Outline

Introduction

Relativistic Hydrodynamics

Heavy ion collisions

Bjorken model

28.1.10:
Experimental investigations of relativistic hydrodynamics and the ideal fluid scenario at RHIC
Historic overview

- **1951** First application of statistical methods to p-p collisions by Fermi
  Phys. Rev. 81, 683 (1951)

- **1953** Landau introduced relativistic hydrodynamics to strongly interacting systems

- **1983** Bjorken scenario

- **2005** “Perfect liquid” found at RHIC

- **Today** many successful applications of rel. hydrodynamics to rel. heavy ion collisions
Relativistic Hydrodynamics
Thermodynamics

Grand-canonical potential:

$$\Omega(T, V, \mu) = E - TS - \mu N$$  \hspace{1cm} (1)

Gibbs-Duhem relation:

$$V dP = S dT + N d\mu$$  \hspace{1cm} (2)

- energy density: $$\epsilon \equiv \frac{U}{V},$$
- entropy density: $$s \equiv \frac{S}{V},$$
- baryon density: $$n \equiv \frac{N}{V}$$

$$\epsilon + P = Ts + \mu n$$  \hspace{1cm} (3)

$$d \epsilon = T ds + \mu dn$$  \hspace{1cm} (4)
Hydrodynamics:

- description of fluids
- continuous medium (collective behavior)
- from particles to fluid elements

Conditions:

- **The system is in local thermodynamic equilibrium!**
  ⇒ equilibrium in some “neighborhood” (fluid element)
- pressure, temperature vary slowly (\( \partial_\mu \) small)
- mean free path \( \ll \) characteristic dimensions of the system
  ⇒ \( Kn \equiv \lambda/R \), so \( Kn \ll 1 \)
The Fluid

- Fluid rest frame of a fluid element: The frame where \( p = 0 \).
- Fluid velocity (collective velocity):

\[
{u} = \left( \frac{1}{\sqrt{1 - \vec{v}^2}}, \frac{\vec{v}}{\sqrt{q - \vec{v}^2}} \right)
\]

(5)

with \( \vec{v} \) the velocity with respect to the laboratory frame.

\[
u_\mu u^\mu = 1
\]

(6)

the lab. system: \( \tilde{u} = (1, 0, 0, 0) \) so that:

\[
u^\mu = \Lambda^\mu_\nu \tilde{u}^\nu \Rightarrow \Lambda^\mu_0 = u^\mu
\]

(7)

and

\[
g^{\mu\nu} \Lambda^\rho_\mu \Lambda^\sigma_\mu = g^{\rho\sigma}
\]

(8)
The energy-momentum tensor I

The conservation of energy and momentum:

\[ \partial_\mu T^{\mu \nu} = 0. \]  

(9)

with the energy-momentum tensor \( T^{\mu \nu} \).

- \( T^{00} \) energy density
- \( T^{0j} \) momentum density
- \( T^{i0} \) energy flux
- \( T^{ij} \) momentum flux

\[ T^{\mu \nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \]
In the fluid rest frame: Isotropy $\Rightarrow T^{i0} = 0$ and $T^{0j} = 0$.

\[
\tilde{T}_{\mu\nu}(x) = \begin{pmatrix}
\epsilon(x) & 0 & 0 & 0 \\
0 & P(x) & 0 & 0 \\
0 & 0 & P(x) & 0 \\
0 & 0 & 0 & P(x)
\end{pmatrix}
\]  

(10)

with Eq.8

\[
T_{\mu\nu} = \Lambda_{\rho}^{\mu} \Lambda_{\sigma}^{\nu} \tilde{T}^{\rho\sigma} = \Lambda^{\mu}_{0} \Lambda^{\nu}_{0} \epsilon + \Lambda^{\mu}_{i} \Lambda^{\nu}_{i} P = (\epsilon + P) u^{\mu} u^{\nu} - g^{\mu\nu} P
\]  

(11)
The baryon current

\[ j_B(x)^\mu = n_B(x) u^\mu(x) \]  \hspace{1cm} (12)

is conserved. So that

\[ \partial_\mu T^{\mu\nu}(x) = 0 \]  \hspace{1cm} (13)

\[ \partial_\mu j_B^\mu(x) = 0 \]  \hspace{1cm} (14)

give five independent equations for the six thermodynamic variables

\[ \epsilon(x), P(x), n_B(x), \bar{v}(x). \]  \hspace{1cm} (15)

⇒ equation of state is needed to solve the system.
Conservation of the entropy current

\[ u_\nu \partial_\mu T^{\mu\nu} = 0 \]  \hspace{1cm} (16)

\[ u_\nu \partial_\mu [ (\epsilon + P) u^\mu u^\nu - g^{\mu\nu} P ] = 0 \]  \hspace{1cm} (17)

with

\[ u_\nu \partial_\mu u^\nu = 0 \]  \hspace{1cm} (18)

this leads to

\[ 0 = \mu_B \partial_\mu (n_B u^\mu) + T \partial_\mu (su^\mu) \]  \hspace{1cm} (19)

in an ideal fluid the entropy current \( s^\mu(x) = s(x)u^\mu(x) \) is conserved.
Relativistic hydrodynamic: Dissipative processes

Modifies the equations:

\[ T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - g^{\mu\nu} P + \tau^{\mu\nu} \]  \hspace{1cm} (20)

\[ n^\mu = nu^\mu + \nu^\mu \]  \hspace{1cm} (21)

with

\[ \tau^{\mu\nu} = \eta \left[ \partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} (\partial_\perp \cdot u) \right] + \zeta \Delta^{\mu\nu} (\partial_\perp \cdot u) \] \hspace{1cm} (22)

\[ \nu^\mu = \kappa \left( \frac{n T}{\epsilon + P} \right)^2 \partial_\perp^\mu \left( \frac{\mu}{T} \right) \] \hspace{1cm} (23)

shear viscosity \( \eta \), bulk viscosity \( \zeta \) and thermal conductivity \( \kappa \)

\[ \partial_\perp^\mu = \partial^\mu - u^\mu (u \cdot \partial) , \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \] \hspace{1cm} (24)
Euler equation

From the transverse projection

\[ (g_{\rho\nu} - u_{\rho} u_{\nu}) \partial_{\mu} T^{\mu\nu} = 0 \]  \hspace{1cm} (25)

we get

\[-\partial_{\rho} P + u_{\rho} u^{\mu} \partial_{\mu} P + (\epsilon + P) u^{\mu} \partial_{\mu} u_{\rho} = 0 \]  \hspace{1cm} (26)

or

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1 - \vec{v}^2}{\epsilon + P} \left[ \nabla P + \vec{v} \frac{\partial P}{\partial t} \right] \]  \hspace{1cm} (27)
The basic equations for rel. hydro. are:

- "energy equation" \[ u^\mu \partial_\mu \epsilon + (\epsilon + P) \partial_\mu u^\mu = 0 \] (28)
- "continuity equation" \[ \partial_\mu j^\mu = 0 \] (29)
- "entropy equation" \[ \partial_\mu (su^\mu) = 0 \] (30)
- "Euler equation" \[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1 - \vec{v}^2}{\epsilon + P} \left[ \nabla P + \vec{v} \frac{\partial P}{\partial t} \right] \] (31)

Together with:

- "equation of state" \[ P(x) = P(\epsilon(x), \mu(x)) \] (32)
Heavy ion collisions
Heavy ion collision

**Relativistic Heavy Ion Collider**

- nucleus-nucleus collision with $\sqrt{s_{NN}} = 200$ GeV
- ion: Au ($A \approx 200$)

Higher energies possible at LHC in the future.

from www.bnl.gov/rhic
central collision at $t = 0$
thermalization at $\tau_0$
hydrodynamic expansion
freeze-out at $\tau_f$
Light-cone diagram of a collision

- Collision at $t = z = 0$
- $\tau_0 \approx 1\text{fm}$ Thermalization
- $\tau_f$ Freeze-out
Bjorken model
Addition law for velocities in relativistic mechanics:

\[ v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \text{ or } \beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \quad \beta = \frac{v}{c} \quad (33) \]

using

\[ \tanh (\alpha + \gamma) = \frac{\tanh (\alpha) + \tanh (\gamma)}{1 + \tanh (\alpha) \tanh (\gamma)} \quad (34) \]

\[ \Rightarrow \tanh^{-1} \beta_1 + \tanh^{-1} \beta_2 = \tanh^{-1} \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \quad (35) \]

the rapidity \( y \):

\[ y = \tanh^{-1} \beta = \frac{1}{2} \ln \frac{1 + \beta}{1 - \beta} \quad (36) \]
Rapidity $y$ II

Lorentz boost

$$y' = y + \tanh^{-1} \beta$$  \hspace{1cm} (37)

shape of $y$ is boost invariant.

For $E \gg m$:

$$y \approx -\ln \left( \tan \frac{\theta}{2} \right) \equiv \eta$$  \hspace{1cm} (38)

pseudo-rapidity $\eta$
New space-time coordinates

From \( z, t \) to proper-time \( \tau \) and space-time rapidity \( \eta_S \):

\[ t = \tau \cosh \eta_S \]
\[ z = \tau \sinh \eta_S \]
\[ \tau = \sqrt{t^2 - z^2} \]
\[ \eta_S = \frac{1}{2} \ln \frac{t+z}{t-z} \]
The Bjorken picture

Assumptions:

▶ fast thermalization
▶ baryon number free fluid
▶ one-dimensional expansion

\[ \epsilon = \epsilon (\tau, y) \] (39)
\[ P = P (\tau, y) \] (40)
\[ T = T (\tau, y) \] (41)
\[ u_\mu = (u_0, 0, 0, z/t) \] (42)

▶ “Plateau”
Plateau I

\[
\text{The same as this just boosted}
\]

⇒ symmetry under Lorentz boosts
⇒ Fluid rapidity equals space-time rapidity:
\[y = \eta_S \Rightarrow \text{initial values independent of } y\]

\[
\begin{align*}
\epsilon (\tau_0, y) &= \epsilon_0 \\
\mathcal{P} (\tau_0, y) &= \mathcal{P}_0 \\
T (\tau_0, y) &= T_0 \\
u_\mu (\tau_0, y) &= \frac{1}{\tau_0} (t, 0, 0, z)
\end{align*}
\]
Plateau II
Boost invariance of $P$

Using Bjorken assumption $\vec{v} = (0, 0, z/t)$ in the Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1 - \vec{v}^2}{\epsilon + P} \left[ \nabla P + \vec{v} \frac{\partial P}{\partial t} \right] = 0$$

(48)

$$\Rightarrow \frac{\partial P}{\partial y} = 0$$

(49)

shows the boost invariance of $P$. 

Plateau III

- invariant initial conditions
- invariant evolution
- invariant entropy leads to “plateau” in $dN/dy$
Plateau IV


Simplified equations (Perfect fluid)

The hydrodynamic equation

\[ u_\mu \partial_\nu T^{\mu\nu} = u^\mu \partial_\mu \epsilon + (\epsilon + P) \partial_\mu u^\mu = 0 \quad \text{and} \quad \partial_\mu (su^\mu) = 0 \quad (50) \]

can be simplified in the Bjorken picture by using

\[
\left( \begin{array}{c} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial z} \end{array} \right) = \left( \begin{array}{cc} \cosh Y & -\sinh Y \\ -\sinh Y & \cosh Y \end{array} \right) \left( \begin{array}{c} \frac{\partial}{\partial \tau} \\ \frac{1}{\tau} \frac{\partial}{\partial Y} \end{array} \right) \quad (51) \]

to

\[ \Rightarrow u_\mu \partial^\mu = \frac{\partial}{\partial \tau}, \quad \partial^\mu u_\mu = \frac{1}{\tau} \quad (52) \]

\[ \frac{d\epsilon}{d\tau} = -\frac{(\epsilon + P)}{\tau} \quad \text{and} \quad \frac{s(\tau)}{s(\tau_0)} = \frac{\tau_0}{\tau} \quad (53) \]
Simple equation of state
(Perfect fluid)

Using a very simple equation of state

\[ P = \lambda \epsilon \] \hspace{1cm} (54)

leads to

\[ s = a T^{1/\lambda}, \quad P = \frac{a}{1 + 1/\lambda} T^{1+1/\lambda} \] \hspace{1cm} (55)

[for an ideal gas: \( \lambda = 1/3 \) and \( a = 4d_{QGP} \pi^2 / 90 \)]

and gives:

\[ \epsilon (\tau) = \epsilon_0 \left( \frac{\tau_0}{\tau} \right)^{1+\lambda} \] \hspace{1cm} (56)

\[ T (\tau) = T_0 \left( \frac{\tau_0}{\tau} \right)^{\lambda} \] \hspace{1cm} (57)

\[ s (\tau) = s_0 \frac{\tau_0}{\tau} \Rightarrow \tau s (\tau) = \tau_0 s_0 \] \hspace{1cm} (58)
From observables to $s_0$ and $\epsilon_0$ I

The volume element of the freeze-out hyper-surface

$$\left(\pi R^2\right) \tau_f dY$$

with the number density of particles $n_f$

$$\frac{dN}{dy} = \pi R^2 \tau_f n_f$$

and the relation (non-interacting particles)

$$s = \xi n \quad \xi = 3.6(\text{bosons}) \text{ or } 4.2(\text{fermions})$$

$$s = \frac{\xi}{\pi R^2 \tau} \frac{dN}{dy} \quad \xrightarrow{s \tau = s_0 \tau_0} \quad s_0 = \frac{\xi}{\pi R^2 \tau_0} \frac{dN}{dy}$$
From observables to $s_0$ and $\epsilon_0$ II

Similarly

$$\frac{dE}{dy} = \pi R^2 \epsilon_0 \tau_0 \left( \frac{\tau_0}{\tau_f} \right)^\lambda \quad (63)$$

$$\Rightarrow$$

$$\epsilon_0 = \frac{1}{\pi R^2 \tau_0} \frac{dE}{dy} \left( \frac{\tau_f}{\tau_0} \right)^\lambda \quad (64)$$

Free streaming:

$$\epsilon_0 = \frac{1}{\pi R^2 \tau_0} \frac{dE}{dy} \quad (65)$$

so $\left( \frac{\tau_0}{\tau_f} \right)^\lambda$ is related to the processes during the hydrodynamic expansion.
Elliptic flow 1

Euler equation for $v_x$:

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\epsilon + P} \frac{\partial P}{\partial x} = -c_s^2 \frac{\partial \ln s}{\partial s}, \quad c_s^2 = \frac{\partial P}{\partial \epsilon}$$  \hspace{1cm} (66)

assuming a gaussian entropy profile

$$s(x, y) \propto \exp \left( -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} \right)$$  \hspace{1cm} (67)

leads to

$$v_x = \frac{c_s^2 x}{\sigma_x^2} t \quad \text{and} \quad v_y = \frac{c_s^2 y}{\sigma_y^2} t$$  \hspace{1cm} (68)
Elliptic flow II

non-central collision:

\[ \sigma_x < \sigma_y \quad \Rightarrow \quad \langle v_x^2 \rangle > \langle v_y^2 \rangle \quad (69) \]

⇒ more particles emitted parallel to the x axis

\[ \frac{dN}{d\phi} \propto 1 + 2v_2 \cos 2\phi \quad (70) \]
Summary

Relativistic Hydrodynamics

▶ strong assumption of local equilibrium
▶ gives set of equations
▶ simple in the case of an ideal fluid
▶ needs initial condition

Bjorken model

▶ assuming “Plateau”
▶ simplifies the equations
▶ allows to estimate the initial conditions

In 2 weeks:
Experimental investigations of relativistic hydrodynamics and the ideal fluid scenario at RHIC
References: