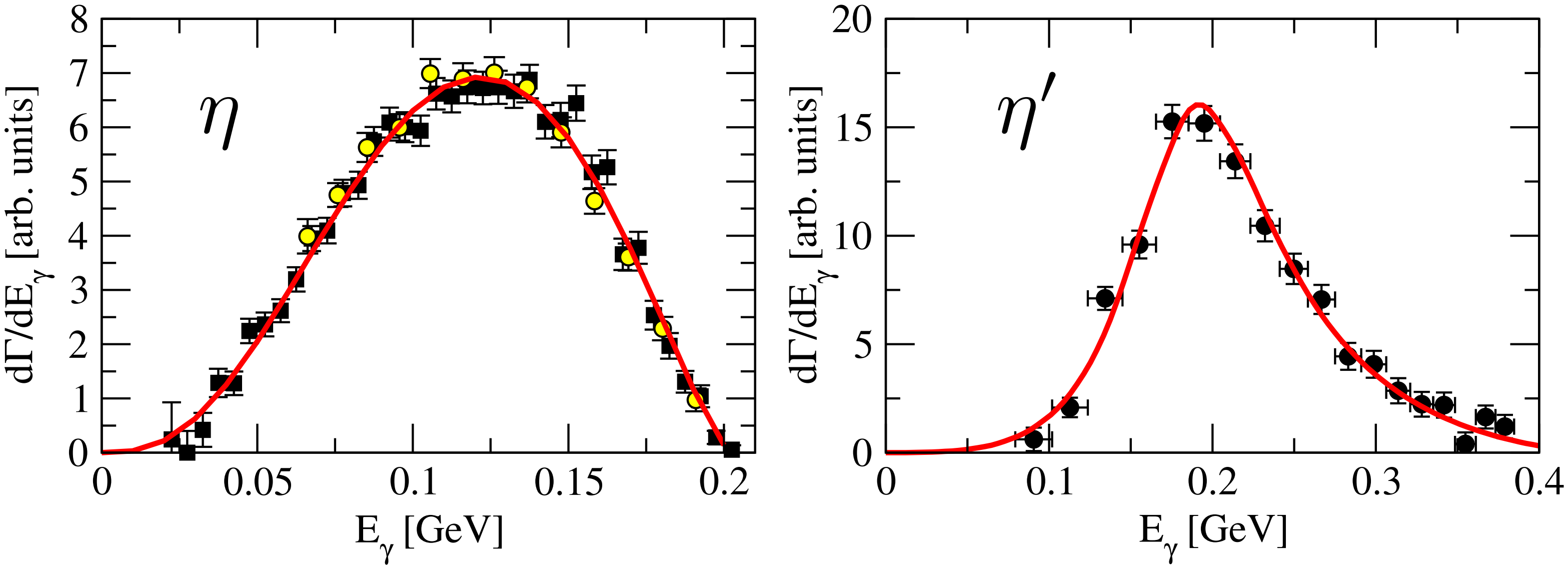
- spectra with fitted normalisation and slope(s) $\alpha^{(\prime)}$



Stollenwerk et al. 2012

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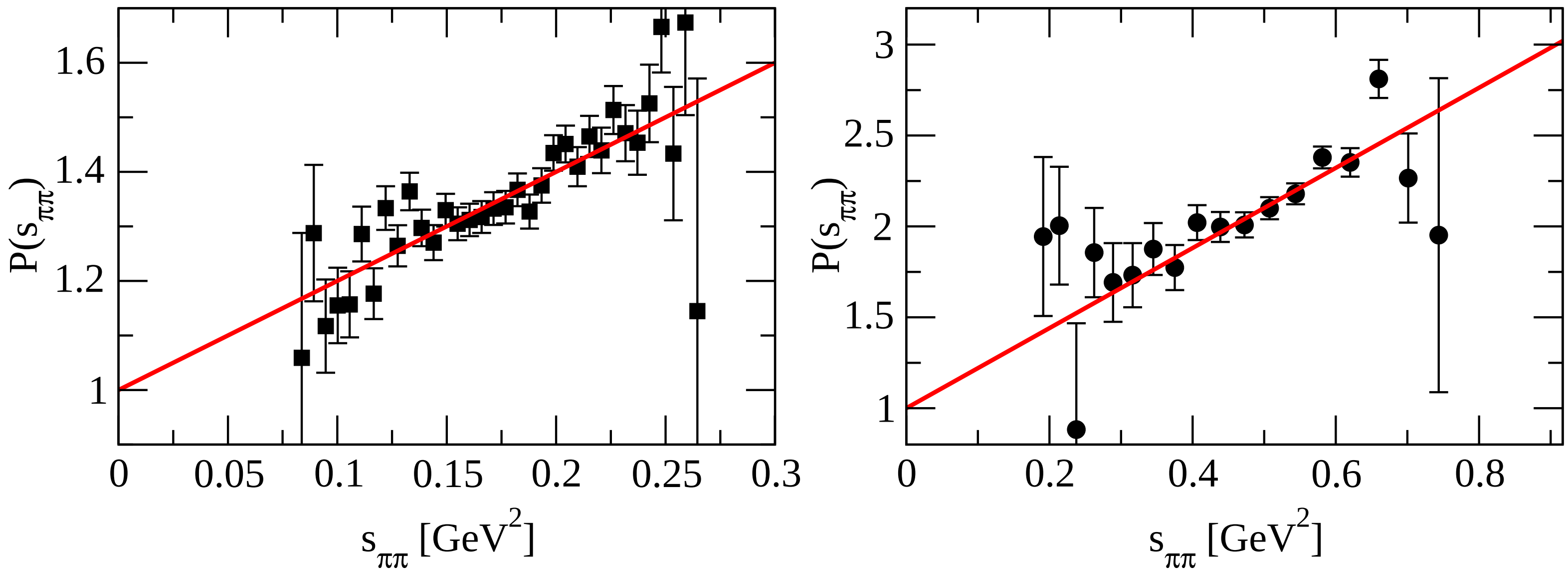
- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave
  $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{A}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(s_{\pi\pi}) \times F^V_\pi(s_{\pi\pi})$, $P(s_{\pi\pi}) = 1 + \alpha^{(\prime)} s_{\pi\pi}$
- divide data by pion form factor $\longrightarrow P(s_{\pi\pi})$



Stollenwerk et al. 2012

$\longrightarrow$ exp.: $\alpha_{\text{WASA}} = (1.89 \pm 0.64)\,\text{GeV}^{-2}$, $\alpha_{\text{KLOE}} = (1.31 \pm 0.08)\,\text{GeV}^{-2}$

$\longrightarrow$ interpret $\alpha^{(\prime)}$ by matching to chiral perturbation theory

$\pi\pi$ scattering constrained by analyticity and unitarity

Roy equations = coupled system of partial-wave dispersion relations
+ crossing symmetry + unitarity

- twice-subtracted fixed-t dispersion relation:

$$T(s,t) = c(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \left\{ \frac{s^2}{s'^2(s'-s)} + \frac{u^2}{s'^2(s'-u)} \right\} \mathrm{Im}T(s',t)$$

- subtraction function $c(t)$ determined from crossing symmetry

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- subtraction function $c(t)$ determined from crossing symmetry
- project onto partial waves $t_J^I(s)$ (angular momentum J, isospin I)
$\longrightarrow$ coupled system of partial-wave integral equations

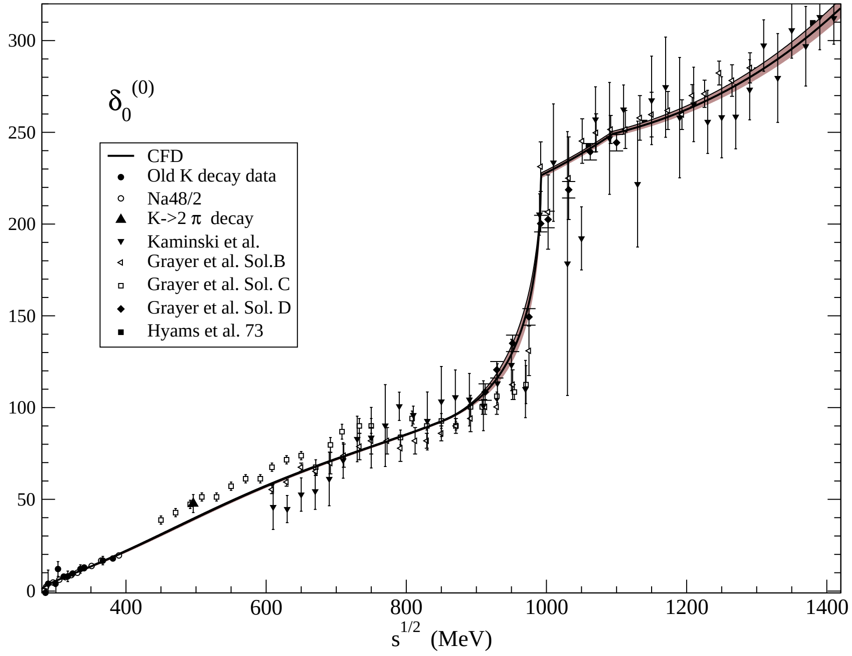
$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s')\mathrm{Im}t_{J'}^{I'}(s')$$

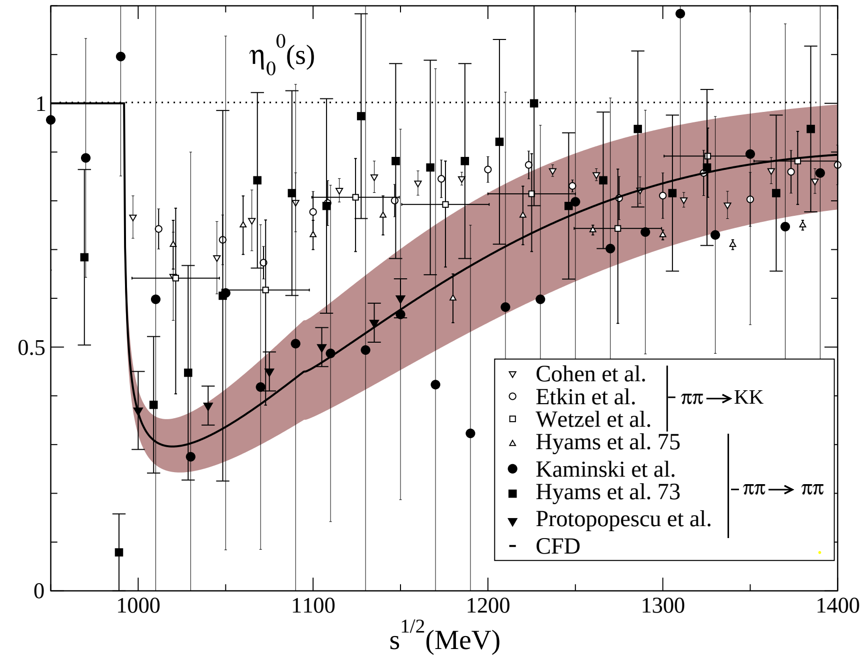
Roy 1971

- subtraction polynomial $k_J^I(s)$: $\pi\pi$ scattering lengths
can be matched to chiral perturbation theory   Colangelo et al. 2001
- kernel functions $K_{JJ'}^{II'}(s,s')$ known analytically

$\pi\pi$ scattering constrained by analyticity and unitarity

- elastic unitarity $\longrightarrow$ coupled integral equations for phase shifts
- modern precision analyses:
  - ▷ $\pi\pi$ scattering Ananthanarayan et al. 2001, García-Martín et al. 2011
  - ▷ πK scattering Büttiker et al. 2004
- example: $\pi\pi$ $I = 0$ S-wave phase shift & inelasticity





García-Martín et al. 2011

- strong constraints on data from analyticity and unitarity!