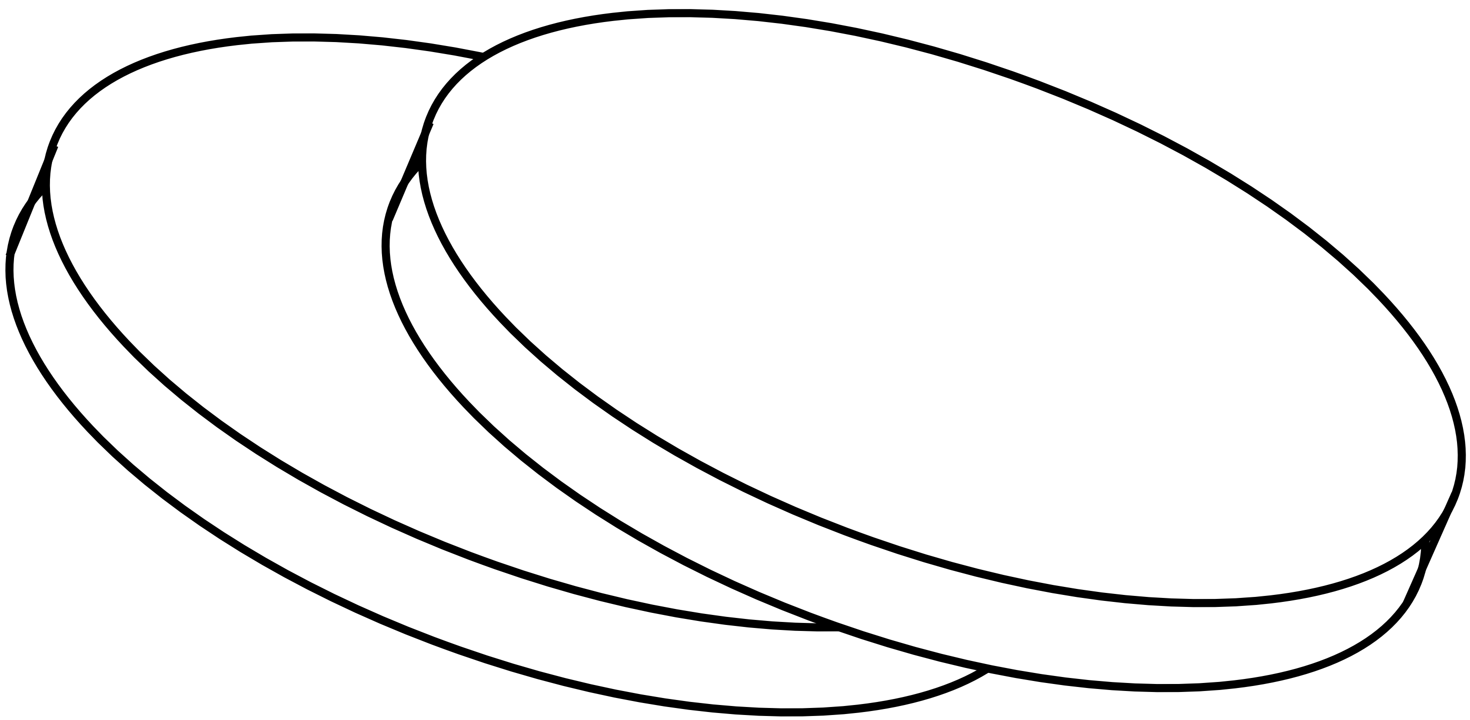
Analytic Structure of $T_{xy}T_{xy}$ at High-T

- Reminder: heavy ion collisions
- Reminder: hydrodynamics
- Issue of analytic structure
- Kinetic theory and collision operators
- Poles vs Cuts with incomplete information

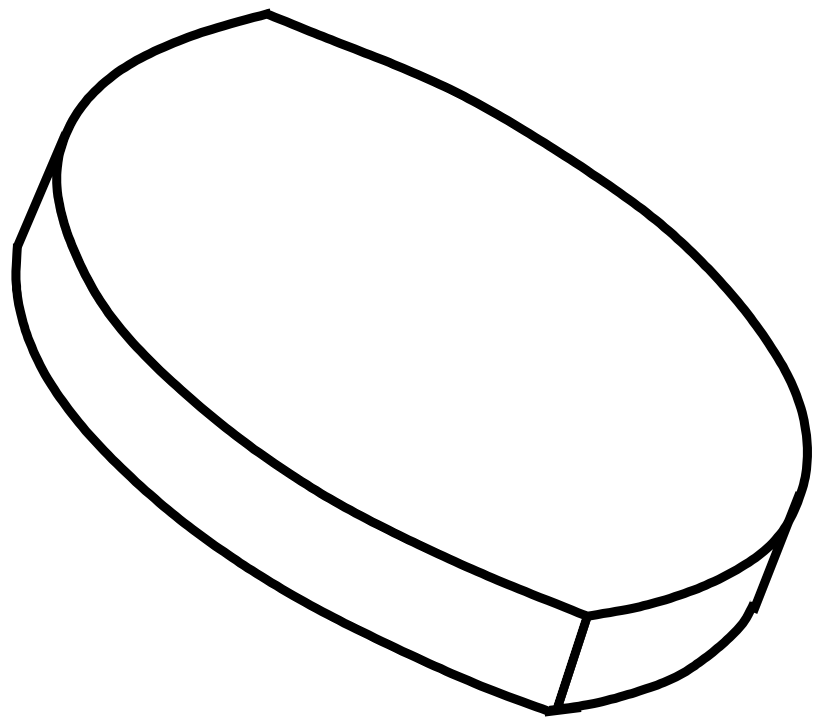
Heavy ion collisions

Accelerate two heavy nuclei to high energy, slam together.



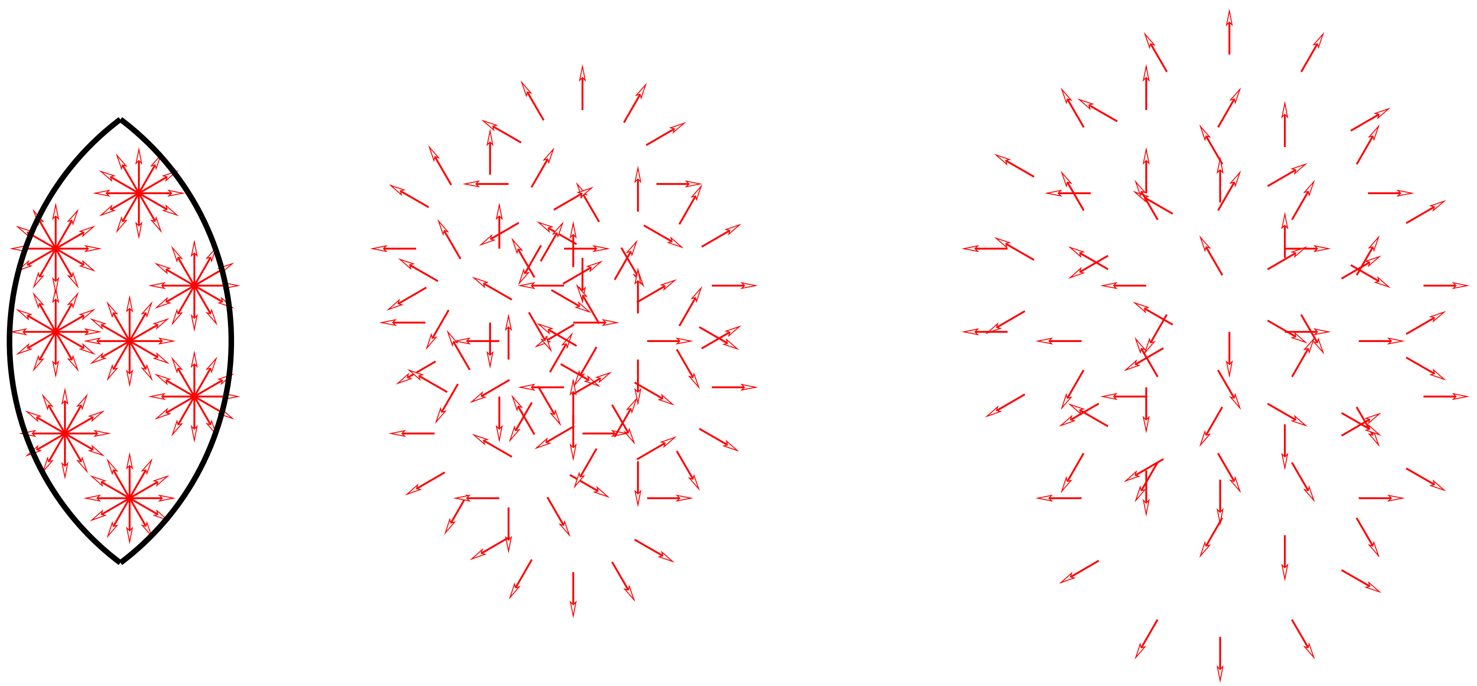
Just before: Lorentz contracted nuclei

After the scattering: region where nuclei overlapped:
"Flat almond" shaped region of $q, \bar{q}, g$ which scattered.



$\sim$10 thousand random $\mathbf{v}$ quarks+gluons

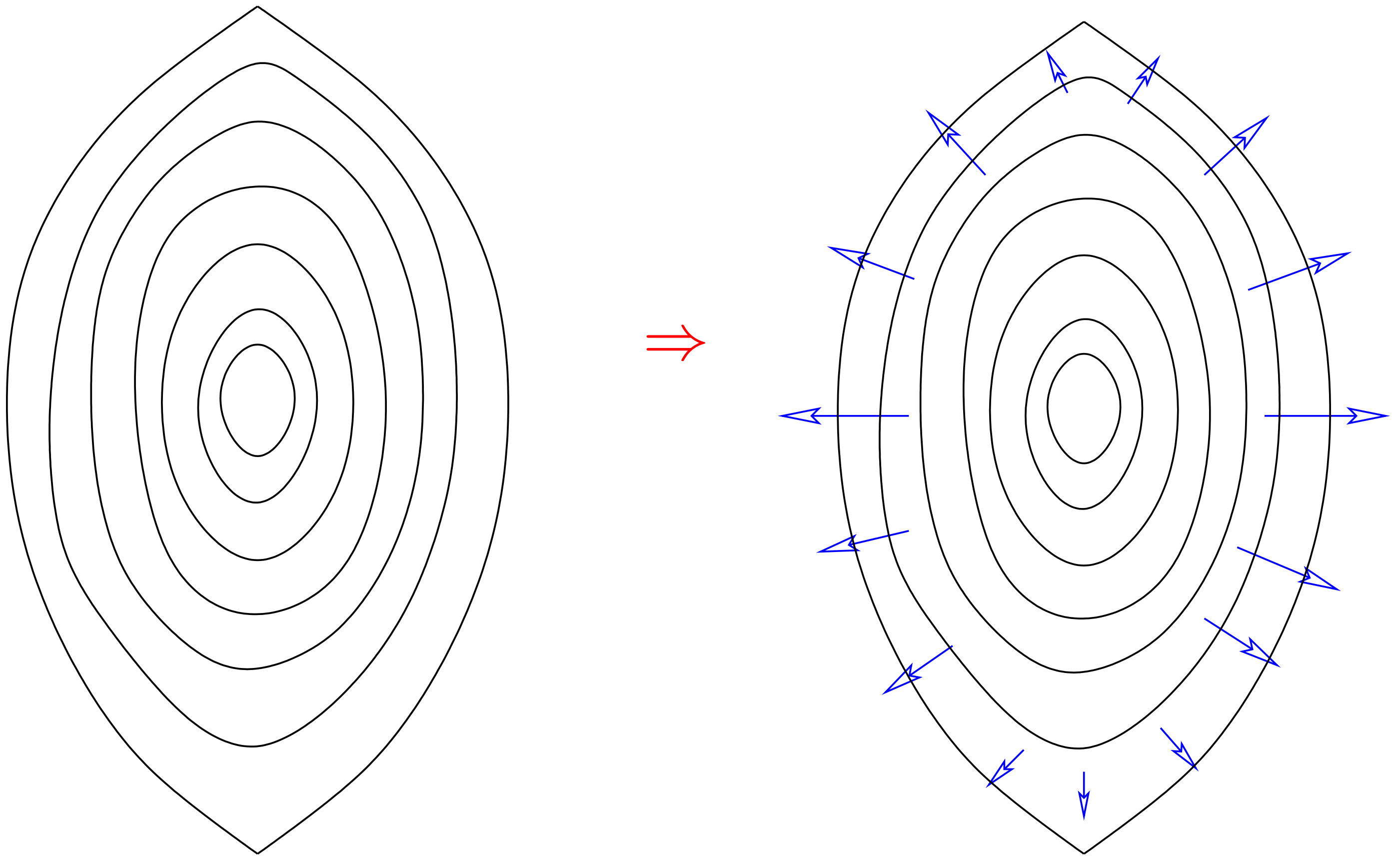
Behavior IF no re-interactions (transparency)



Just fly out and hit the detector.

Detector will see xy plane *isotropy*

Opposite limit: Ideal Hydrodynamics



Pressure contours Expansion pattern

Anisotropy $\to$ anisotropic flow, $v_2 \equiv (p_x^2 - p_y^2)/(p_x^2 + p_y^2)$.

Which limit works best?

Significant elliptical flow is observed.

Ideal hydro *overpredicts* flow

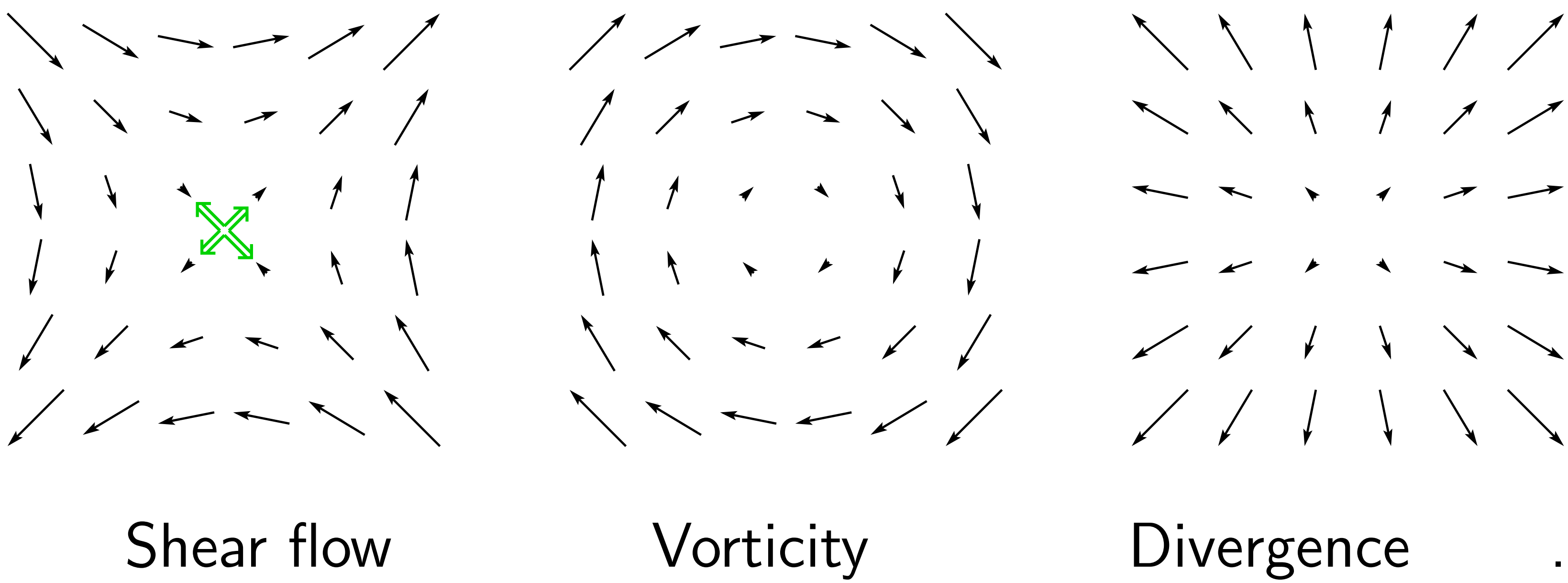
Nonuniform flow should lead to *viscous corrections*

Good description requires these viscous effects

We need to understand viscous corrections.

Nonuniform flow patterns: $\partial_i v_j \neq 0$

$\partial_i v_j$ is rank-2 tensor. $\ell = 2, 1, 0$ components:



Shear flow $\rightarrow T_{ij}$ stress: $\quad T_{ij} = -\eta(\partial_i v_j + \partial_j v_i)_{\ell=2}$.

How do I (theorist) get shear flow?

Start with equilibrium: Schwinger-Keldysh

$$\begin{aligned}\langle \mathcal{O}(x,t)\rangle &= \mathrm{Tr}\, e^{iHt}\mathcal{O}(x)e^{-iHt}e^{-\beta H} \\ &= \int \mathcal{D}[\Phi_1,\Phi_2,\Phi_3]\mathcal{O}[\Phi_2]\exp\left(-i\int_0 d^4y\,\mathcal{L}(\Phi_1)\right) \\ &\quad\times \exp\left(+i\int_0 d^4y\,\mathcal{L}(\Phi_2)\right)\exp\left(-\int_0^\beta d^4y\mathcal{L}_{\mathrm{Eucl}}(\Phi_3)\right)\end{aligned}$$

Squeeze my geometry! $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ with $\partial_t h_{ij}=\partial_i v_j$

$h_{\mu\nu}$ couples to $T^{\mu\nu}$, so we get

$$\langle T_{ij}(x,t)\rangle_{\mathrm{flow}} = \int d^3y dt'\; ih_{kl}\left\langle \Big[T_{ij}(x,t)\,,\,T_{kl}(y,t')\Big]\right\rangle \Theta(t-t')$$

Shear flow and correlators

Therefore, viscosity η determined by retarded correlator

$$\eta = i\partial_\omega G_{\text{ret}}^{T_{xy}T_{xy}}(\omega, k)|_{k=0,\omega\to 0}$$

Stable hydro code: $T_{xy} = -\eta(\partial_i u_j + \partial_j u_i)$ *implemented* as

$$\tau_\pi \partial_t T_{ij} = -\eta(\partial_i v_j + \partial_j v_i)_{\ell=2} - T_{ij}$$ Israel Stewart 1976,1979

Amounts to an *Ansatz* for the analytic form of G_{ret}

$$G_{\text{ret}}^{TT}(\omega, k=0) = \frac{-\eta}{\tau_\pi - i\tau_\pi^2 \omega}$$

Single pole at $\omega = -i/\tau_\pi$. Residue$= -i\eta\tau_\pi^{-2}$.

What does $G^{TT}_{\rm ret}(\omega, k=0)$ really look like?

What is actual analytic structure of $G^{TT}_{\rm ret}$?
True QCD: I have no idea. Probably not 1 pole.

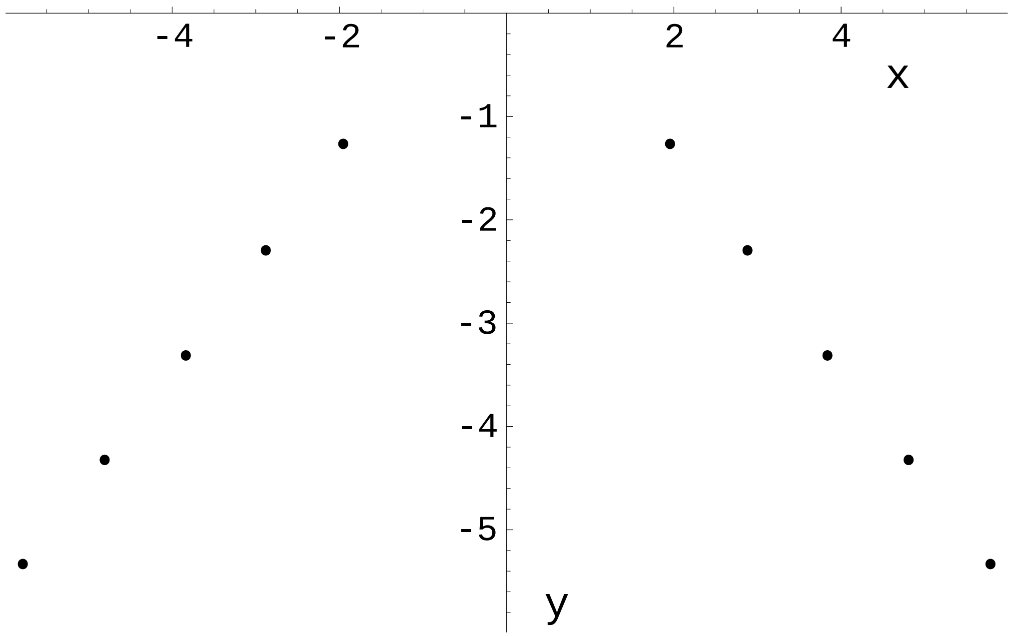
Theories we can solve: usual suspects:

- Strongly coupled $\mathcal{N}=4$ SYM theory (Holography)
- Weakly coupled relativistic field theory

Let's see what we can learn in each!

Theories with Holographic Dual

Kovtun Starinets 2005:
$(x, y) = (\mathrm{Re}, \mathrm{Im})$ of $\omega/2\pi T$ plotted
$k = 2\pi T$ not 0,
but $k = 0$ similar.



Many well-isolated poles, Re and Im parts
Interpreted in terms of BH Quasinormal Modes

Weak Coupling in Scalar $\lambda\phi^4$

Easier to study Wightman correlator

$$
\begin{aligned}
G^{>}_{T_{xy}T_{xy}}(\omega, k=0) &= \int d^3x\, dt\; e^{i\omega t}\, \langle T_{xy}(x,t) T_{xy}(0,0) \rangle \\
&= \frac{1}{1-e^{-\omega/T}} \mathrm{Im}\, G^{\mathrm{ret}}_{T_{xy}T_{xy}}(\omega, k=0)
\end{aligned}
$$

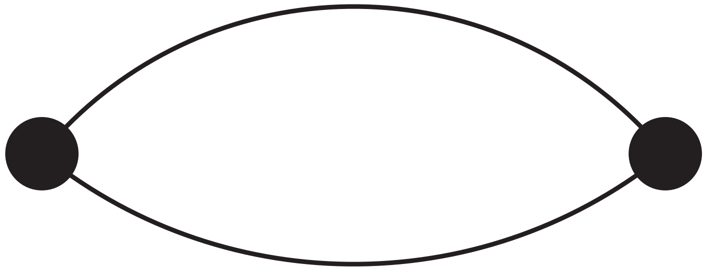
What is functional dependence on ω?

Are there distinct poles? Purely imaginary, or real parts?

Or are there cuts? Where, what discontinuity?

Or both? What is nonanalyticity nearest the real axis?

Why we need resummations



Simplest diagram: 1 loop

Blobs are T_{xy} insertions

Propagators carry

4-momentum $\pm P^\mu$

Propagators are "cut", eg,

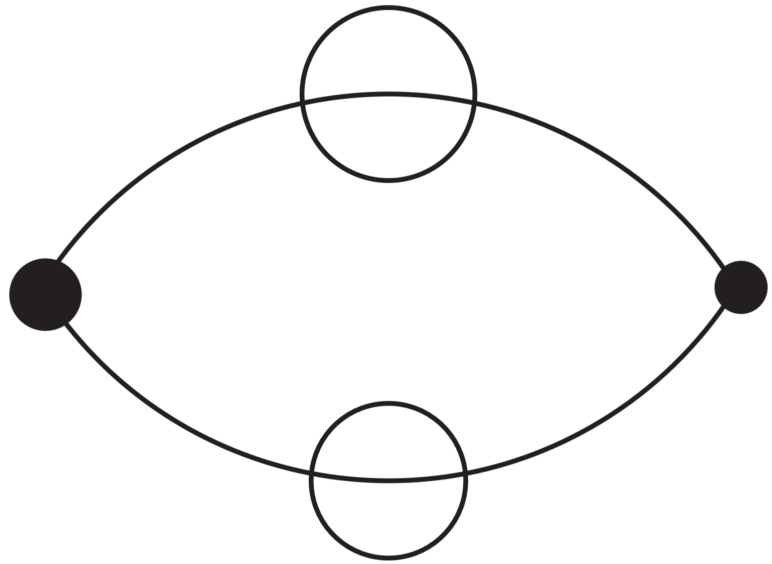
$$\Delta(p) = 2\pi[1+f(p)]\delta(p^2)$$

on-shell Delta function (at free level). Divergent:

$$\propto \int d^4p \; f(p)[1+f(p)] \, \delta(p^2) \, \delta(p^2)$$

Therefore you need

To get finite answer you **MUST** include scattering, width: on-shell δ becomes Lorentzian



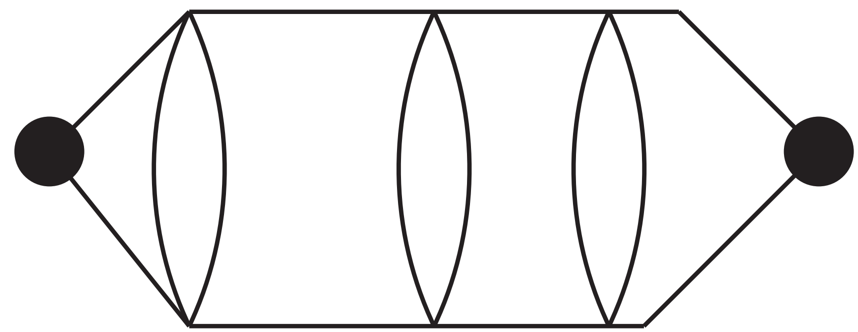
$$\int d^4p f[1+f]\Big(\delta(p^2)\Big)^2 \Longrightarrow \int d^4p f[1+f]\left(\frac{\Gamma p^0}{(p^2)^2+\Gamma^2 p_0^2}\right)^2$$

Divergence becomes $T^5/\Gamma \sim T^4/\lambda^2$ (Γ is 2-loop, $\propto \lambda^2$)

Except then $\partial_\mu \langle T^{\mu\nu}(x) T^{\alpha\beta}(0)\rangle \neq 0$

Ladder resummation

Higher loops involve more powers of $1/\Gamma$. Compensate λ^2 loop "cost". Also restore stress-tensor conservation.



Each "rail" at different (matching pair of) momentum than last. Each rail $\propto \lambda^{-2}$, each "rung" $\propto \lambda^2$. Jeon hep-ph/9409250; Jeon Yaffe hep-ph/9512263

Neglecting these gets answer wrong by factor $\simeq 3$.

Effective kinetic theory

Effective theory resums these ladders.

Contribution of rung-pair described by

$$\delta f(k,t) = f_0(k)[1 + f_0(k)]\chi(k,t)$$

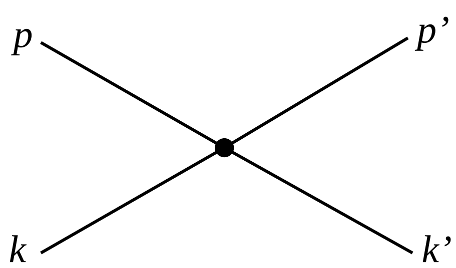
(f_0 Bose distribution). Time evolution Boltzmann Eq

$$\begin{aligned} \partial_t \chi(k,t) &= S(k)\delta(t) - \mathcal{C}[\chi] \qquad \text{($\mathcal{C}$ is integral operator)} \\ &= S(k)\delta(t) - \int d^3p\, \mathcal{C}_{p,k}\, \chi(p) \\ &= S(k)\delta(t) - \int d^3p \Big[\Gamma_k \delta^3(p-k) - \mathcal{C}_{k\to p}\Big]\chi(p) \end{aligned}$$

First(loss), second(gain) term in [ ] from rails/rungs.

What is collision operator $\mathcal{C}$?

Represents possibility for particle p to change occupancy due to particle l. For scalar theory,



$$\mathcal{C}(p,l) = \int \frac{d^3k d^3p' d^3k'}{(2\pi)^9 2p2p'2k2k'}$$
$$\times |\mathcal{M}(\mathbf{p},\mathbf{k},\mathbf{p}',\mathbf{k}')|^2 (2\pi)^4 \delta^4(p+k-p'-k')$$
$$\times f(p)f(k)[1\pm f(p')][1\pm f(k')]$$
$$\times \Big(\delta(p-l) + P_2(\cos\theta_{pk})\delta(k-l)$$
$$-P_2(\cos\theta_{pp'})\delta(p'-l) - P_2(\cos\theta_{pk'})\delta(k'-l) \Big)$$

p, k incoming, $p'k'$ outgoing energies, $|\mathcal{M}|^2 = \lambda^2$

with $P_2(x) = (3x^2-1)/2$ the $\ell = 2$ Legendre polynomial

p-space nonlocal, multiple integrals. Ugh!

Connection to η

Correlator $\langle T_{xy}(t) T_{xy}(0)\rangle$ given by

$$T_{xy}(t) = \int d^3k \; \chi(k) S(k) f_0[1{+}f_0] \quad \equiv \; \langle \chi | S \rangle$$

$\mathcal{C}$ is positive symmetric operator under inner product

$$\langle \chi | \phi \rangle \equiv \int d^3k \; \chi(k) \phi(k) f_0[1{+}f_0]$$

In terms of inner product, Boltzmann equation is

$$\partial_t |\chi\rangle = \delta(t) |S\rangle - \mathcal{C} |\chi\rangle$$

and

$$\eta = \frac{1}{6T} \int dt \langle S | \chi(t) \rangle = \frac{1}{3T} \langle S | \mathcal{C}^{-1} | S \rangle$$

Only problem: $\mathcal{C}$ is a nasty integral operator. Need $\mathcal{C}^{-1}$!

Eigenspectrum of $\mathcal{C}$

Space of $|\chi\rangle$ is $\mathcal{L}^2$: ∞-dimensional.

Any positive symmetric operator has eigenspectrum

$$\mathcal{C} = \sum_i \lambda_i |\xi_i\rangle\langle\xi_i| + \int_D d\lambda' \lambda' |\xi(\lambda')\rangle\langle\xi(\lambda')|$$

discrete (pole) plus continuous (cut) spectrum, D the portion of $\Re^+$ which is cut.
Eigenvectors obey orthogonality

$$\langle\xi_i|\xi_j\rangle = \delta_{ij}\,, \quad \langle\xi_i|\xi(\lambda')\rangle = 0\,, \quad \langle\xi(\lambda')|\xi(\lambda'')\rangle = \delta(\lambda' - \lambda'')$$

Spectral decomposition solves Boltzmann equation:

$$|\chi(t)\rangle = \sum_i e^{-\lambda_i t}|\xi_i\rangle\langle\xi_i|S\rangle \; + \int_D d\lambda' e^{-\lambda' t}|\xi(\lambda')\rangle\langle\xi(\lambda')|S\rangle$$

Value of η is

$$3T\eta = \sum_i \lambda_i^{-1}\Big(\langle S|\xi_i\rangle\Big)^2 + \int_D d\lambda' \lambda'^{-1}\Big(\langle S|\xi(\lambda')\rangle\Big)^2$$

Retarded function has poles at $\omega = -i\lambda_i$, residue $\Big(\langle\xi_i|S\rangle\Big)^2$, and cuts along $-iD$ with discontinuity $\Big(\langle\xi(\lambda')|S\rangle\Big)^2$

If only I could find this decomposition explicitly.

Test function method

Work in finite-dimensional subspace spanned by test functions:

$$|\chi\rangle = \sum_{i=1}^{N} c_i |\phi_i\rangle$$

Test functions I will use:

$$\phi_{i,\mathrm{Yaffe}}(k) = \frac{k^{i+1} T^{M-i-2}}{(k+T)^{M-1}}\,, \qquad i = 1, \ldots, N\,, \quad N \geq M$$

Need to orthonormalize (easy). Large M: basis more complete everywhere. Large $N-M$: more complete UV. AMY used $N = M$ but we don't have to.

Test function method

Find "vector"

$$S_i = \langle S|\phi_i\rangle = \int d^3pS(p)\phi_i(p)f_0[1+f_0]$$

Find "matrix" (Hard! $\mathcal{C}$ = multi-dimensional integration)

$$C_{ij} = \langle\phi_i|\mathcal{C}|\phi_j\rangle = \int d^3pd^3k\phi_i(p)\phi_j(k)\mathcal{C}_{k,p}f_0[1+f_0]$$

Eigenspectrum of C_{ij}: matrix. Automatically discrete spectrum

Test function method

Discontinuities purely on negative imaginary axis.
Always in kinetic theory! (Im parts higher in g^2)

Method automatically "predicts" discrete spectrum of poles.
Just because we work in finite-dimensional subspace.

Try to tell if it's really poles or cuts by varying basis size,
seeing whether poles stay put or "fill in" denser and denser.

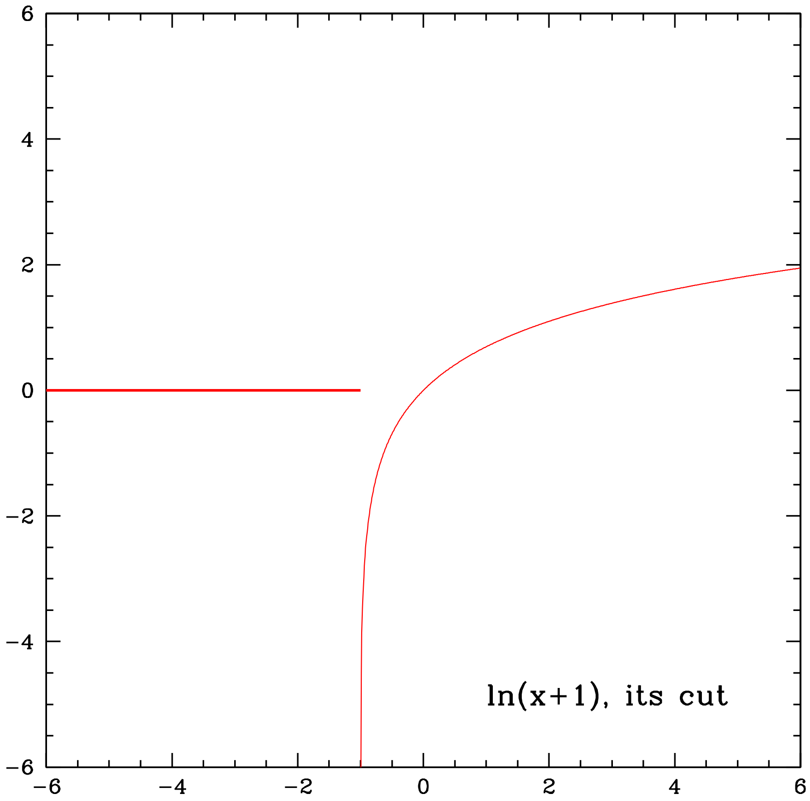
What does a cut look like

if I can only see poles + zeros?

Consider the function

$$f(x) = \ln(1 + x)$$

Cut from -1 to $-\infty$

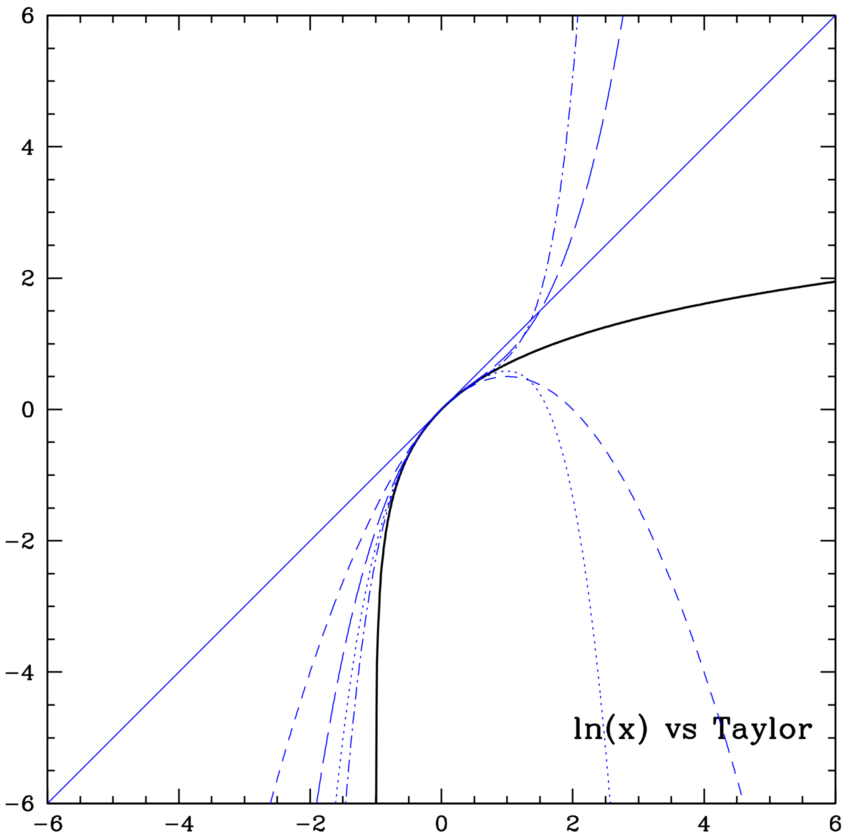


Suppose we only had information from the point $x = 0$.

Taylor:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

Does terrible job!



Why so bad?

Taylor is same as assuming function has n zeros and no poles.

Not good description of a cut!

Assume instead that function has 1 more zero than pole: Padé

$$P_{N,N-1}(x) = \frac{\sum_{n=1}^{N} d_n x^n}{1 + \sum_{n=1}^{N-1} c_n x^n}$$

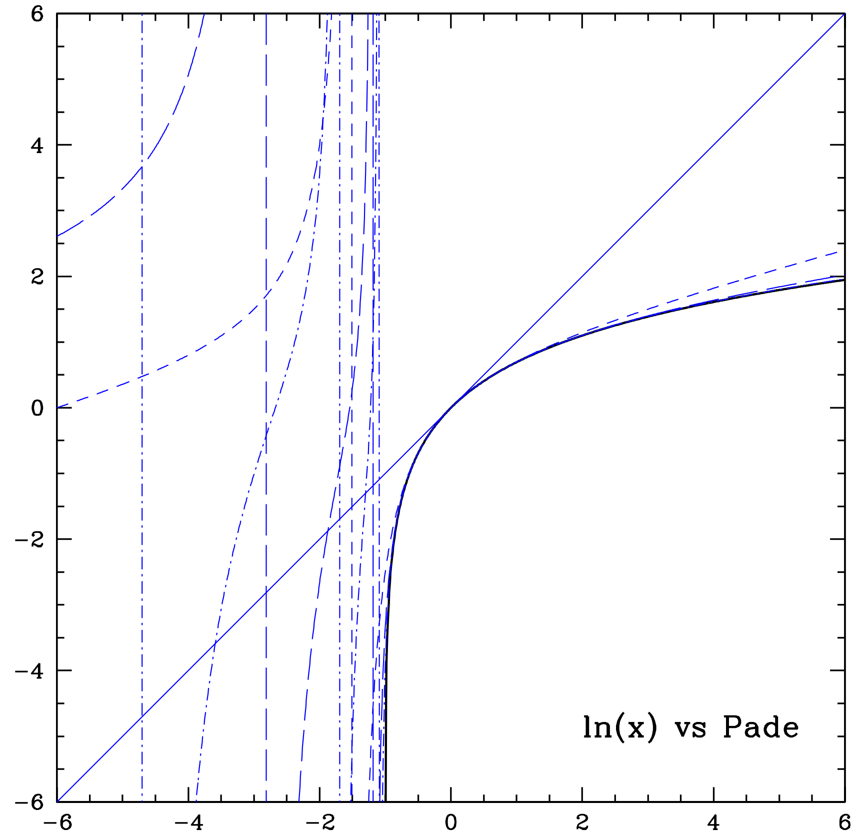
Taylor expand $P(x)$ to order $2N-1$

Choose unique d_n, c_n such that Taylor series of P and Taylor series of $\ln(1+x)$ agree through $2N-1$ terms

Does a far better job!

Padé Approximations of $\ln(1+x)$

Here are $(1,0)$, $(2,1)$, $(3,2)$, and $(4,3)$ Padé approximants of $\ln(1+x)$.



What is this mess at $x < -1$?

Padé is:

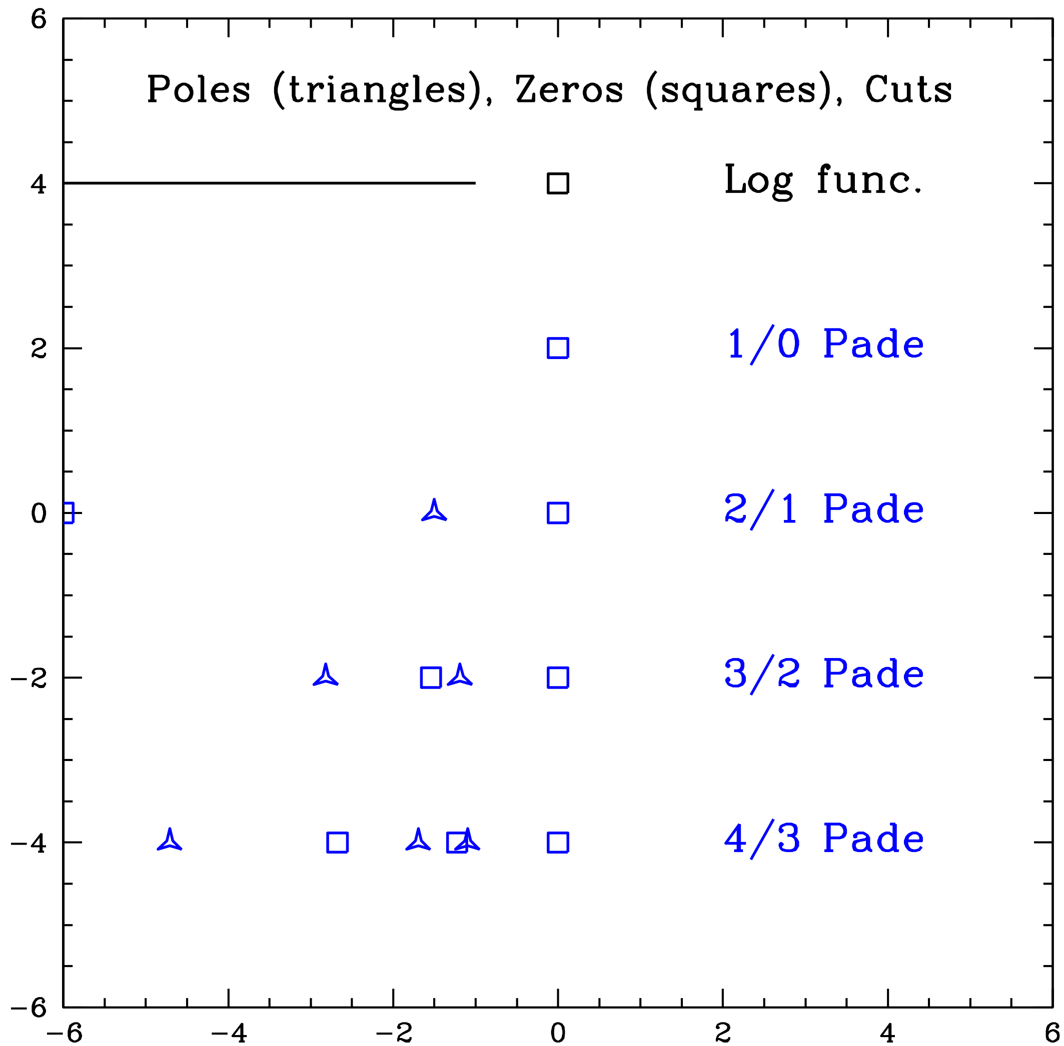
$$\frac{d_1 x + d_2 x^2 + \ldots}{1 + c_1 x + c_2 x^2 + \ldots} = \frac{A(x - z_1)(x - z_2)\ldots}{(x - p_1)(x - p_2)\ldots}$$

product of zeros and poles, at $z_1, \ldots$ and $p_1, \ldots$

Cut got replaced by series of zeros and poles.

Trying to describe a cut as a series of zeros and poles.

For last two, one zero is off edge of plot.



What if there is also a true pole?

Consider function

$$f(x) = \frac{\sqrt{(x+2)(x+3)/6}}{x+1}$$

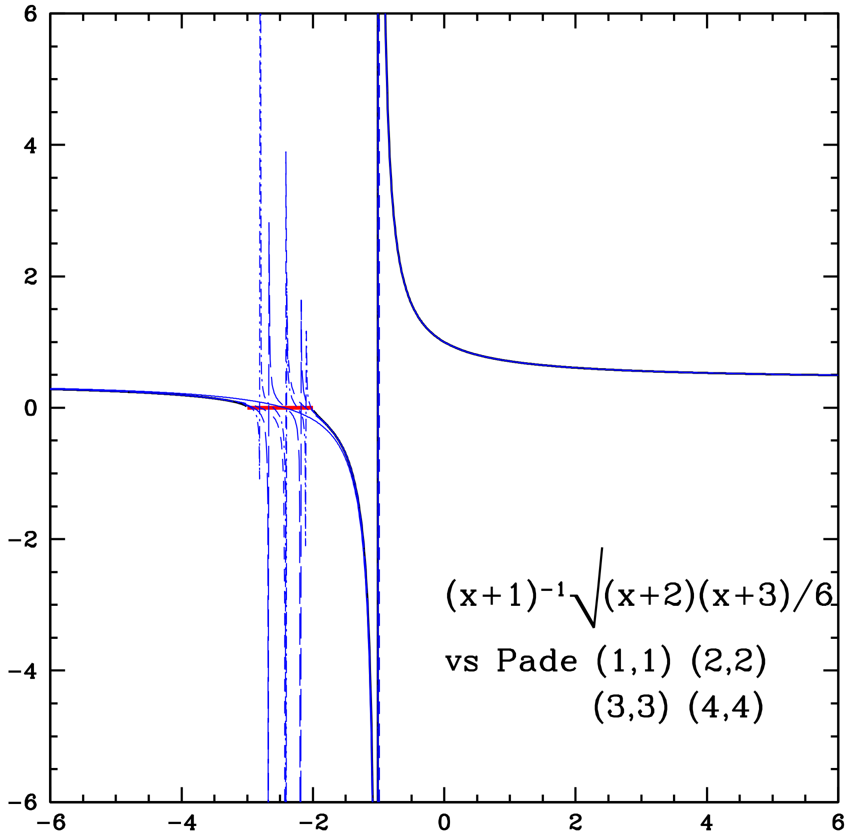
Pole at $x = -1$

Cut from $x = -2$ to $x = -3$

Fit it with an (N, N) Padé approximant

(Taylor series is, once again, crap)

Pole/zero fitting of a pole and cut

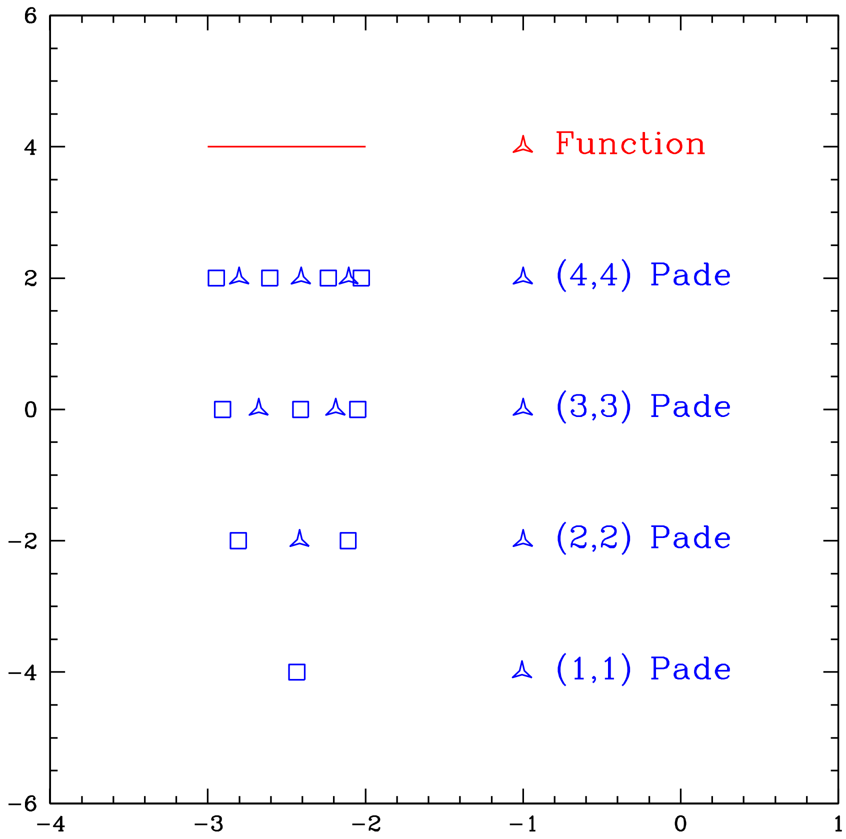


Even (1,1) Padé is great!

Pole treated as pole.

Cut $= N$ zeros, $N-1$ poles

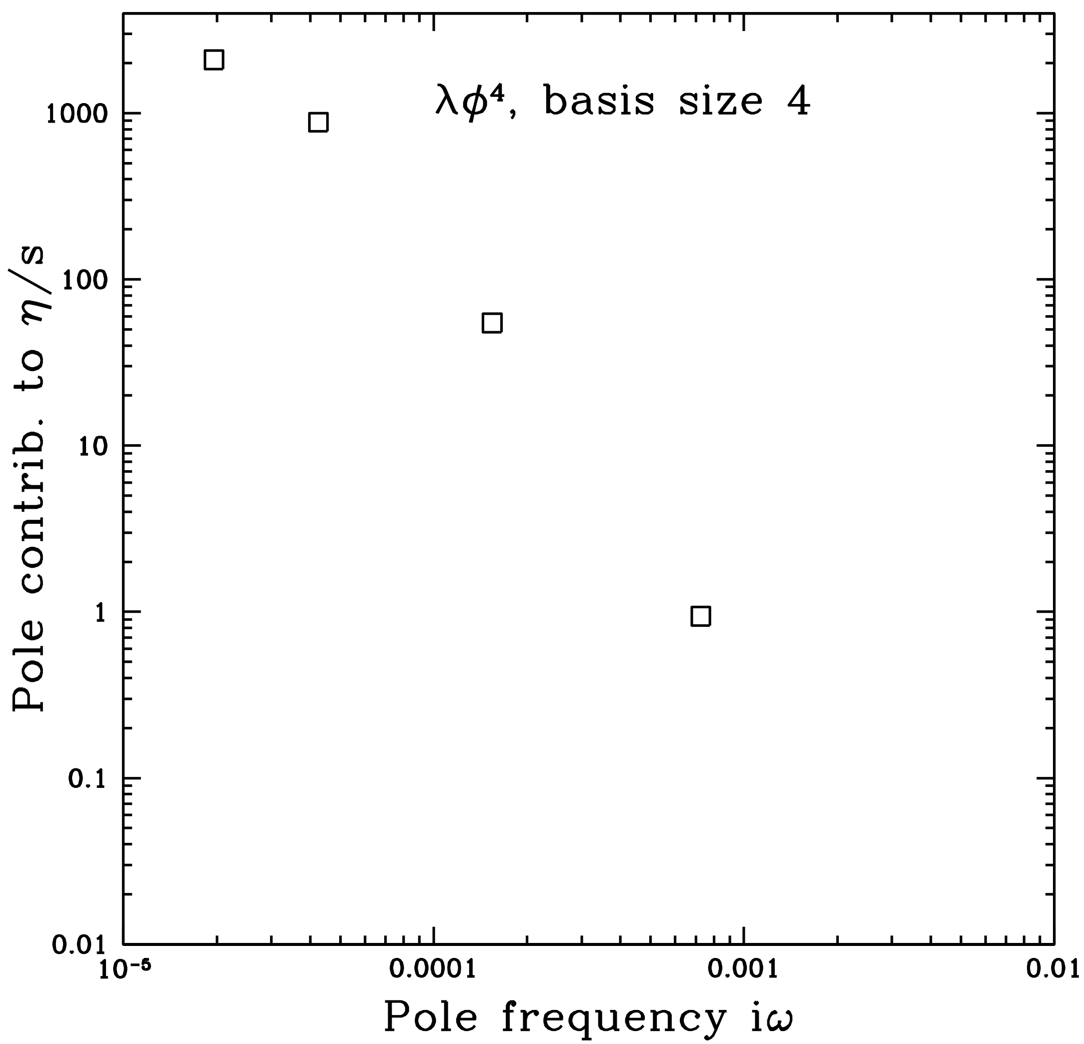
Pole/zero fitting of a pole and cut



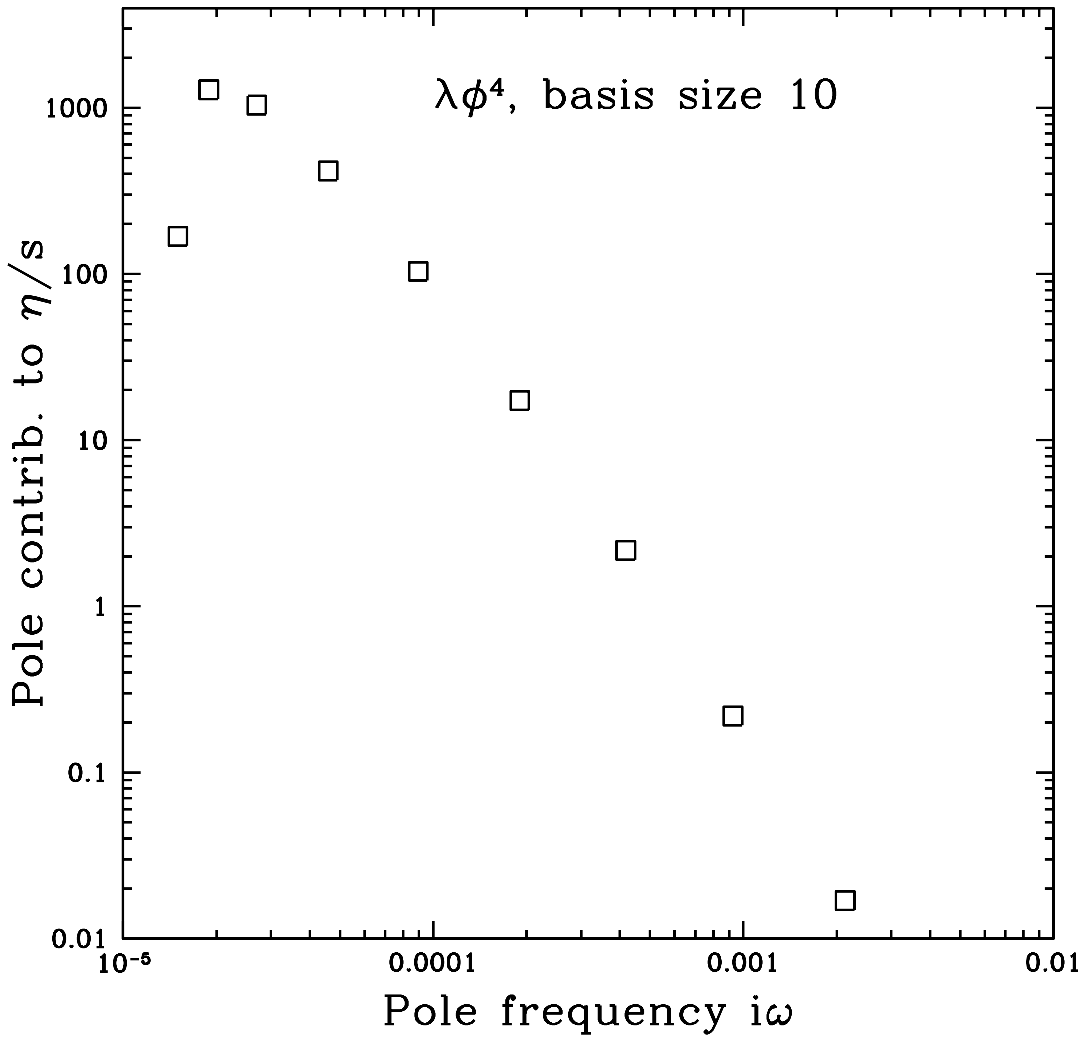
Pole stays put as increase Padé size.
zeros/poles get tighter together.
Note: not evenly spaced

I can tell that there is an isolated pole in front of cut!

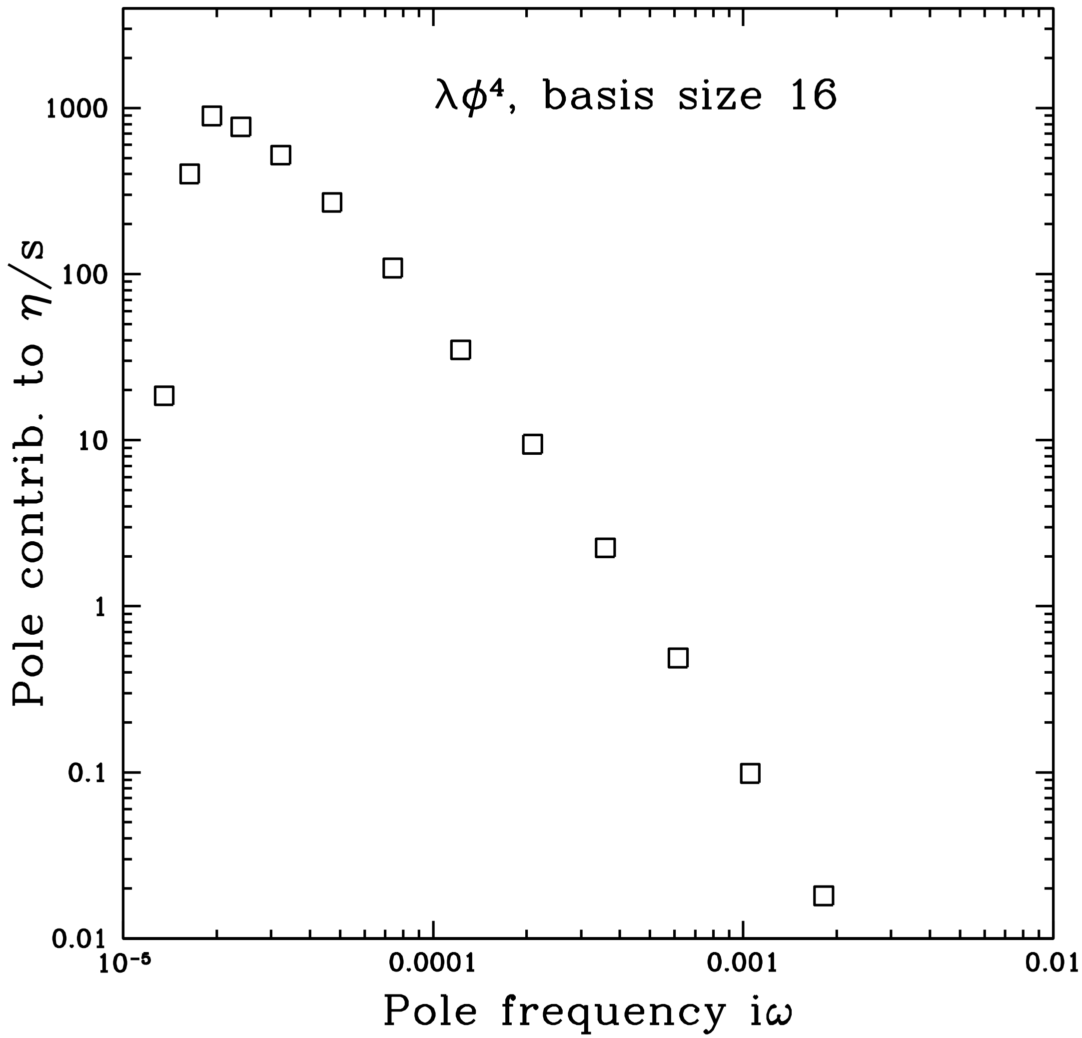
4 basis elements



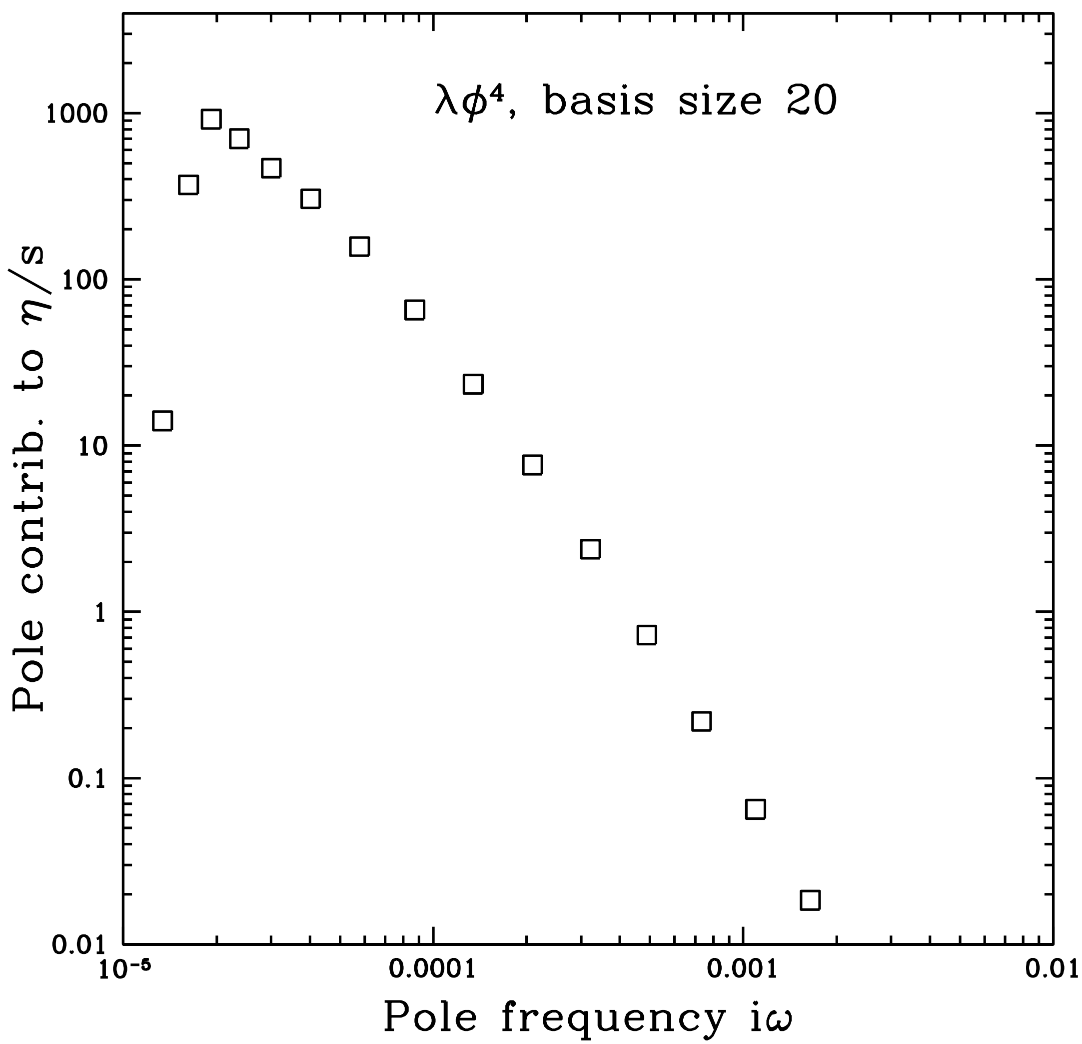
10 basis elements



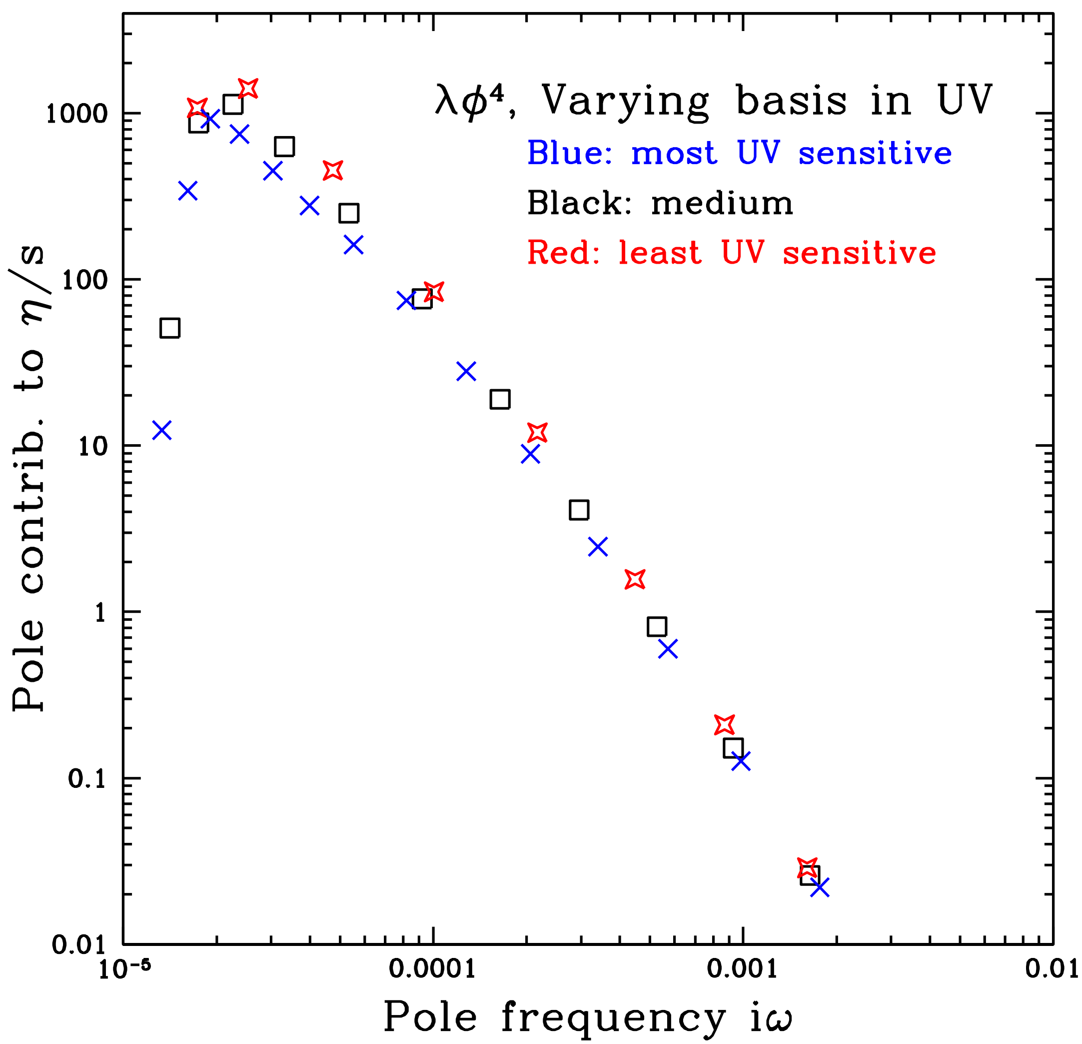
16 basis elements



20 basis elements



Bases, different UV sensitivity



My interpretation

- Looks to me like a cut!
- Dominant contribution from one scale
- Cut discontinuity falls fast at smaller ω
- Discontinuity also falls fast at larger ω
- Large/small ω from small/large-k particles (?)

Summary

- Considering $\langle T_{xy} T_{xy}(\omega \ll T, k = 0) \rangle$
- $\lambda\phi^4$ theory at weak coupling has a cut at strictly imaginary ω
- Cut has a narrow region of large discontinuity
- Extends to larger ω with small discontinuity (forever? yes at small λ, cut off by thermal mass...)
- Extends to small ω with small discontinuity (all the way to $\omega = 0$? If so, exponentially small)

But note!

Actually $G_{\mathrm{ret}}^{TT}(\omega, k=0)$ should be non-analytic right up to $\omega = 0$ due to long-time tails

Perturbation theory misses them: g^4 vs g^{-4}

Holography misses them: N_c^0 vs N_c^2

Disc. of cut should involve half-int power of ω

Can N_c^{-2} corrections be computed in AdS/CFT?