Ultracold degenerate Fermi gases
Seminar 'Relativistic heavy ion physics'

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Outline

1. Introduction
2. Production of ultracold Fermi gases
3. Ideal Fermi gas in harmonic trap
4. Two-body collisions
5. The many-body problem at equilibrium: Uniform gas
6. Interacting Fermi gas in harmonic trap
Motivation

- Elementary fermions: Quarks and Leptons
- Compound particles with even/odd number of constituents: Bosons/Fermions
- Atomic constituents: $p, n, e^- \rightarrow$ All fermions!

- 'Ordinary' Fermi gases/liquids
  - Electrons in metals/semiconductors
  - Neutron stars
  - Nucleus: Protons and Neutrons

- Enables experimental research of quantum effects!
Historical overview

- **1995**: First realization of Bose-Einstein condensation (Anderson et al., Bradley et al., Davis et al.)
- **1998**: Investigation of Feshbach resonances in bosonic systems (Courteille et al., Inouye et al.)
- **1999**: First ultracold fermionic gas (De Marco and Jin, JILA)
- **2001**: Observation of quantum degeneracy effects in $^6$Li (Truscott et al., Schreck et al.)

Last years of research:
- Dynamic behaviour of ultracold fermionic systems (Superfluidity)
- Spin polarized configurations
- Fermi gases in periodic potentials
Bosons vs. Fermions

**BOSONS**
- Integer spin
- Bose-Einstein statistics
  \[ \langle n_{BE} \rangle = \frac{1}{e^{\beta(E-\mu)} - 1} \]
- Same q.s. preferred

**FERMIONS**
- Half-integer spin
- Fermi-Dirac statistics
  \[ \langle n_{FD} \rangle = \frac{1}{e^{\beta(E-\mu)} + 1} \]
- Pauli’s exclusion principle

**Behavior for T→0**

Quantum degeneracy

- Fermionic ensemble in arbitrary potential $V$

- $T > 0$
  Width of Fermi edge $\sim 2k_B T$

- $T = 0$
  Maximum particle energy: Fermi energy $E_F = k_B T_F$

- Fermi temperature $T_F$ marks crossover of classical $\leftrightarrow$ quantum regime

$$k_B T_F \sim \frac{\hbar^2 n^{2/3}}{m}$$

(Typically for atomic gases: $T_F \lesssim 1\mu K$)
Production of ultracold Fermi gases

Production of BECs

Laser cooling (MOT) and evaporative cooling
Seminar talk 'Experimental production of BECs' (A. Kalweit)
⇒ Particle collisions play crucial role, for example keeping the gas in thermal equilibrium

Problem

Pauli’s exclusion principle!
⇒ s-wave-scattering inhibited for fermions in the same internal quantum state at ultracold temperatures + Fermi blocking

⇒ New cooling techniques required!
World's first ultracold fermionic gas
Brian DeMarco and Deborah Jin (JILA)

**Introduction**

Production Ideal FG 2-Body Collisions Uniform Gas Interacting FG

**World's first ultracold fermionic gas**

Brian DeMarco and Deborah Jin (JILA)

**Idea**

Trapping of two different spin states \((m_f = +9/2, +7/2)\) of \(^{40}\text{K}\) in the \(f = \frac{9}{2}\) hyperfine ground state

1. Collecting sample of \(^{40}\text{K}\) from a room-temperature vapour
2. Cooling in MOT \(\rightarrow \sim 500\) million atoms @ 150\(\mu\)K
3. Loading atoms in two different internal spin-states into a magnetic trap
4. Evaporative cooling \(\rightarrow <300nK\)

'One of the top 10 scientific breakthroughs in 1999’
(Science magazine)
Further method: Sympathetic cooling
Hulet et al. (Rice University), Salomon et al. (ENS)

Idea

Cooling gas ‘cocktail’, containing two different isotopes (one fermionic, one bosonic) - e.g. $^6$Li and $^7$Li

- Apply established cooling methods to bosonic part of the gas
- Fermionic part cools simply by being in thermal contact

$\Rightarrow$ New cooling method for Fermi gases + enables study of boson-fermion mixtures!
Ideal Fermi gas in harmonic trap

Keywords

Fermionic ensemble, harmonic potential, no interactions!

- Harmonic potential
  \[ V_{ho} = \frac{1}{2} m\omega_x^2 x^2 + \frac{1}{2} m\omega_y^2 y^2 + \frac{1}{2} m\omega_z^2 z^2 \]
- Fermi distribution function
  \[ f(r, p) = \frac{1}{\exp[\beta(p^2/2m + V_{ho}(r) - \mu)] + 1} \]
- Normalization condition
  \[ N_\sigma = \frac{1}{(2\pi\hbar)^3} \int dr dp \ f(r, p) \]
  \[ = \int_0^\infty \frac{g(\epsilon) d\epsilon}{\exp[\beta(\epsilon - \mu)] + 1} \]

- Large N
- Many single-particle states occupied
- \( \Rightarrow \) Semiclassical approach

\( g(\epsilon) \): Single-particle density of states
• Energy dependence of $g(\epsilon)$

$$g(\epsilon) = \frac{\epsilon^2}{2(\hbar \omega_{ho})^3} \quad \omega_{ho} = (\omega_x \omega_y \omega_z)^{1/3}$$

• $\Rightarrow$ Thermodynamic functions, e.g. energy $E(T)$

$$E(T) = \int_0^\infty d\epsilon \frac{\epsilon g(\epsilon)}{\exp[\beta(\epsilon - \mu)] + 1}$$

• At $T = 0$: $\mu$ coincides with the Fermi energy $E_{F}^{ho}$:

$$E_{F}^{ho} \equiv k_B T_{F}^{ho} = (6N_\sigma)^{1/3} \hbar \omega_{ho}$$

$$\Rightarrow E(0) = \frac{3}{4} E_{F}^{ho} N_\sigma$$
Density and momentum distributions

- Use Fermi energy $E_F^{ho}$ to define typical length/momentum scales

**Thomas-Fermi radius $R_i^0 \ (i = x, y, z)$**

$$R_i^0 = \sqrt{2E_F^{ho}/m\omega_i^2} = a_{ho}(48N_\sigma)^{1/6}\frac{\omega_{ho}}{\omega_i}$$

$$a_{ho} = \sqrt{\hbar/m\omega_{ho}}$$

$\Rightarrow$ width of the density distribution $n_\sigma(r)$ at $T = 0$

$$n_\sigma(r) = \frac{8}{\pi^2} \frac{N_\sigma}{R_x^0 R_y^0 R_z^0} \left[1 - \left(\frac{x}{R_x^0}\right)^2 - \left(\frac{y}{R_y^0}\right)^2 - \left(\frac{z}{R_z^0}\right)^2\right]^{3/2}$$
Density and momentum distributions

Fermi wavevector $k_F^0$

\[
k_F^0 \equiv \frac{p_F^0}{\hbar} = \sqrt{\frac{2mE^0_F}{\hbar}} = \frac{1}{a_{ho}} (48N_\sigma)^{1/6}
\]

⇒ width of the momentum distribution $n_\sigma(p)$ at $T = 0$

\[
n_\sigma(p) = \frac{8}{\pi^2} \frac{N_\sigma}{(p_F^0)^3} \left[1 - \left(\frac{p}{p_F^0}\right)^2\right]^{3/2}
\]

$n_\sigma(r), n_\sigma(p)$: Thomas-Fermi distributions
Experimentally accessible values
Absorption images (TOF experiments)

- Switch off potential, observe propagation of particles
  ⇒ Momentum/Energy distribution
- Non-interacting gas: ballistic law, \( f_0 := f(t = 0) \)

\[
f(r, p, t) = f_0(r - pt/m, p)
\]

\[
\langle r_i^2 \rangle = \frac{E(T)}{N_\sigma} \frac{1}{3m\omega_i^2} (1 + \omega_i^2 t^2)
\]

- Release energy (→ Equipartition theorem, ideal gas in HO)

\[
E_{rel} = \frac{E}{2} \quad E(T) : \text{total energy}
\]

- At low temperature:
  Energy per particle deviates from classical value \(3k_B T\)
Evidence for quantum degeneracy
DeMarco, Papp and Jin (2001), JILA

Graphics reference: (1) 'Theory of ultracold atomic Fermi gases' (S. Giorgini, L. Pitaevskii, S. Stringari), (2) 'A Fermi gas of atoms' (Deborah Jin, JILA), physicsworld.com 04/2002
Interaction effects in quantum degenerate, dilute Fermi gases can be accurately modeled by a small number of parameters characterizing the physics of two-body collisions.

Low temperatures, large mean particle distance

- $R_0 \ll \lambda_T = \sqrt{\frac{2\pi \hbar^2}{mk_B T}}$, $\lambda_T$: Thermal wavelength
- $R_0 \ll k_F^{-1}$, $k_F$: Fermi wavevector
- $R_0$: Spatial range of interatomic potential

⇒ Main contribution to scattering from states with $l=0$

Remember: Only particles with different spin can interact!
Theory of elastic scattering (s-wave channel)

- Schroedinger equation for relative motion

- Energy $\epsilon > 0$, asymptotic region $r \gg R_0$
  
  **s-wave wavefunction** $\Psi_0(r) \propto \sin[kr + \delta_0(k)]/r$
  
  $k = \sqrt{2m_r\epsilon/\hbar}$, $r = |r_1 - r_2|$, $\delta_0(k)$: s-wave phaseshift

- **s-wave scattering amplitude** $f_0(k) = \left[-k \cdot \cot\delta_0(k) + ik\right]^{-1}$
  
  **s-wave scattering length** $a = -\lim_{k \to 0} f_0 = -\lim_{k \to 0} \frac{\delta_0}{k}$

**Expansion of $\delta_0(k)$**

$$\Rightarrow f_0(k) = -\frac{1}{a^{-1} - k^2 R^*/2 + ik} \quad R^* : \text{effective range}$$

- For $a \to \infty$ ('unitary limit') and $k \ll 1/|R^*|$:  
  
  universal law $f_0(k) = i/k$
Many-body problem

- Microscopic potential $V \rightarrow$ effective potential $V_{\text{eff}}$

**Attractive square-well potential**

$$V_{\text{eff}}(r) = \begin{cases} 
-V_0 & (r < R_0) \\
0 & (r > R_0) 
\end{cases}$$

\[ a = R_0 \left[ 1 - \tan(K_0 R_0)/(K_0 R_0) \right] \]

\[ K_0 = \sqrt{2m_r V_0/\hbar^2} \]

\[ R^* = R_0 - R_0^3/3a^2 - 1/K_0^2 a \]

**Regularized zero-range pseudo-potential (Huang and Yang, 1957)**

$$V_{\text{eff}}(r) = g \delta(r) \frac{\partial}{\partial r} r$$

\[ g = 2\pi \hbar^2 a/m_r \]

Range $R_0 = 0$, $f(k) = -[a^{-1} + ik]^{-1}$
Fano-Feshbach resonances

Take place when

\[ E_{\text{open}} \approx E_{\text{closed}} \]

+ Coupling between channels
  - Open channel: scattering of the two particles
  - Closed channel: formation of weakly bound state (dimer)

Interatomic potentials

Theory of Feshbach resonances

Seminar talk 'Fermion-Fermion and Boson-Boson Interaction at low T' (M. Freudenberger)
Fano-Feshbach resonances

- If magnetic moments in open/closed channel are different:
  ⇒ 'Artificial' resonance tuned by magnetic field $B$
  ⇒ Selective modification of the interaction!
- Parametrization of the scattering length: $a(B) = a_{bg} \left(1 - \frac{\Delta B}{B-B_0}\right)$

- Broad ($k_F |R^*| \ll 1$) and narrow ($k_F |R^*| \gg 1$) resonances
- If $a < 0$: Attraction between fermions! ⇒ BCS-Theory
- ⇒ Feshbach resonance forms BCS-BEC crossover

Graphics reference: S. Giorgini, L. Pitaevskii, S. Stringari
BCS-BEC crossover

Seminar talk 'Molecular BECs and fermionic condensates of Cooper pairs - from BEC to BCS ' (S. Huber) - January 29, 2009
The many-body problem at equilibrium: Uniform gas

Keywords

Two-component Fermi gas, Uniform configuration \((V_{\sigma,\text{ext}} = 0)\)

- Ideal gas model provides good description of cold spin polarized Fermi gas
- For atoms occupying different spin states \(\sigma\):
  Interactions deeply affect solution of the many-body problem!

Grand canonical many-body Hamiltonian

\[
\hat{H} = \sum_{\sigma} \int d\mathbf{r} \ \hat{\Psi}^{+}_{\sigma}(\mathbf{r}) \left( -\dfrac{\hbar^2 \Delta^2}{2m_\sigma} + V_{\sigma,\text{ext}}(\mathbf{r}) - \mu_\sigma \right) \hat{\Psi}_{\sigma}(\mathbf{r}) \\
+ \int d\mathbf{r} d\mathbf{r}' \ V(\mathbf{r} - \mathbf{r}') \ \hat{\Psi}^{+}_{\uparrow}(\mathbf{r})\hat{\Psi}^{+}_{\downarrow}(\mathbf{r}')\hat{\Psi}_{\downarrow}(\mathbf{r}')\hat{\Psi}_{\uparrow}(\mathbf{r})
\]
The many-body problem at equilibrium: Uniform gas

- Uniform configuration
  - One-body potential $V_{\uparrow, \text{ext}} = V_{\downarrow, \text{ext}} = 0$ (!)
  - Number of atoms $N_{\uparrow} = N_{\downarrow} = N/2$
  - Atomic masses $m_{\uparrow} = m_{\downarrow} = m$
  - Atomic densities $n_{\uparrow} = n_{\downarrow} = \frac{1}{2} n$

- Fermi wavevector $k_F = (3\pi^2 n)^{1/3}$
- Fermi energy $E_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$

- Important cases where many-body problem for the interacting Fermi gas can be solved exactly:
  Repulsive gas, weakly attractive gas, gas of composite bosons
Interactions: Pseudo-potential, positive scattering length $a$

Perturbation theory with small parameter $k_F a \ll 1$ (diluteness condition of the gas)

Expansion of energy per particle ($T = 0$):

$$\frac{E}{N} = \frac{3}{5} E_F \left( 1 + \frac{10}{9\pi} k_F a + \frac{4(11 - 2\log2)}{21\pi^2} (k_F a)^2 + \ldots \right)$$

(Huang and Yang (1957), Lee and Yang (1957))
Weakly attractive gas

- Interaction with negative scattering length \( (k_F |a| \ll 1) \)
- Formation of bound states (Cooper pairs)
- Many-body solution: diagonalization of \( \hat{H} \) by applying Bogoliubov transformation to the Fermi field operators

- Critical temperature \( T_C \):

\[
T_C = \left( \frac{2}{e} \right)^{7/3} \frac{\gamma}{\pi} T_F e^{\pi/2k_F a} \approx 0.28 T_F e^{\pi/2k_F a}
\]

\[\gamma = e^C \approx 1.781, \ C: \text{Euler constant}\]
Weakly attractive gas

Ground-state energy per particle

\[
\frac{E}{N} = \frac{E_{\text{normal}}}{N} - \frac{3\Delta^2_{\text{gap}}}{8E_F}
\]

\[
\Delta_{\text{gap}} = \frac{\pi k_B T_C}{\gamma} \approx 1.76k_B T_C
\]

\(E_{\text{normal}}\): Perturbation expansion of 'Repulsive gas' with \(a < 0\)
Gas of composite bosons

- Tuning \((a < 0) \rightarrow (a > 0)\) through Feshbach resonance

  \[\Rightarrow \text{Formation of dimers (bosons), } \epsilon_b \simeq -\frac{\hbar^2}{ma^2}\]

- Behavior of dilute gas of dimers (BEC limit) described by theory of Bose-Einstein condensed gases

- Critical temperature \(T_C\):

  \[T_C = \frac{2\pi\hbar^2}{k_Bm} \left( \frac{n_d}{\xi(3/2)} \right)^{2/3} = 0.218 T_F\]

  \(n_d: \text{ density of dimers, } \xi(3/2) \simeq 2.612\)
Gas of composite bosons

- **Interaction between molecules:**
  - Atom-dimer scattering length $a_{ad} \simeq 1.18a$
    (Skorniakov and Ter-Martirosian (1956), Petrov (2003))
  - Dimer-dimer scattering length $a_{dd} \simeq 0.60a$
    (Petrov, Salomon and Shlyapnikov (2004))

- **Energy per dimer:**
  \[
  \frac{E}{N} = \frac{\epsilon_b}{2} + \frac{k_F a_{dd}}{6\pi} \left[ 1 + \frac{128}{15\sqrt{6\pi}^3} (k_F a_{dd})^{3/2} \right] E_F
  \]
  (Lee, Huang and Yang (1957), Leyronas and Combescot (2007))
Gas at unitarity

- More difficult problem: Behavior of many-body system for \( k_F |a| \gtrsim 1 \) (i.e. scattering length larger than interparticle distance)?

- Application of approximate schemes, numerical simulations
  \( \Rightarrow \) Gas stable and superfluid for \( T < T_C \)

- \( k_F |a| \rightarrow \infty \): unitary regime \( (f_0(k) = i/k) \)
  - Chemical potential \( (T = 0) \): \( \mu = (1 + \beta)E_F \)
  - Energy per particle: \( E/N = (1 + \beta)3E_F/5 \)
  - Pressure: \( P = (1 + \beta)2nE_F/5 \)

- Finite temperatures: Transition temperature \( T_C = \alpha T_F \)
  QMC methods: \( \alpha = 0.157(7) \) (Burovski et al., 2006)
Interacting Fermi gas in harmonic trap

Keywords
Local density approximation (LDA) ⇒ Density profiles

- Solution of many-body problem for non-uniform configurations
  ⇒ Numerical calculations

- Experimentally relevant case: \( N \approx 10^5 - 10^7 \), harmonic potential
  ⇒ Local density approximation (LDA)

- Profits of the knowledge of equation of state of uniform matter
Local density approximation at $T = 0$

- Equation of state of uniform gas provided by density dependence $\epsilon(n)$ of the energy density
- LDA: System behaves \textbf{locally} like a uniform gas

\[\epsilon(n) = n \frac{E(n)}{N} \quad \frac{E(n)}{N}: \text{energy/atom (uniform matter)}\]

- Total energy of trapped system

\[E = \int d\mathbf{r}\{\epsilon[n(\mathbf{r})] + V_{ho}(\mathbf{r})n(\mathbf{r})\}\]

\[n(\mathbf{r}) = n_{\uparrow}(\mathbf{r}) + n_{\downarrow}(\mathbf{r})\]: total density profile
Local density approximation at $T = 0$

- Value of $n(r)$ at equilibrium (implicit) determined by variational relation: $\delta(E - \mu_0 N)/\delta n(r) = 0 \Rightarrow$ Thomas-Fermi equation

$$\mu_0 = \mu[n(r)] + V_{ho}(r)$$

$\mu_0$: Chemical potential of trapped gas
$\mu(n) = \frac{\partial \epsilon(n)}{\partial n}$: Local chemical potential

- Ideal gas: Density dependence of local chemical potential

$$\mu(n) = (3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{2/3}$$

$$n_{\sigma}(r) = \frac{8}{\pi^2} \frac{N_{\sigma}^0}{R_x^0 R_y^0 R_z^0} \left[ 1 - \left( \frac{x}{R_x^0} \right)^2 - \left( \frac{y}{R_y^0} \right)^2 - \left( \frac{z}{R_z^0} \right)^2 \right]^{3/2}$$
Local density approximation at $T = 0$

- Shape of density profile modified by interactions
- At unitarity: Same density dependence of equation of state like ideal gas (with dimensionless renormalization factor: $1 + \beta$)
- $\Rightarrow$ Thomas-Fermi radii

\[ R_i = (1 + \beta)^{1/4} R_i^0 = (1 + \beta)^{1/4} a_{ho} (24N)^{1/6} \frac{\omega_{ho}}{\omega_i} \]

$\Rightarrow$ Oscillator energy of trapped gas

\[ E_{ho} = (1 + \beta)^{1/2} E_{ho}^0 \]

\[ E_{ho}^0 = \frac{3}{8} N E_F^{ho} \quad \text{(Ideal gas value)} \]
Local density approximation at $T = 0$

- BCS regime (small, negative scattering length)
- Calculation of first correction to non-interacting density profile via perturbation theory

$$R_i = \sqrt{\frac{2\mu_0}{m\omega_i^2}} = R_i^0 \left(1 - \frac{256}{315\pi^2} k_F^0 |a|\right)$$

- BEC limit (positive scattering length)
- Interaction between dimers: meanfield term $\mu_d = g_d n/2$ with $g_d = 2\pi \hbar^2 a_{dd}/m \Rightarrow$ inverted parabola profile (Dalfovo et al.)

$$R_i = a_{ho} \left(\frac{15}{2} N \frac{a_{dd}}{a_{ho}}\right) \frac{\omega_{ho}}{\omega_i}$$
Density profiles along BEC-BCS crossover of $^6\text{Li}$
Bartenstein et al., 2004 (Univ. Innsbruck)

Very good agreement between theory and experiment at unitarity!

\[ n_z^{(1)} = \int dx dy \, n(r) = \frac{N}{R_z} \frac{16}{5\pi} \left( 1 - \frac{z^2}{R_z^2} \right)^{5/2} \]

\[ R_z = (1 + \beta)^{1/4} R_z^0 = (1 + \beta)^{1/4} a_{ho} (24N)^{1/6} \frac{\omega_{ho}}{\omega_z} \]

\[ \Rightarrow \beta = -0.73^{+0.12}_{-0.09} \]

- Most recent exp. data:
  \[ \beta = -0.54(5) \]
  Partridge et al., 2006 (Rice University)
Release energy and virial theorem

- Further source of information: Release energy

\[ E_{rel} = E_{kin} + E_{int} \]

\[ E_{rel} = \int d\mathbf{r} \, \epsilon[n(\mathbf{r})] \quad \text{(within LDA)} \]

- Derivation of general relationship between release/potential energy using the 'virial theorem'
Release energy and virial theorem

- Virial theorem holds for polytropic dependence of energy density on the density: \( \epsilon(n) \propto n^{\gamma+1} \)
  - BEC limit: \( \gamma = 1 \)
  - Unitary limit: \( \gamma = 2/3 \)

- Theorem is derived by applying number conserving transformation
  \[ n(r) \rightarrow (1 + \alpha)^3 n[(1 + \alpha)r] \]
  to density of gas at equilibrium
  \[ \Rightarrow 3\gamma E_{rel} = 2E_{ho} \]

At unitarity: \( E_{rel} = E_{ho} \)
Virial theorem at unitarity in $^6\text{Li}$
Thomas, Kinast and Turlapov (2005)

Thanks for your attention!

Talk based on:
'Theory of ultracold atomic Fermi gases'
Stefano Giorgini, Lev P. Pitaevskii, Sandro Stringari
(CNR-INFM BEC Center, Trento, Italy)