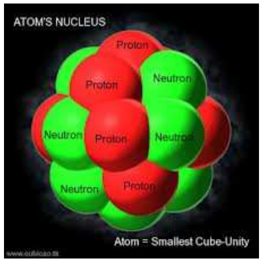
Relativistic Hydrodynamics and Heavy Ion Collisions

- Studying the strong interactions in a new regime
- Hydrodynamics for hot strongly-interacting matter
- Shear viscosity and its role
- Rigorous definitions of hydro coefficients
- A (new) limit on how small shear viscosity can be
- Conclusions

1932: nucleus of protons, neutrons
bound by a strong force



1940's to 1960's: more strongly-interacting particles found

$\pi^{\pm}, \pi^0, K^{\pm}, K^0_{l,s}, \eta, \rho^{\pm 0}, \phi, \ldots$ n-spin "Mesons"

$p, n, \Lambda^0, \Delta^{++,+,0,-}, \Sigma^{\pm 0}, \Xi, \Omega, \ldots$ $n+\frac{1}{2}$ spin, conserved #: "Baryons"

1970's: All originate from substructure.

- Quarks: $uds(cbt)$ spin-$\frac{1}{2}$ charge $+\frac{2}{3}, -\frac{1}{3}$ and 3 colors
- Gluons: stick quarks together into $q\bar{q}$ or qqq states

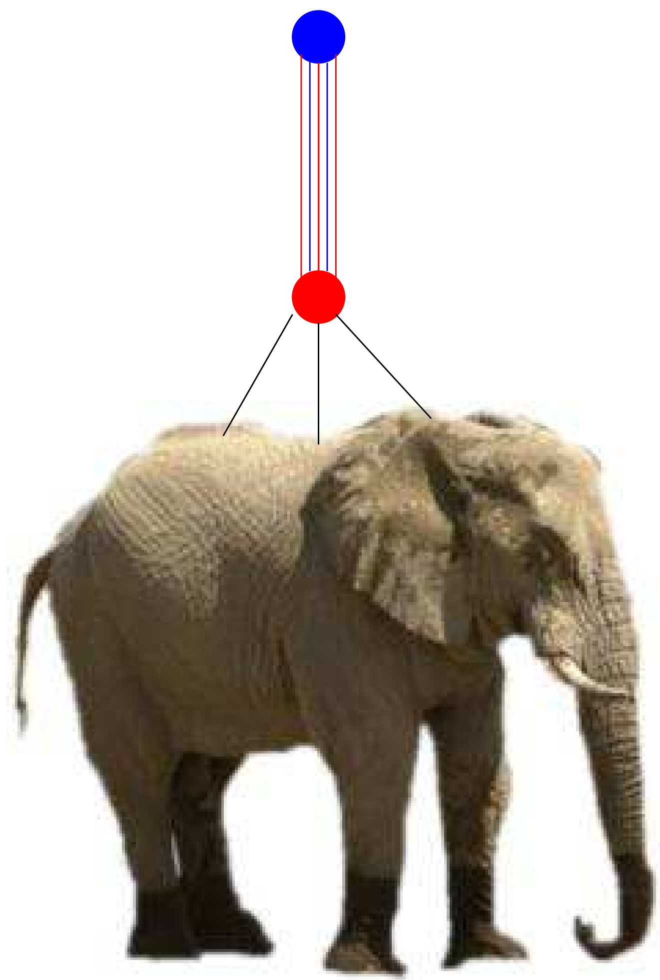
Charges, spins and masses explained

Linear confinement

Electromagnetism: force falls with distance

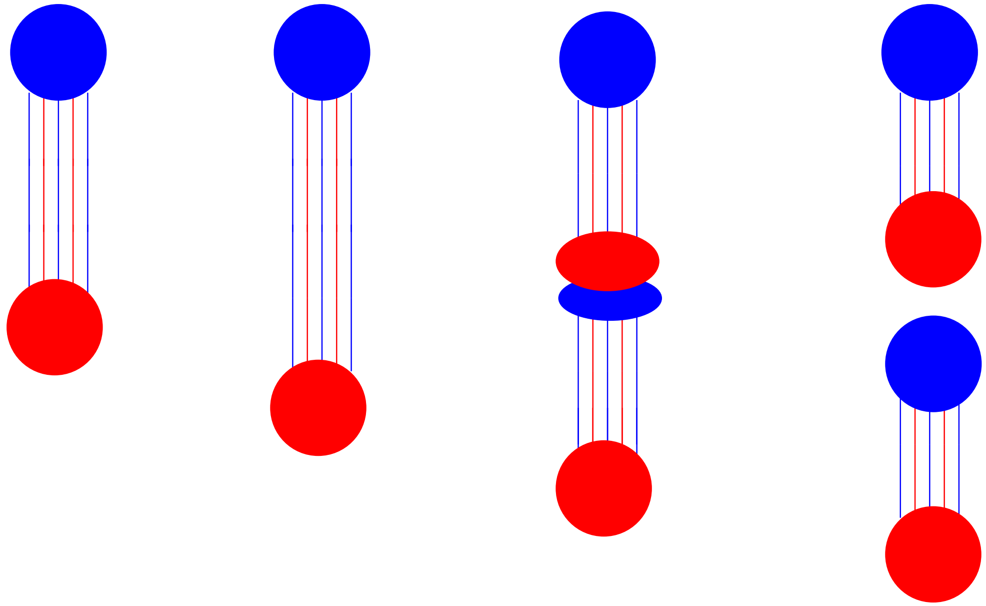
$$F = \frac{e_1 e_2 \hat{r}}{4\pi r^2}$$

Strong force: beyond 10^{-15}m, independent of distance between quark and antiquark. Anchor q, hang something from "rope" between it and $\bar{q}$:



Strong force between $q, \bar{q}$ enough to hold up an elephant!

What would really happen . . . .



Potential Energy starts to exceed mc^2 to make $q\bar{q}$ pair. Pair spontaneously forms, preventing need for "long string" connecting original $q, \bar{q}$. *hadronization*

confinement, and impossibility to see isolated quark.

Asymptotic freedom

At short distance $r \ll 10^{-15}$m, $1/r^2$ law:

$$\text{inter-gluon force} \quad F_{q\bar{q}} \sim \frac{g^2}{4\pi r^2}\,, \quad \text{except} \quad g^2 = g^2(r)$$

at small r, $g^2(r)$ gets small $g^2/4\pi\hbar c \ll 1$

Potential energy $V \sim g^2(r)/4\pi r$.

Heisenberg: $p \geq \hbar/r$. Relativity $E \simeq cp \geq \hbar c/r \gg V$

Short distances: Kinetic Energy $\gg V$. "Free particles"

and lets us compute with "perturbation theory"

Heavy ion collisions

Can we hope to see this free-particle behavior?

Heat nuclear matter hot enough and you should.

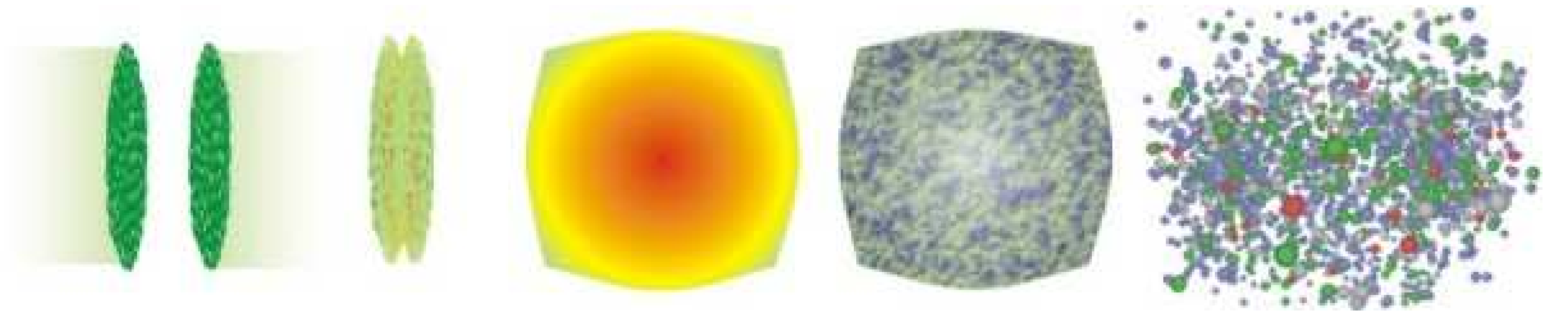
Relevant temperature $\frac{\hbar c}{10^{-15}\mathrm{m}} = 200MeV= 2.4 \times 10^{12}$Kelvin.

Only way to get there is to accelerate nuclei to very high energy and collide them. *heavy ion collisions*

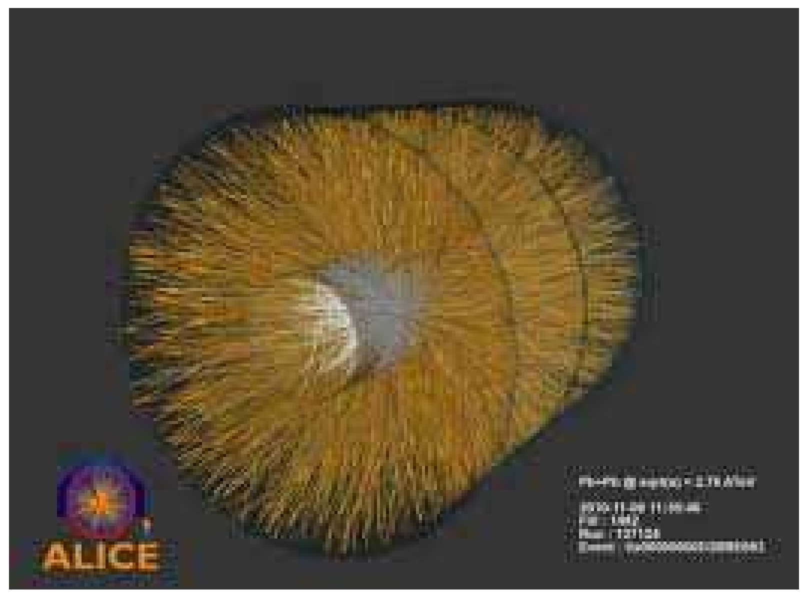
Bad news: most energy in each nucleus goes through the other. But we still get high T: LHC $(1.38 \times 10^6$MeV per nucleon) gets to $T \sim 500$MeV.

Achieving a new state of matter?!

LHC collides lead nuclei ($82p + 126n = 208$ nucleons)

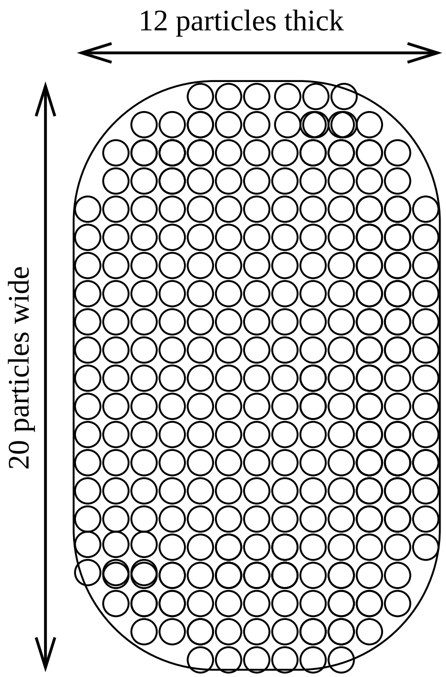


leading to 3200 charged,
> 1600 neutral particles
between $\theta = 40^\circ$ and
$\theta = 140^\circ$ $(-1 < \eta < 1)$



Each n, p gets "torn open," spilling out many $g, q, \bar{q}$ inside

Hot ball of 5000 excitations

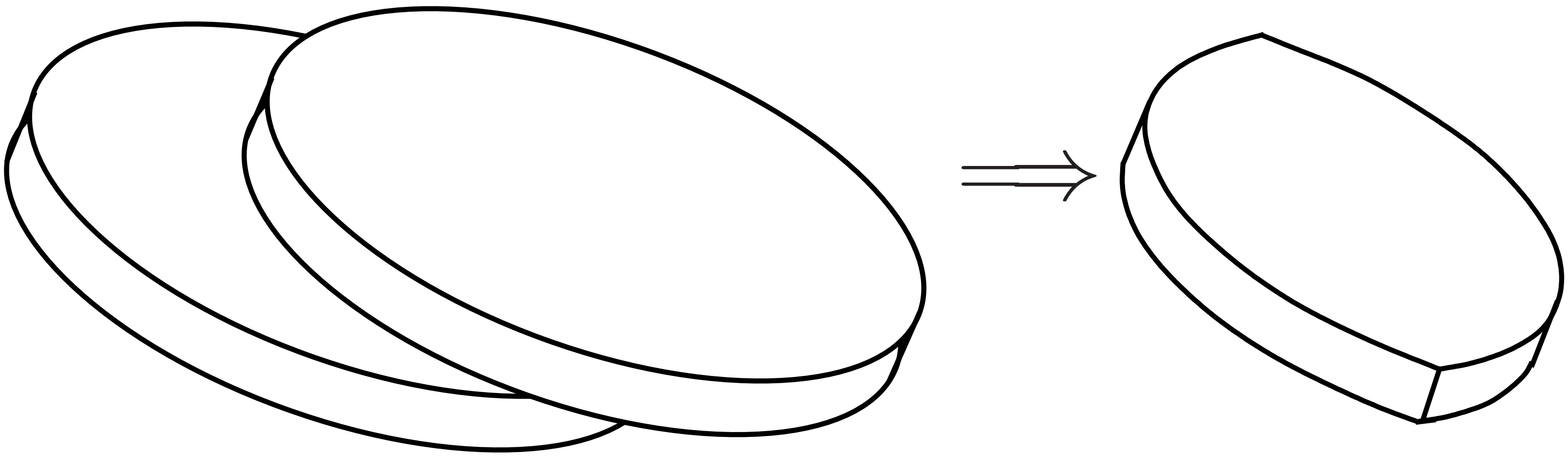


5000 excitations is around $20 \times 20 \times 12$ across. Enough to show collective or "fluid" behavior?

Hydrodynamics: Many "subsystems" big enough for *local* equilibration in each (Different regions with different $T, \vec{v}$,...). Not obvious but plausible

Testing for local equilibration

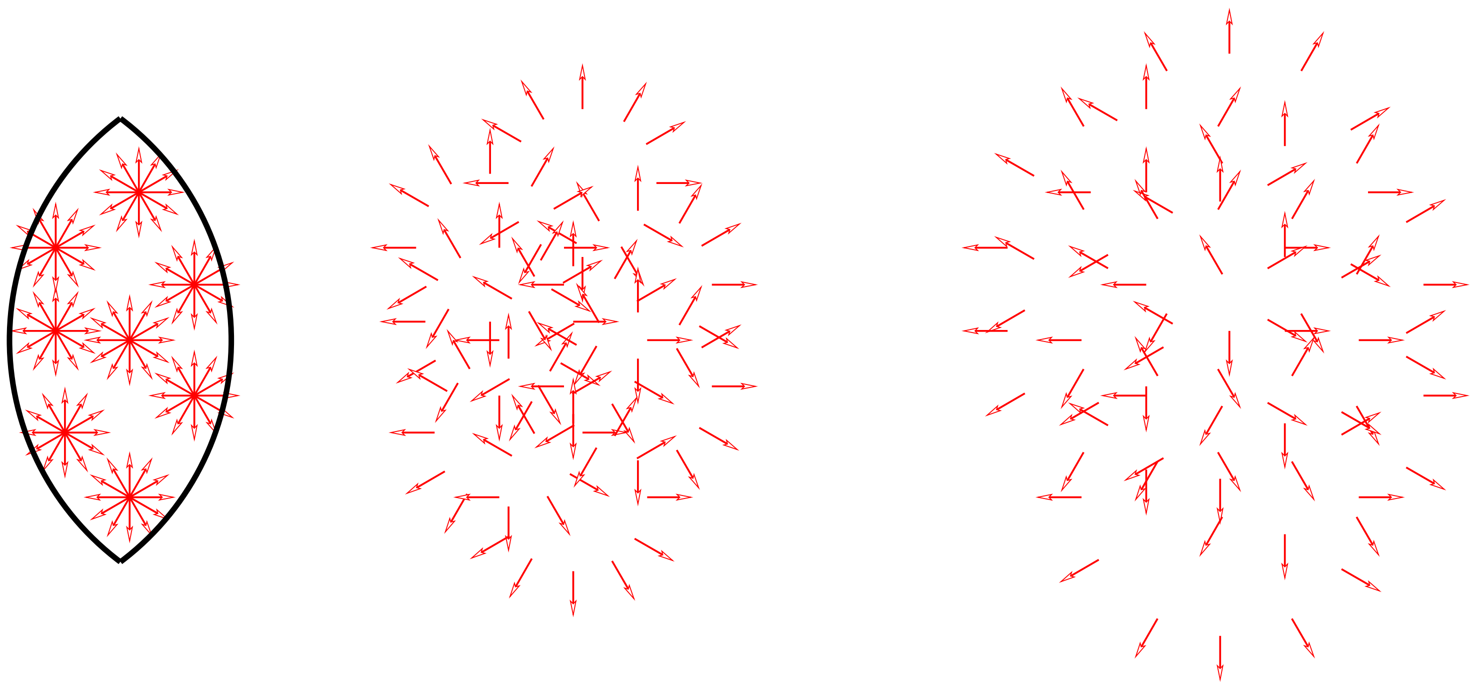
Nuclei generically strike off-center



leading to irregular shaped region of plasma

"Almond sliver" with long axis, short axis, and very short initial thickness along beam direction.

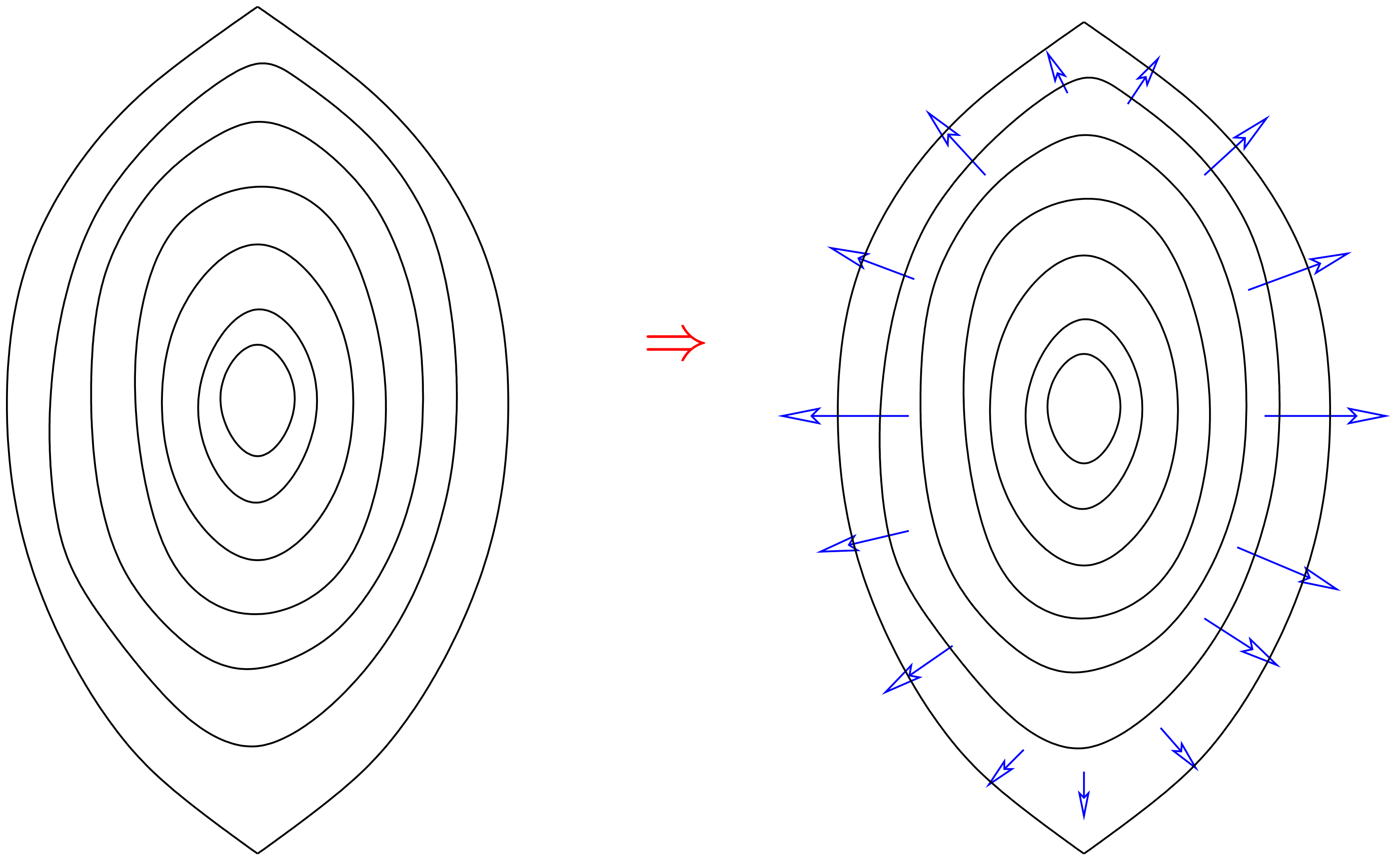
Behavior IF no re-interactions (transparency)



Just fly out and hit the detector.

Detector will see xy plane *isotropy*

local CM motions



Pressure contours Expansion pattern

Anisotropy leads to anisotropic (local CM motion) flow.

Free particle propagation:

- System-average CM flow velocities $\langle v^2_{x,\mathrm{CM}} \rangle > \langle v^2_{y,\mathrm{CM}} \rangle$
- Must have local CM $\langle p^2_x \rangle < \langle p^2_y \rangle$ so total $\langle p^2_x \rangle = \langle p^2_y \rangle$

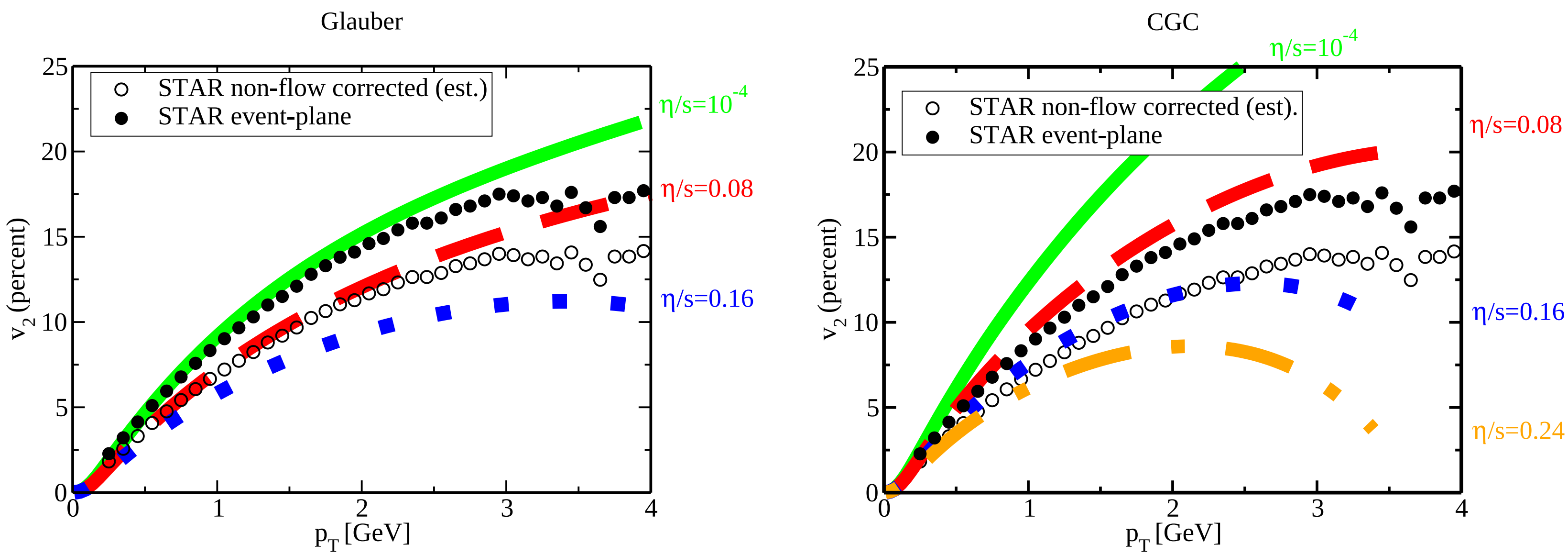
Efficient re-interaction:

- System-average CM flow still has $\langle v^2_{x,\mathrm{CM}} \rangle > \langle v^2_{y,\mathrm{CM}} \rangle$
- system changes *locally* towards $\langle T^{xx}_{\mathrm{local\ CM}} \rangle = \langle T^{yy}_{\mathrm{local\ CM}} \rangle$
- Adding these together, $\langle T^{xx}_{\mathrm{tot,labframe}} \rangle > \langle T^{yy}_{\mathrm{tot,labframe}} \rangle$

Net "Elliptic Flow" $v_2 \equiv \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2}$ measures re-interaction

Measured elliptic flow vs. theory fits

Hydrodynamic fits – based on assuming much rescattering



Luzum Romatschke 0804.4015, data STAR min-bias

Elliptic flow, differential in particle transverse momentum.

Two guesses at initial conditions (left and right),

Perfect rescattering (top) vs incomplete re-scattering (lower)

Perfect Rescattering: Ideal Hydrodynamics

System in local equilibrium. List all conserved quantities:

$$E, \quad \vec{p}, \quad Q_{\rm el}, \quad B \text{ [baryon number]}$$

Define local densities e, π, ρ, n, space varying. Local properties fixed by Equation of State:

$$-\Omega = P = P(e, \pi, \rho, n)$$ Only conserved quantities determine equilibrium.

Use Ω, thermodynamics to find conserved currents:

$$T^{\mu\nu}, \quad J^{\mu}_{Q}, \quad J^{\mu}_{\rm B}$$

Current conservation equations: 1 condition per unknown.

Ideal Hydrodynamics

Relativity: write $(e, \pi) = \frac{\varepsilon}{\sqrt{1-v^2/c^2}}(u^0, \vec{u})$: u^μ flow 4-vector,

$u^0 = \frac{1}{\sqrt{1-v^2/c^2}}$, $\vec{u} = \frac{\vec{v}/c}{\sqrt{1-v^2/c^2}}$

At rest, $T_{00} = \varepsilon$ and $T_{ij} = P\delta_{ij}$. Relativity:

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

with $g^{\mu\nu}$ the metric tensor. Conservation:

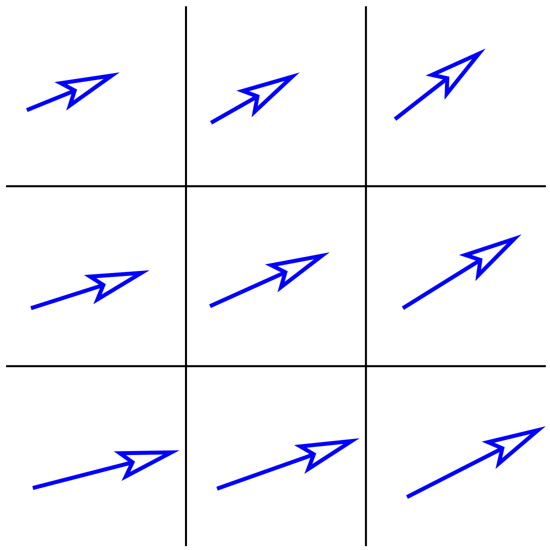
$$\nabla_\mu T^{\mu\nu} = 0$$

small $\vec{v}/c$: turns into usual Euler fluid eq.

Nonideal Hydro

Each region feels information about neighboring regions diffusing across its boundary.

$\vec{v}$ nonuniformity means nonvanishing $\nabla_i v_j$ which will influence center region (diffusion of information)

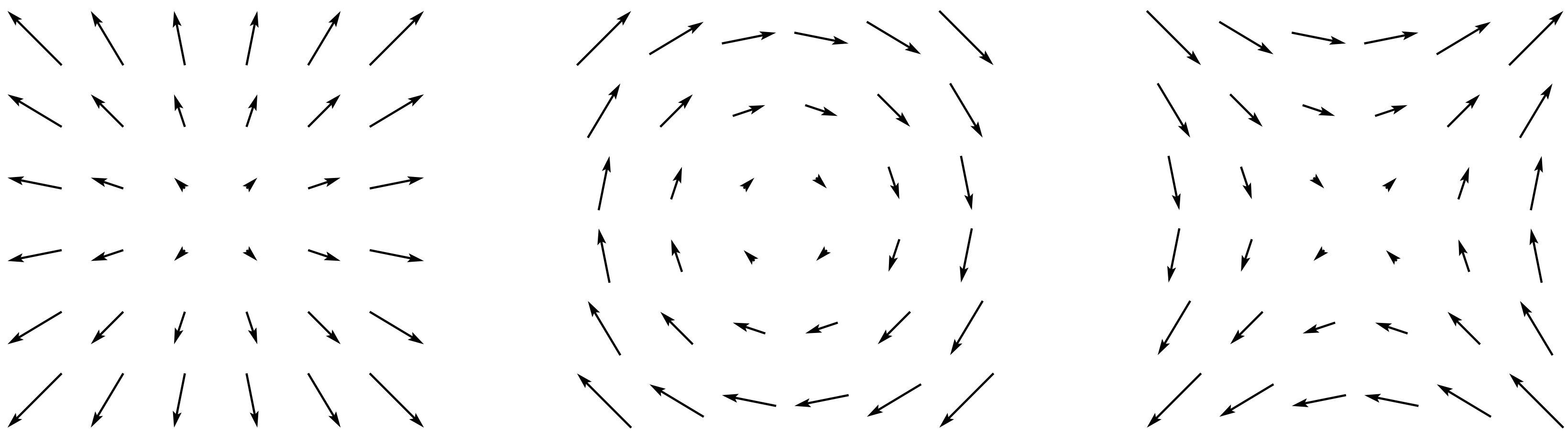


Decompose: scalar, antisymm, traceless symm tensor

$$\nabla_i v_j = \frac{\delta_{ij}}{3}\nabla\cdot v + \frac{1}{2}(\nabla_i v_j - \nabla_j v_i) + \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i - \frac{2\delta_{ij}}{3}\nabla\cdot v\right)$$

What each tensor piece means

$\delta_{ij}\nabla\cdot v$ $\quad$ $\nabla_i v_j - \nabla_j v_i$ $\quad$ $\nabla_i v_j + \nabla_j v_i - ..$



Divergence $\quad$ Vorticity $\quad$ Shear flow

scalar divergence can change scalar pressure $P \Rightarrow P_{\text{equil.}} - \zeta\nabla\cdot v$

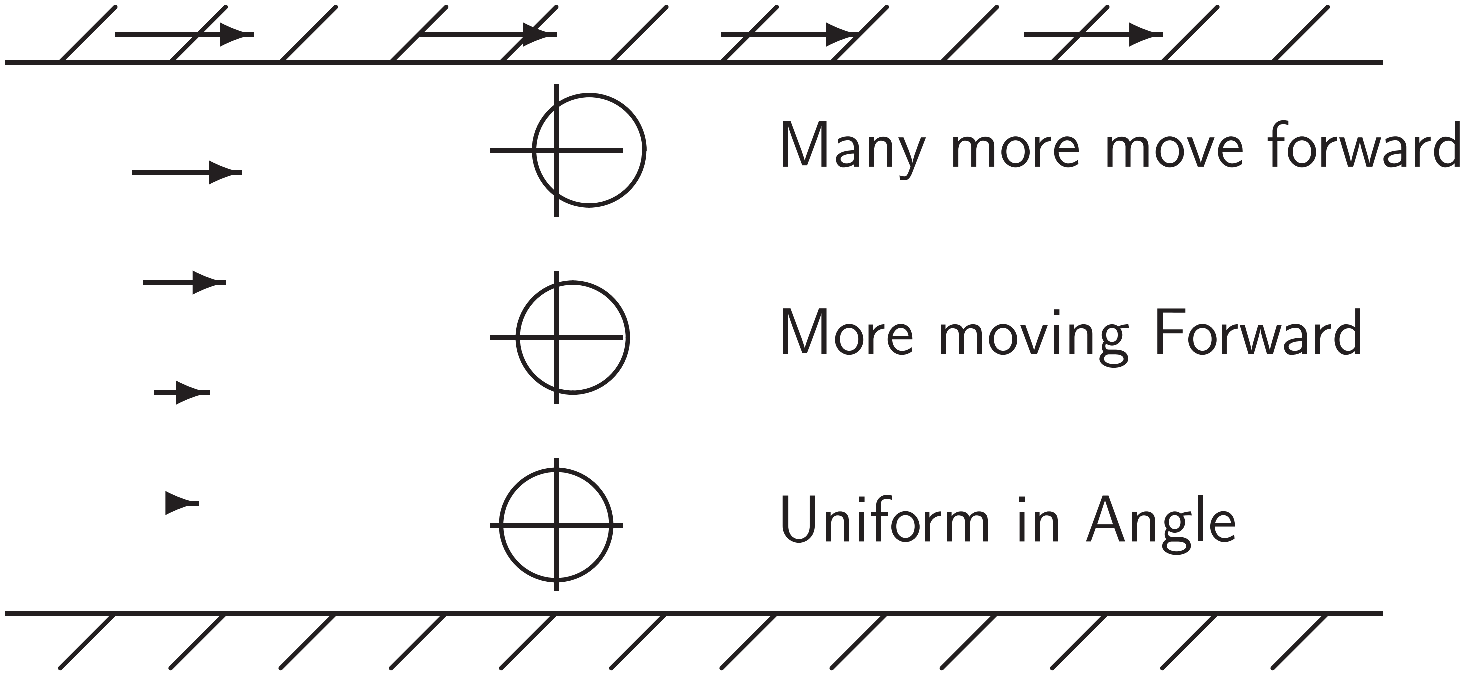
symm. tensor shear flow can change symm. tensor stress tensor

$T_{ij} \Rightarrow T_{ij,\text{equil.}} - \eta(\nabla_i v_j + \nabla_j v_i - ..)$

pseudovector vorticity cannot change either

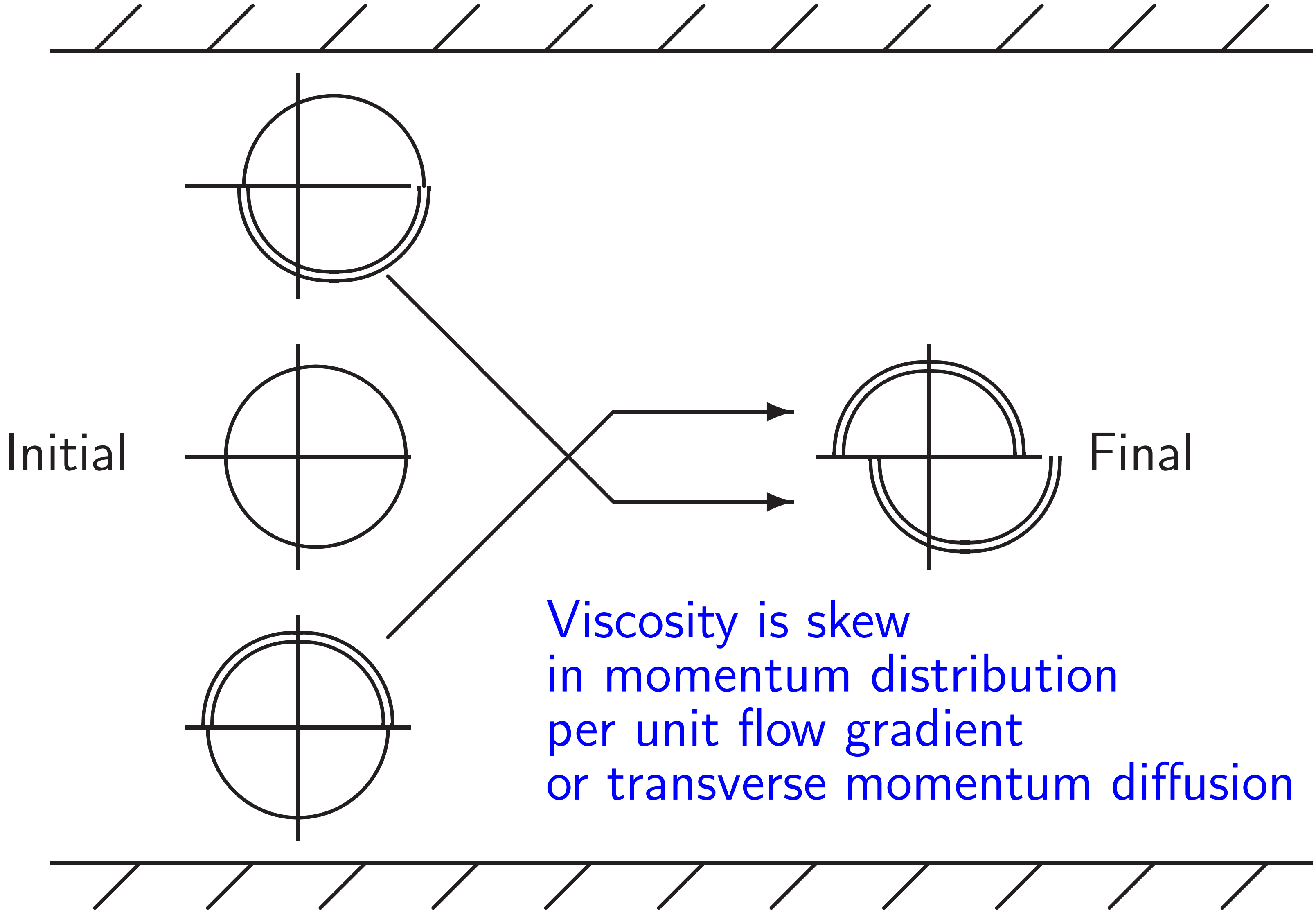
What is shear viscosity?

Consider non-uniform fluid flow of *particles*:



(Plotting ϕ=direction, $r = \#$ particles in that ϕ)

Propagation by Mean Free Path



Viscosity is skew
in momentum distribution
per unit flow gradient
or transverse momentum diffusion

How big do I expect viscosity to be?

MKS: $\eta \sim \frac{\text{Force/Area}}{dv/dt} \sim \frac{300\,\mathrm{MeV} \times 10^{-24}\,\mathrm{s}}{(10^{-15}\,\mathrm{m})^3} = 5 \times 10^{10}\,\mathrm{Pa\,s}$ Huge!

"Specific" viscosity $\eta/\hbar s$ should be ~ 1 Water STP: $\eta/\hbar s = 33$

Hi-T: nearly-free q, g:

kinetic-theory calculation

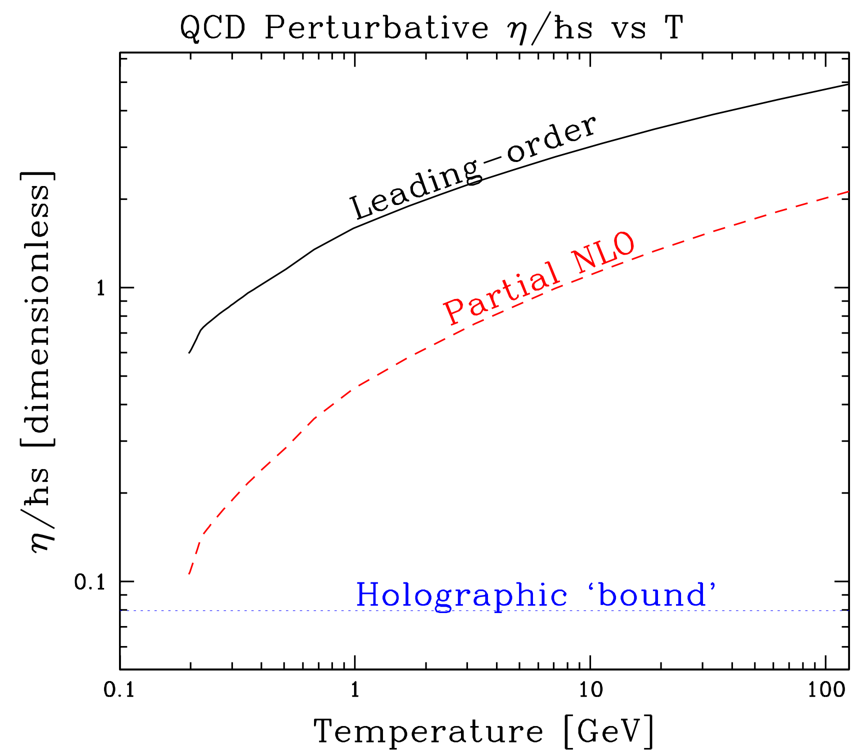
Leading-order Arnold,GM,Yaffe 2003

partial next-order GM, to appear

η/s small at 200MeV

but pert. expansion

shows poor convergence



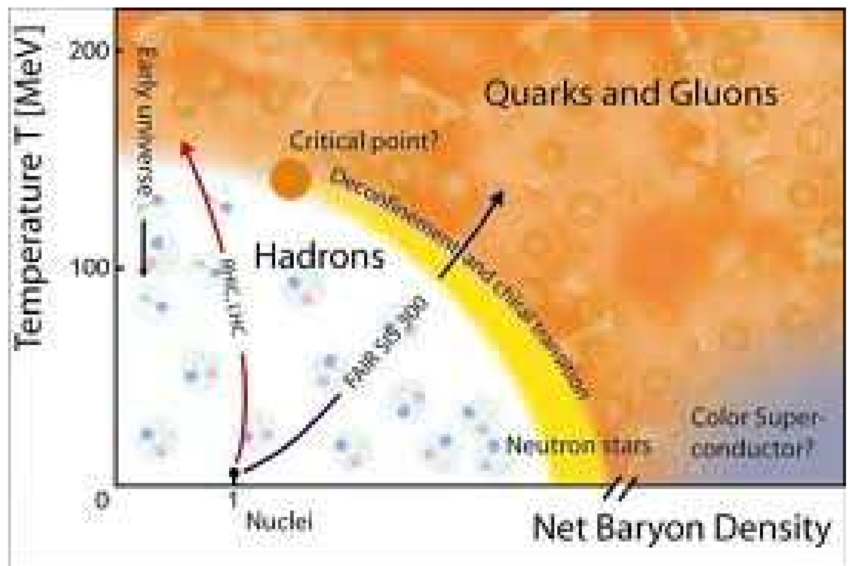
And bulk viscosity?

High-T: perturbatively ζ/s very small Arnold Dogan GM, hep-ph/0608012

Phase transition: $\zeta \to \infty$.

$near$ transition, ζ grows in predictable way based on universality arguments

Hohenberg Halperin '77, Onuki '97



If QCD has transition line, 2-order endpoint in Ising class

H-dynamic universality class Son Stephanov hep-ph/0401052

and ζ diverges as $t^{-z\nu+\alpha} \simeq t^{-2}$ GM and Saremi 0805.4201

Could be important near 150MeV but not as well explored as η

Hydrodynamic Modeling of Heavy Ion Collisions

Navier-Stokes: $\nabla_\mu T^{\mu\nu} = 0$ but with

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P - \zeta \nabla \cdot v)(g^{\mu\nu} + u^\mu u^\nu) - \eta\, \mathrm{shear}^{\mu\nu}$$

Problems!!!

- Acausal: shear viscosity is transverse momentum diffusion. Diffusion $\partial_t P_\perp \sim \nabla^2 P_\perp$ has instantaneous prop. speed. Müller 1967, Israel+Stewart 1976
- Unstable: $v > c$ prop + non-uniform flow velocity $\rightarrow$ propagate from future into past, exponentially growing solutions. Hiscock 1983

Numerical implementations really face these problems

Israel-Stewart approach

Add one second order term: Israel Stewart 1976

$$T^{\mu\nu} = T^{\mu\nu}_{\text{equil.}} + \Pi^{\mu\nu}\,, \quad \Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\pi\, u^\alpha\partial_\alpha\sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \to -\Pi$ in order-2 term:

$$\tau_\pi\, u^\alpha\partial_\alpha\Pi^{\mu\nu} \equiv \tau_\pi\,\dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

Relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$. "Telegraph Equation"

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_π)

But why only one 2'nd order term???

Second order hydrodynamics

It is more consistent to include all possible 2'nd order terms.

Simplifying assumptions: 5 possible terms Baier *et al*, [arXiv:0712.2451]

$$\Pi^{\mu\nu}_{2\text{ ord.}} = \eta\tau_\pi \left[u^\alpha\partial_\alpha\sigma^{\mu\nu} + \frac{1}{3}\sigma^{\mu\nu}\partial_\alpha u^\alpha\right] + \lambda_1 \left[\sigma^\mu_\alpha \sigma^{\nu\alpha} - (\text{trace})\right]$$
$$+\lambda_2 \left[\frac{1}{2}(\sigma^\mu_\alpha\Omega^{\nu\alpha} + \sigma^\nu_\alpha\Omega^{\mu\alpha}) - (\text{trace})\right]$$
$$+\lambda_3 \left[\Omega^\mu{}_\alpha\Omega^{\nu\alpha} - (\text{trace})\right] + \kappa\left(R^{\mu\nu} - \ldots\right)\,,$$

$\Omega_{\mu\nu}$ vorticity, $\sigma_{\mu\nu}$ shear flow

Starting point for most numerical studies

What are these coefficients?

Rigorously defined in terms of "Kubo formulae"

Example: diffusion of particle number.

If number density ρ, chem. potential μ nonuniform $\vec{\nabla}\mu \neq 0$, particles should "flow" from large-μ to small-μ.
Flow $\vec{j}$ must be related to $\vec{\nabla}\mu$.

Kubo: balance $\vec{\nabla}\mu$ with compensating (fictitious) $\vec{E}$-field.
Response to $\vec{\nabla}\mu$ **same** as response to $\vec{E}$.
Easier to consider μ constant, $\vec{E} \neq 0$, and study $\vec{j} = \sigma\vec{E}$.

Now do Quantum Mechanics with this fake $\vec{E}$-field

Consider state $|\psi\rangle$, evolves in modified $H = H_0 + H_I$
with $H_I = \int d^3x \vec{j} \cdot \vec{A} \quad (\vec{E} = d\vec{A}/dt)$

Evaluate $\vec{j}$ after E has been "turned on"

$$\begin{aligned} \langle \vec{j}(t) \rangle &= \langle \psi |\, e^{i \int_0^t H_I dt'} \;\; \vec{j} \;\; e^{-i \int_0^t H_I dt'} \, | \psi \rangle \\ &\simeq \int_0^t dt' \int d^3x A_i(t') \;\; \langle \psi |\, i \Big[j_i(x,t') \, , \, \vec{j}(t) \Big] \, | \psi \rangle \end{aligned}$$

Conductivity, particle diffusion determined by a *retarded* correlation function of the current with itself, *in equilibrium*

Fluctuation-dissipation: dissipative σ determined by unequal-time fluctuations

$\nabla_i n$ vector, current $\vec{j}$ couple to E-field

$\nabla_i v_j$ tensor, stress T_{ij} couple to *spacetime curvature* h_{ij}

Viscosity determined by stress-stress correlator

$$\eta = \frac{d}{d\omega} \int^t dt' e^{-i\omega(t-t')} \int d^3x \, \langle\psi| \, i\Big[T_{xy}(x,t') \, , \, T_{xy}(t,0) \Big] \, |\psi\rangle$$

All higher-order coefficients $\tau_\pi, \kappa, \lambda_1, \lambda_2, \lambda_3$ defined in terms of similar correlators, with more derivatives and (some terms) with more stress tensors GM Sohrabi PRL 2011

$\lambda_{1,2,3}$ involve $\displaystyle\int^t dt_1 \int^{t_1} dt_2 \, \langle\psi| \, \Big[T_{xy}(t_2) \, , \, [T_{xz}(t_1) \, , \, T_{yz}(t)] \Big] \, |\psi\rangle$

Values of these "extra" coefficients

Same problems computing them as computing η.

But certain dimensionless ratios may be more robust

Hi-T using kinetic theory York and GM 0811.0729

Related strongly-coupled theory $\mathcal{N}=4$ SYM, two groups, 0712.2451, 0712.2456

Ratio	QCD value	SYM value
$\frac{\eta\tau_\pi(\varepsilon+P)}{\eta^2}$	5 to 5.9	2.6137
$\frac{\lambda_1(\varepsilon+P)}{\eta^2}$	4.1 to 5.2	2
$\frac{\lambda_2(\varepsilon+P)}{\eta^2}$	-10 to -11.8	-2.77
$\frac{\kappa(\varepsilon+P)}{\eta^2}$	small	4
$\frac{\lambda_3(\varepsilon+P)}{\eta^2}$	small	0

Is second-order hydro consistent?

Hydrodynamics predicts long-lived shear,sound modes.
solving Navier-Stokes, dispersion of these modes:

$$\omega_{\rm shear} = i\frac{\eta}{\varepsilon + P}k^2\,, \qquad \omega_{\rm sound} = v_{\rm sound}k + i\frac{2\eta}{3(\varepsilon + P)}k^2$$

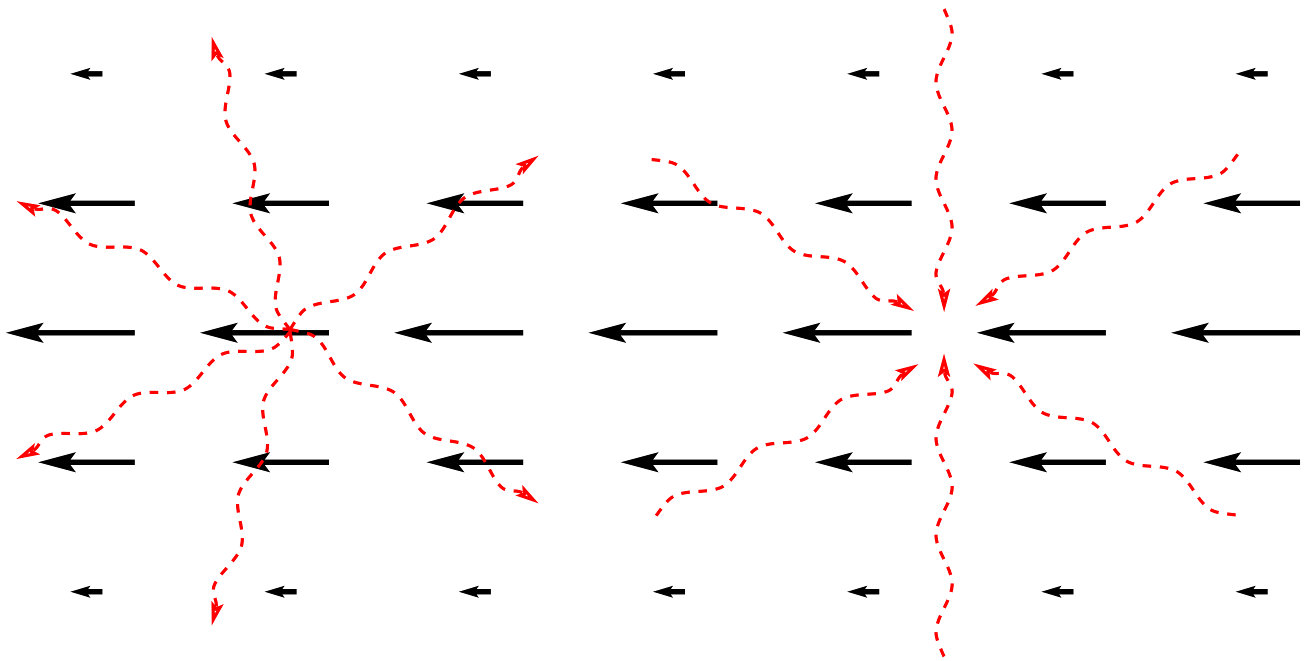
Equivalently, lifetime $\tau \sim \frac{\varepsilon+P}{\eta}\lambda^2$ grows as λ^2

Hydro modes propagate long distances, survive long times
Imperils *hidden assumption* that equilibration is fast, local process (allowed expansion in derivatives)

Causes *long time tails*, well known in gas kinetics community

Hydro Waves Contribute to Viscosity!

Consider shear flow:



Sound waves transport x-momentum in y-direction, out of flowing region. Transport of x-momentum in y-direction *is* viscosity! Effects are calculable, Kovtun GM Romatschke 1104.1586

Naive estimate: sound waves as "particles"

"stumbling drunk" with step length λ, time between steps $t = \lambda/c$: diffusion constant is $\lambda^2/t \sim c\lambda$.

Mean free path of a sound wave $\lambda \sim (\epsilon + P)/(\eta k^2)$.

Equipartition: Energy $T/2$, Mom. $T/2c$ per sound mode

Contribution of sound modes to momentum diffusion

$$\eta_{\rm from\,soundwaves} \sim \int d^3k\, T\lambda \sim \int d^3k \frac{T(\epsilon + P)}{k^2 \eta}$$

Finite. Time scale to establish this contribution:

$$\eta\tau_\pi \sim \int d^3k\, T\lambda t \sim \int d^3k \frac{T(\epsilon + P)^2}{k^4 \eta^2} \qquad \text{Divergent!!}$$

More detailed computation

Contribution to viscosity

$$\eta_{\rm new} \simeq \frac{17}{240\pi^2} \frac{T^3}{\hbar^2 c^3} \left(\frac{\hbar s}{\eta} \right)^2$$

Irrelevant if $\eta/\hbar s$ large or $s \gg (T/\hbar c)^3$

But grows as η gets small, setting *floor* on η.

For QCD, that floor is around $\eta/\hbar s = 0.1$.

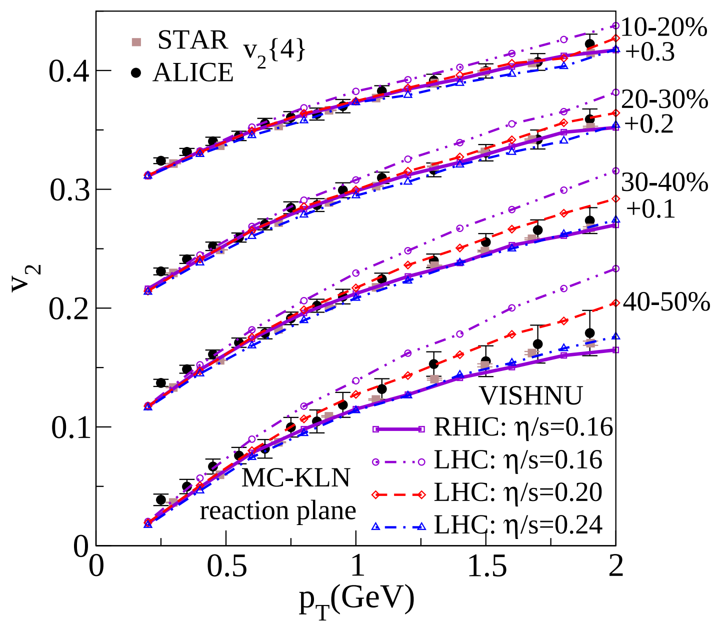
Stress-stress correlator is non-analytic. Higher-order coefficients $\tau_\pi, \ldots$

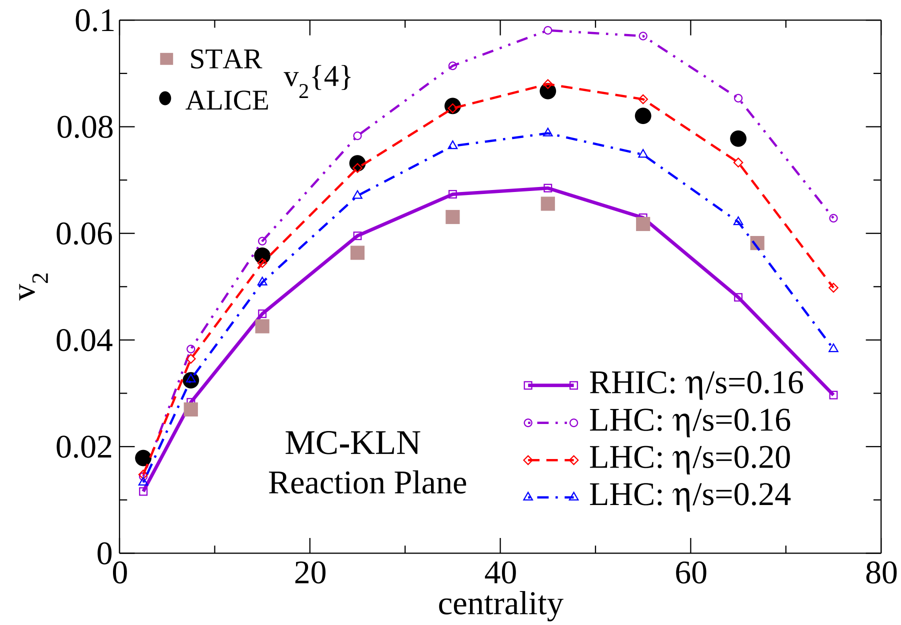
scale-dependent Wilsonian coefficients. Below some scale they show power-law

scale dependence. 2-order hydro is OK if that scale is long – OK if $\eta/\hbar s = 0.2$

but not if $\eta/\hbar s = 0.1$.

Comparison of Hydro with Experiment





Several groups here Song Bass Heinz 1103.2380 are doing hydro fits.

$\eta/\hbar s \sim 0.1$—0.2 depending on initial condition model

More observables (v_3, v_4, p_T, centrality) may constrain initial condition model. Active field of research!

Conclusions

- Hydro seems sensible framework in heavy ion coll.
- Need 2'nd order Hydro, 6 hydro coefficients!
- Pert. computation of 2'nd order Hydro: dim'less ratios same order as $\mathcal{N}{=}4$SYM, differ in detail
- Kubo relations for nonlinear coefficients found.
  κ, λ_3 special (really thermodynamic)
- Hydro waves contribute to hydro coefficients!
- Self-consistency issues if η too small, and very low freq.