

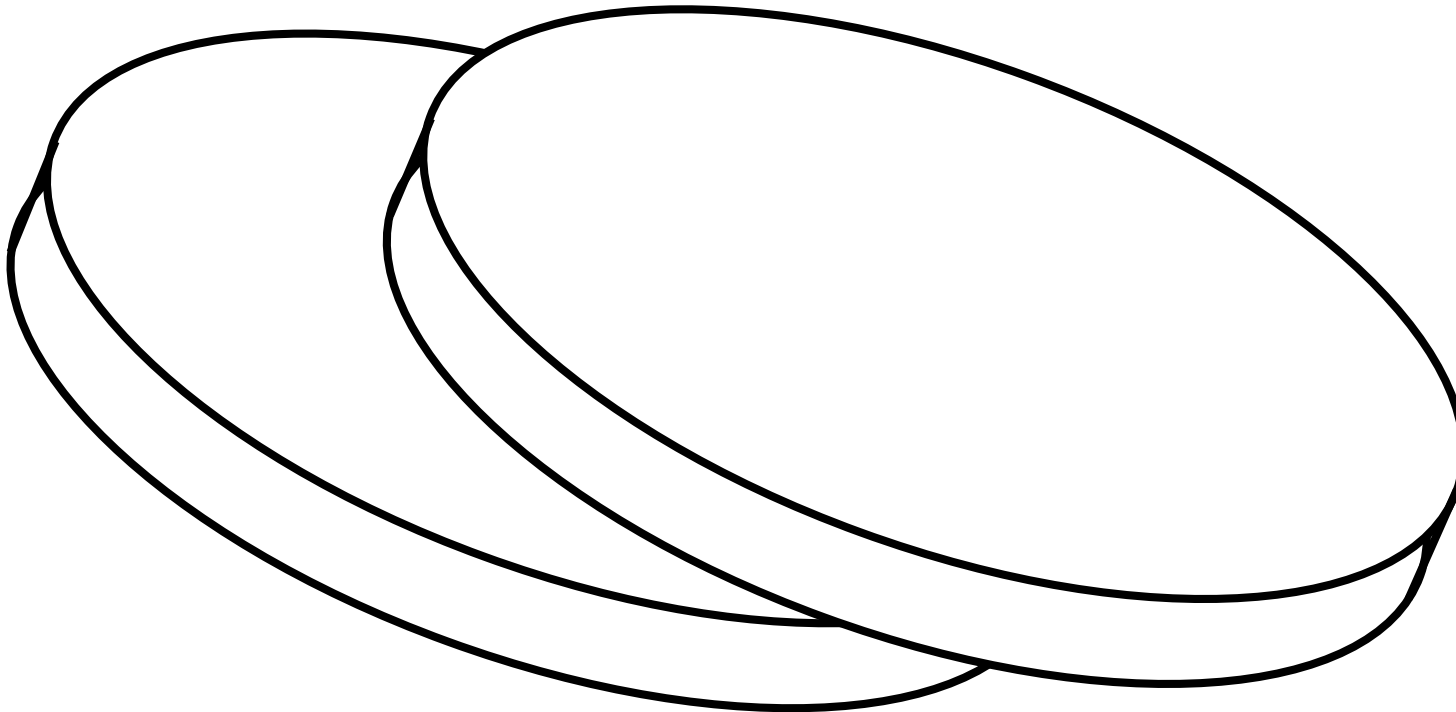
Second Order Hydro in QCD

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- Why do hydrodynamics in QCD?
- Why find 2'nd order coefficients and what are they?
- Kinetic theory: setup
- Kinetic theory: details
- Interesting physics along the way
- Conclusions

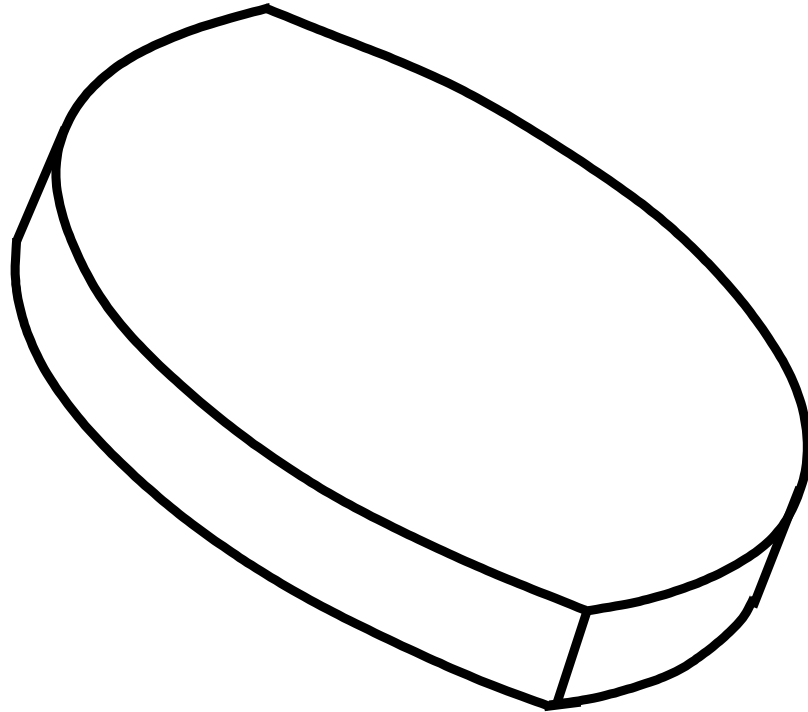
Heavy ion collisions

Accelerate two heavy nuclei to high energy, slam together.



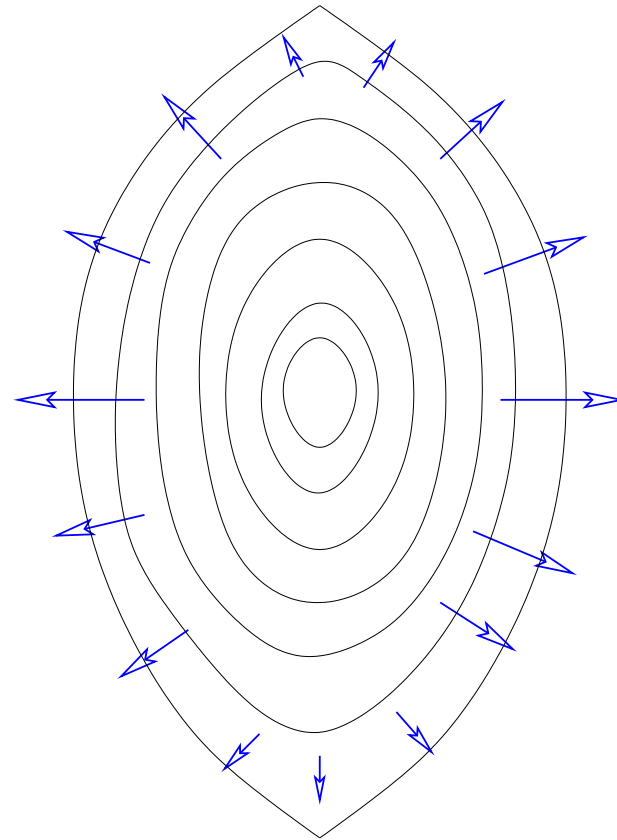
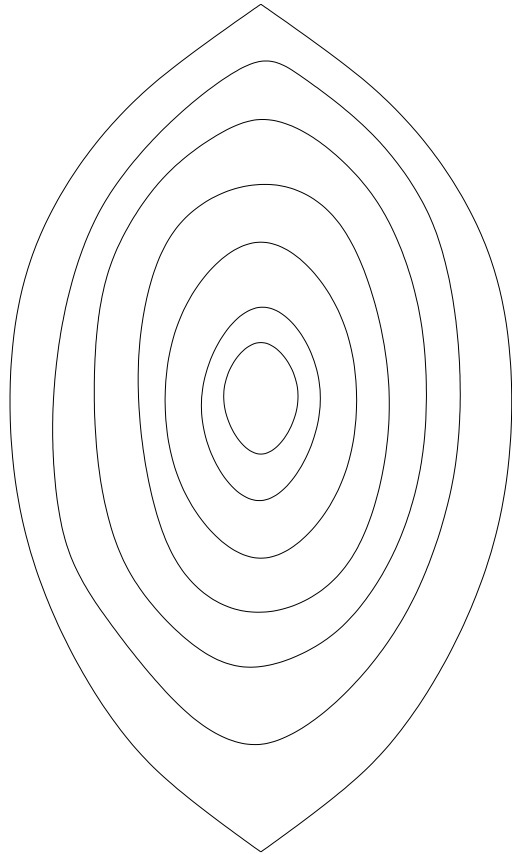
Just before: Lorentz contracted nuclei

After the scattering: region where nuclei overlapped:
“Flat almond” shaped region of q, \bar{q}, g which scattered.



~ 2 thousand random v quarks+gluons: isotropic in xy plane

local CM motions



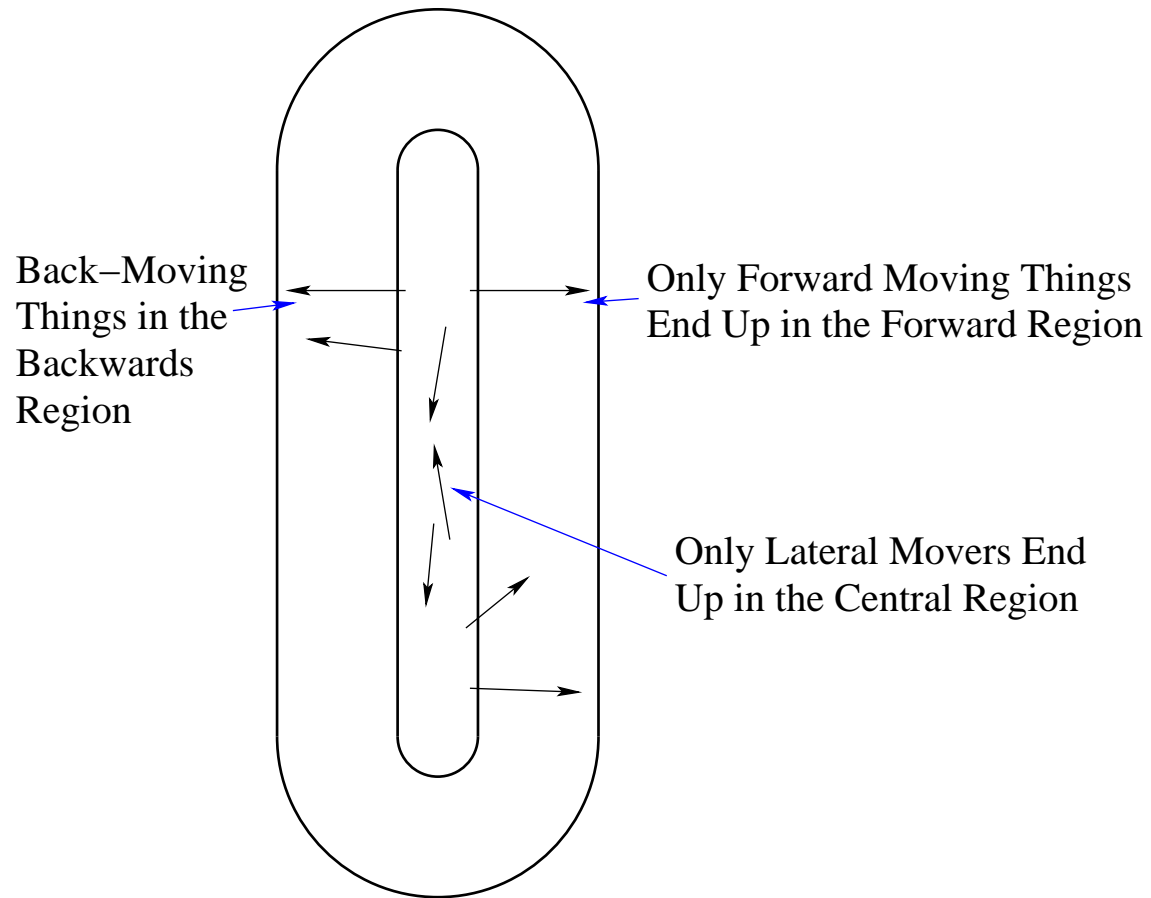
Pressure contours

Expansion pattern

Anisotropy leads to anisotropic (local CM motion) flow.

Momentum Selection

Side-on view of the flat almond as it expands



Space aniso. \rightarrow aniso. of "particle" p distrib.

Free particle propagation:

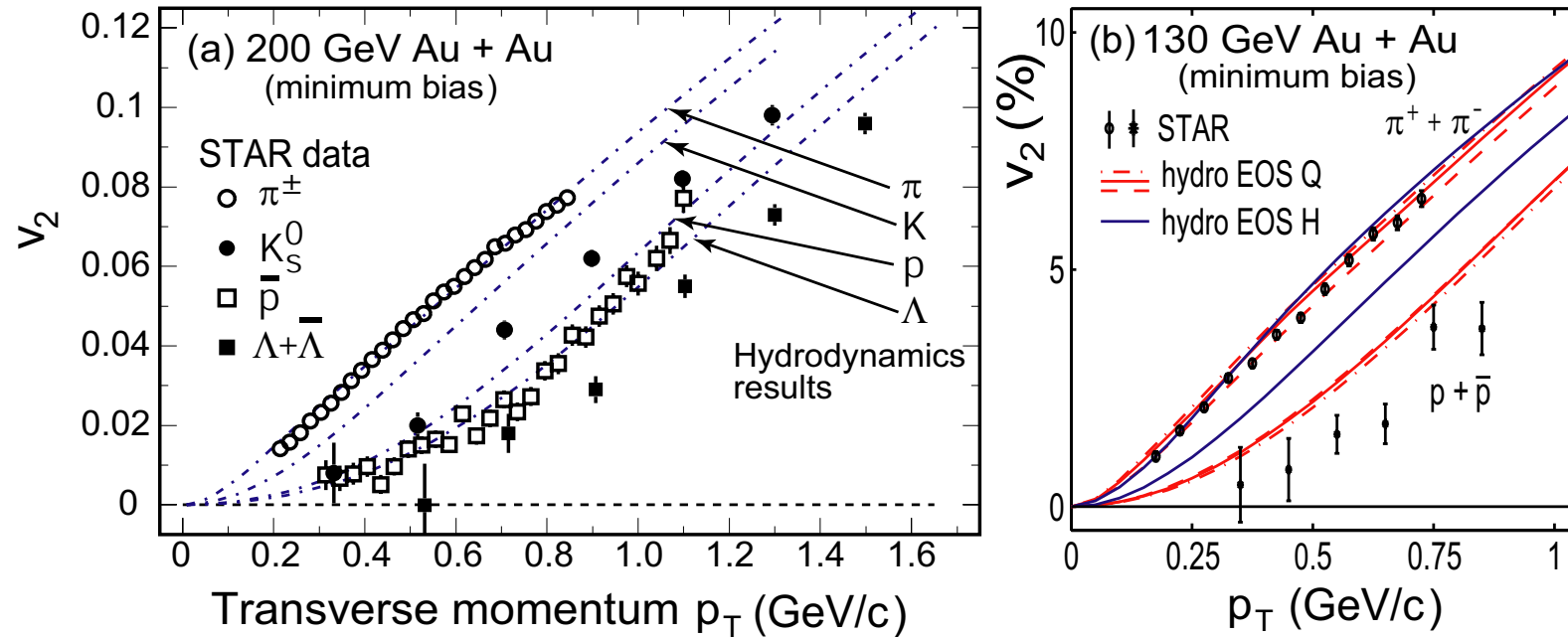
- Particle distributions locally triaxial, $\langle v_x^2 \rangle < \langle v_y^2 \rangle$, but
- System-average CM flow velocities $\langle v_{x,\text{CM}}^2 \rangle > \langle v_{y,\text{CM}}^2 \rangle$
- Total particle distribution $\langle v_x^2 \rangle = \langle v_y^2 \rangle$

Efficient scattering:

- Drives system *locally* towards $\langle v_{x,\text{relative}}^2 \rangle = \langle v_{y,\text{relative}}^2 \rangle$
- System-average CM flow still has $\langle v_{x,\text{CM}}^2 \rangle > \langle v_{y,\text{CM}}^2 \rangle$
- Adding these together, $\langle v_{x,\text{tot}}^2 \rangle > \langle v_{y,\text{tot}}^2 \rangle$

Net “Elliptic Flow” $v_2 \equiv \frac{v_x^2 - v_y^2}{v_x^2 + v_y^2}$ measures scattering

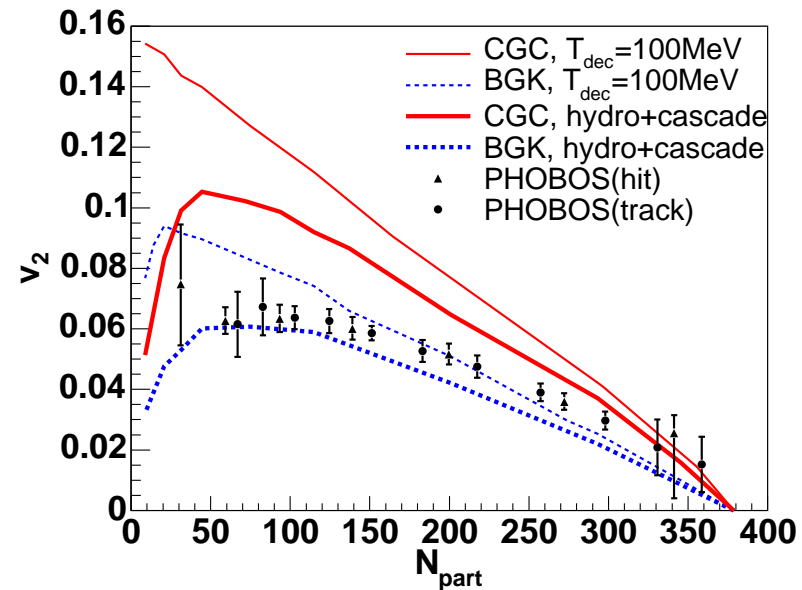
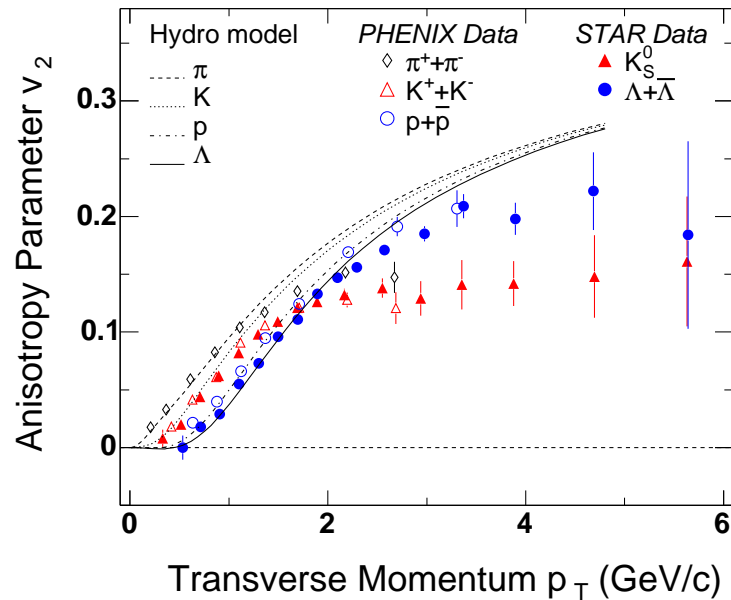
Elliptic flow is measured



STAR experiment, minimum bias...

We should try to understand it theoretically.

First attempt: ideal hydro



Works, DEPENDING on initial conditions.

Corrections to ideality exist, but are “small” (?)

Can we quantify that?

Ideal Hydrodynamics

Ideal hydro: stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0 \quad (4 \text{ equations, } 10 \text{ unknowns})$$

plus local equilibrium *assumption*:

$$\begin{aligned} T^{\mu\nu} &= T_{\text{eq}}^{\mu\nu} = \epsilon u^\mu u^\nu + P(\epsilon) \Delta^{\mu\nu}, \\ u^\mu u_\mu &= -1, \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \end{aligned}$$

depends on 4 parameters (ϵ , 3 comp of u^μ): closed.

Ideal hydro works well: corrections *eg, viscosity* small

Claim: “Most Perfect Liquid” exotic behavior. **Quantify!**

Nonideal Hydro

Assume that ideal hydro is “good starting point,” look for small systematic corrections.

Near equilibrium iff $t_{\text{therm}} \ll t_{\text{vary}}, l_{\text{vary}}/v$ (so ∂ small)

Allows expansion of corrections in gradients:

$$T^{\mu\nu} = T_{\text{eq}}^{\mu\nu} + \Pi^{\mu\nu}[\partial, \epsilon, u]$$

$$\Pi^{\mu\nu} = \mathcal{O}(\partial\mu, \partial\epsilon) + \mathcal{O}(\partial^2\mu, (\partial\mu)^2, \dots) + \mathcal{O}(\partial^3 \dots)$$

For Conformal theory $T_{\mu}^{\mu} = 0 = \Pi_{\mu}^{\mu}$, 1-order term unique:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta} \left(\partial_{\alpha}u_{\beta} + \partial_{\beta}u_{\alpha} - \frac{2}{3}g_{\alpha\beta}\partial \cdot u \right)$$

Coefficient η is shear viscosity.

Viscous hydro

So why not consider (Navier-Stokes)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} - \eta \sigma^{\mu\nu} \quad ?$$

Because in **relativistic** setting, it is

- **Acausal:** shear viscosity is transverse momentum diffusion. Diffusion $\partial_t P_\perp \sim \nabla^2 P_\perp$ has instantaneous prop. speed. Müller 1967, Israel+Stewart 1976
- **Unstable:** $v > c$ prop + non-uniform flow velocity \rightarrow propagate from future into past, exponentially growing solutions. Hiscock 1983

Problem only on short length scales where $\eta|\sigma| \sim P$. But numerics must treat these scales (or “numerical viscosity” which exceeds η is present)

Israel-Stewart approach

Add one second order term:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_{\Pi} u^{\alpha} \partial_{\alpha} \sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \rightarrow -\Pi$ in order-2 term:

$$\tau_{\Pi} u^{\alpha} \partial_{\alpha} \Pi^{\mu\nu} \equiv \tau_{\Pi} \dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

Relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$.

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_{Π})

But why only one 2'nd order term???

Second order hydrodynamics

It is more consistent to include all possible 2'nd order terms.

Assume *conformality* and *vanishing chem. potentials*:

5 possible terms [Baier et al, \[arXiv:0712.2451\]](#)

$$\begin{aligned}\Pi_{2\text{ ord.}}^{\mu\nu} = & \eta\tau_{\Pi} \left[u^{\alpha} \partial_{\alpha} \sigma^{\mu\nu} + \frac{1}{3} \sigma^{\mu\nu} \partial_{\alpha} u^{\alpha} \right] + \lambda_1 \left[\sigma_{\alpha}^{\mu} \sigma^{\nu\alpha} - (\text{trace}) \right] \\ & + \lambda_2 \left[\frac{1}{2} (\sigma_{\alpha}^{\mu} \Omega^{\nu\alpha} + \sigma_{\alpha}^{\nu} \Omega^{\mu\alpha}) - (\text{trace}) \right] \\ & + \lambda_3 \left[\Omega_{\alpha}^{\mu} \Omega^{\nu\alpha} - (\text{trace}) \right] + \kappa (R^{\mu\nu} - \dots) , \\ \Omega_{\mu\nu} \equiv & \frac{1}{2} \Delta_{\mu\alpha} \Delta_{\nu\beta} (\partial^{\alpha} u^{\beta} - \partial^{\beta} u^{\alpha}) \quad [\text{vorticity}] .\end{aligned}$$

Now, besides η , we have 5 more unknown coefficients.

Second order: philosophy

(nonideal) hydro only consistent if η makes small corr.
These τ_{Π} , $\lambda_{1,2,3}$ make smaller corrections. κ irrelevant.

Make reasonable estimate for τ_{Π} , λ_{123} , test sensitivity

Guy argues: Ratios should be relatively robust

- Forget $\frac{\eta}{s}$. Think of $\frac{\eta}{P+\epsilon} = t_{\eta}$ a *timescale*. Pert: $1/g^4 T$
- Next order: $\frac{\lambda_1}{P+\epsilon} = t_{\lambda}^2$, $\frac{\eta\tau_{\Pi}}{P_{\epsilon}} = t_{\pi}^2$ Pert: $1/g^8 T^2$
- All are thermalization times. One expects $t_{\lambda} \sim t_{\eta} \sim t_{\pi}$.

Determine ratios where you can, **use as priors** in fit

Two toy models of QCD

To date, coeff's computed in *toy model* for QCD:

$\mathcal{N}=4$ SYM theory at $N_c, g^2 N_c \rightarrow \infty$

(conformal, vast number DOF, many scalars, infinite coupling,...) [Baier et al \[arXiv:0712.2451\]](#), [Tata group, \[arXiv:0712.2456\]](#)

I know another *toy model* for QCD:

Weakly coupled $N_c = 3, N_f = 0, \dots 6$ QCD in pert. theory!

(asymptotically free, mass gap, right number DOF, finite coupling...)

Leading order calculation: theory conformal, same coeff's

Toolkit for calculation: kinetic theory (valid at leading order)

Kinetic theory

Weak coupling: IR-safe corr. funcs nearly Gaussian.
Adequate description in terms of 2-point function.

Value of 2-pt function has interpretation as particle number:
 $\phi^\dagger \phi$ is $\frac{1}{2} + \hat{N}$ number operator of free th.

Leading-order: free propagation. Scatterings “rare”.

Allows extra approximation: $\Delta x \sim 1/p \sim 1/T$ small
compared to free path $\lambda \sim 1/g^2 T$. Propagation classical,
 $[x, p] \simeq 0$ “classical phase space” behavior.

Kinetic theory

State, all measurables described by particle distrib. $f_a(x, p)$:

$$T^{\mu\nu}(x) = \sum_a \int_p 2P^\mu P^\nu f_a(x, p), \quad \int_p \equiv \int \frac{d^3p}{(2\pi)^3 2p^0}$$

(Assumes weak coupling, slow x^μ dependence, little else)

Dynamics: Boltzmann equation (Schwinger-Dyson eq):

$$2P^\mu \partial_{\mu x} f(x, p) = -\mathcal{C}[p, f(x, q)]$$

LHS: particle propagation. $p^0 \equiv \sqrt{\mathbf{p}^2} \equiv p$

RHS: scattering (Im self-energy). Local in x but not p .

Theory dependence all contained in detailed form of $\mathcal{C}[f]$.

In our case, described in detail in [AMY5: hep-ph:0209353](#)

Two gradient expansions

Hydrodynamics description relies on

$$(t_{\text{therm}}, vt_{\text{therm}}) \ll (t_{\text{vary}}, l_{\text{vary}})$$

slow variation in time,space. Expandable order-by-order.

Kinetic theory description relies on

$$(t_{\text{deBroglie}} \sim T^{-1}, \lambda_{\text{deBroglie}}) \ll (\Gamma^{-1}, \lambda_{\text{mfp}})$$

where $\Gamma^{-1} \leq t_{\text{therm}}$ is inverse scatt rate.

We don't know how to do this expansion order-by-order!

g^2 corrections lie outside kinetic description.

Expansion in (hydro) gradients

$$f(x, p) = f_0(\beta(x), u(x), p) + f_1(\partial, \beta, u, p) + f_2(\partial^2, \beta, u, p)$$

Subscript counts order in derivatives. β, u and ϵ, \vec{P} dual

LHS of Boltzmann has 1 deriv: RHS has 0.

$$\mathcal{O} 0: \quad \mathcal{C}[p, f_0(x, q)] = 0 \quad \rightarrow \quad f_{0,a} = \frac{1}{\exp(-\beta u^\mu P_\mu) \pm 1}$$

$$\mathcal{O} 1: \quad 2P^\mu \partial_{\mu x} f_0(\beta(x), u(x), p) = -\mathcal{C}_1[p, f(q)]$$

where \mathcal{C}_1 is \mathcal{C} expanded to lin. order in f_1 .

$$\mathcal{O} 2: \quad 2P^\mu \partial_{\mu x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, f q]$$

with \mathcal{C}_{11} 2 order in f_1 , \mathcal{C}_2 lin. order in f_2

First order in expansion

Organize it as

$$f_1(q) = -\mathcal{C}_{1,qp}^{-1} 2P^\mu \partial_\mu f_0(-\beta u^\nu P_\nu)$$

Gradients of free-theory distribution act as **source** for f_1 .

$$2P^\mu \partial_\mu f_0 = -f_0' \beta P^\mu P^\nu (\partial_\mu u_\nu + u_\mu \partial_\nu \beta)$$

Organize source in spherical harmonics. $\ell = 0, 1$ determine u, β :

$$\partial_t \beta = \frac{\beta}{3} \partial_i u_i \quad \text{and} \quad \partial_t u_i = \frac{1}{\beta} \partial_i \beta$$

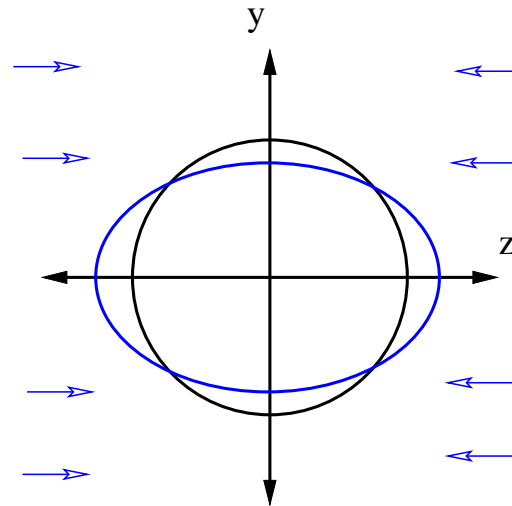
Remaining term is nontrivial:

$$f_1(q) = \mathcal{C}_{1,qp}^{-1} (p_i p_j - p^2 \delta_{ij}/3) \beta \sigma_{ij} f_0'$$

Solution always of form $f_1(p) = \frac{\beta^3}{2} \sigma_{ij} (p_i p_j - p^2 \frac{\delta_{ij}}{3}) \chi(-\beta u_\mu P^\mu)$

Physics so far: viscosity

Consider system under compression: momentum distribution becomes prolate spheroidal (limited by scattering)



Viscosity measures $T_{\mu\nu}$ of this distortion, so $\eta/P = 4\eta/(P + \epsilon)$ measures extent of prolateness.

Prolateness can differ at different $|p|$; $\chi(\beta p)$ tells how this varies, η gives some average.

Second order Boltzmann Equation

$$2P^\mu \partial_{\mu x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, fq]$$

Organize it as

$$f_2 = -\mathcal{C}_1^{-1} \left(2P^\mu \partial_\mu f_1 + \mathcal{C}_{11}[f_1] \right)$$

Term on right acts like a source for 2'nd order departure f_2 .

Two pieces: effect of inhomogeneity on 1'st order departure, nonlinearity of collision operator in departure from equilib.

Determining Π^{ij} only requires $\ell = 2$ moment of f_2 , which simplifies calculation: only need $\ell = 2$ of RHS.

Term $2P^\mu \partial_\mu f_1$

Inhomogeneous flow when f already skewed. Consider

$$2P^\mu \partial_\mu \sigma_{\alpha\beta} (P^\alpha P^\beta - g^{\alpha\beta} p^2/3) \beta^3 \chi (-\beta u^\gamma P_\gamma)$$

Two types of terms: ∂_μ acts on $\sigma_{\alpha\beta} \beta^3$, or on χ .

First term:

$$P^\mu P^\nu P^\alpha (\partial_\mu \sigma_{\nu\alpha} + 3\sigma_{\nu\alpha} \partial_\mu \ln \beta) \beta^3 \chi$$

Only contributions to Π^{ij} when 2 P 's space, one time.

$$E p_i p_j (\partial_0 \sigma_{ij} + 2\partial_i \sigma_{0j} + 3\sigma_{ij} \partial_0 \ln \beta)$$

Contributes to τ_{Π} , λ_2 , λ_1 . Second term: contributes to λ_1 .

What we get so far

One contribution to τ_{Π} , λ_2 , and λ_1 .

Automatically in ratio $1 : -2 : -1$. Therefore

$$\lambda_2 = -2\eta\tau_{\Pi}$$

Extra independent (positive) contribution to λ_1 .

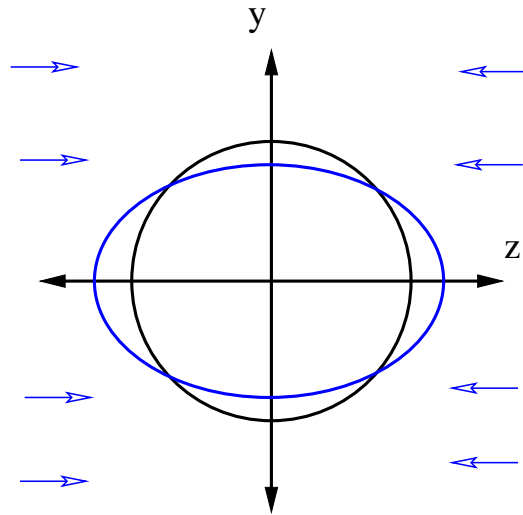
Detailed values depend on functional form of $\chi(\beta E)$.

Specific *Ansatz* (Grad 14-moment) gives specific values:

$$\frac{\eta\tau_{\Pi}}{\epsilon + P} = \frac{6\zeta(4)\zeta(6)}{\zeta^2(5)} \left(\frac{\eta}{\epsilon + P} \right)^2, \quad \frac{\lambda_1}{\eta\tau_{\Pi}} = -\frac{1}{3}.$$

But we *solve* for $\chi(\beta E)$ —slightly different value

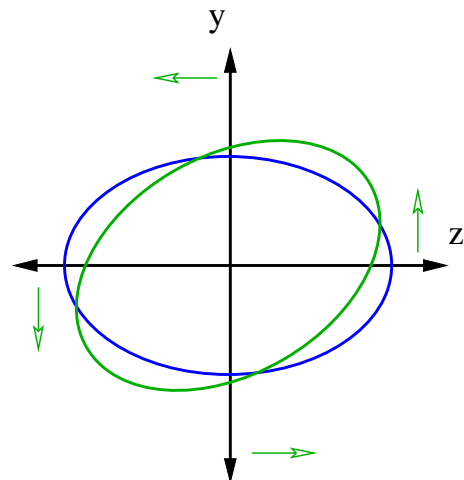
What do these coefficients mean?



Contraction \rightarrow prolateness.

τ_{II} : how long prolateness lasts after contraction ends.

λ_1 : induced prolateness depends on how prolate it already is!



(Vorticity) rotation

Vorticity means rotation.

λ_2 : how much prolateness gets rotated in a rotating system.

Depends on how long prolateness lives.

Hence relation $\lambda_2 = -2\eta\tau_{II}$.

Hard part: \mathcal{C}_{11}

Nonlinear term in collision operator:

“extra” particles scattering from other “extra” particles

Theory dependent. Consider $2 \leftrightarrow 2$ scattering (present in most theories):

$$\mathcal{C}[p, f[q]] = \int_{k p' k'} (2\pi)^4 \delta^4(P+K-P'-K') |\mathcal{M}^2| \times \\ \left(f(p)f(k)[1 \pm f(p')][1 \pm f(k')] - f(p')f(k')[1 \pm f(p)][1 \pm f(k)] \right)$$

First order expansion: define $\bar{f}_1 = f_0(1 \pm f_0)f_1$.

$$\left(f(p)f(k)[1 \pm f(p')][1 \pm f(k')] - f(p')f(k')[1 \pm f(p)][1 \pm f(k)] \right) \\ = 0 + f_0(p)f_0(k)[1 \pm f_0(p')][1 \pm f_0(k')] \left(\bar{f}_1(p) + \bar{f}_1(k) - \bar{f}_1(p') - \bar{f}_1(k') \right)$$

That's what we needed in defining \mathcal{C}_1 . Used twice already!

Next order: $f_0(p)f_0(k)[1\pm f_0(p')][1\pm f_0(k')]$ times

$$\begin{aligned} & \bar{f}_1(p)\bar{f}_1(k)f_0(p)f_0(k)(e^{\frac{p+k}{T}} - 1) + \bar{f}_1(p')\bar{f}_1(k')f_0(p')f_0(k')(1 - e^{\frac{p+k}{T}}) \\ & + \left[\bar{f}_1(p)\bar{f}_1(p')f_0(p)f_0(p') \left(e^{\frac{p}{T}} - e^{\frac{p'}{T}} \right) + (p' \rightarrow k') \right. \\ & \quad \left. + (p \rightarrow k) + (p, p' \rightarrow k, k') \right] \end{aligned}$$

Note that $\bar{f}_1(p) \propto \sigma_{ij}(p_i p_j - \delta_{ij} p^2 / 3)$.

In evaluating $\langle S_{ij} | \mathcal{C}_{11}[f_1] \rangle$ we meet angular integrations:

$$\begin{aligned} \text{defining } p_{\langle i q j \rangle} &= \frac{3p_i q_j + 3q_i p_j - 2p \cdot q \delta_{ij}}{6}, \quad x_{pq} \equiv p \cdot q, \\ & \sigma_{lm} \sigma_{rs} \int d\Omega_{\text{global}} p_{\langle i p j \rangle} q_{\langle l q m \rangle} r_{\langle r s \rangle} \\ &= \frac{4}{105} \left(\sigma_{il} \sigma_{jl} - \frac{\delta_{ij}}{3} \sigma_{lm} \sigma_{lm} \right) \\ & \quad \times \left(3x_{pq} x_{pr} x_{qr} - x_{pp} x_{qr}^2 - x_{qq} x_{pr}^2 - x_{rr} x_{pq}^2 + 2x_{pp} x_{qq} x_{rr} / 3 \right) \end{aligned}$$

Using these, one can bludgeon \mathcal{C}_{11} term to death. Contributes only to λ_1 .

Subtlety!

Preceding assumed that matrix element $|\mathcal{M}^2|$ is f independent.

In gauge and Yukawa theories, f enters \mathcal{M} through screening!

Change in $f_0 \rightarrow f_0 + f_1$ changes screening, leading to correction to $|\mathcal{M}|^2$ linear in f_1 .

This is where things get hard.

So I won't tell you about it except one detail.

Antiscreening and plasma instability

Plasmas screen all interactions *except* $\omega \rightarrow 0$ magnetic.

Deep consequence of dimensional reduction: in 3D theory, rotation+gauge inv mean F^2 , not A^2 op can appear.

Nonzero f_1 : plasma is *not* rotation non-invariant. Former (equilibrium!) argument fails: magnetic “mass” possible.

But \int_{Ω} of screening effect is zero!

Angular average zero, value in some directions nonzero \rightarrow

Screening is negative in some directions (plasma instabilities)

Plasma instability for us

We are working perturbatively – won't see full instability.

$$G^{\mu\nu} = \frac{1}{G_0^{-1} - \delta\Pi} \simeq G_0^{\mu\nu} + G_0^{\mu\alpha} \delta\Pi_{\alpha\beta} G_0^{\beta\nu}$$

Now $\delta\Pi$ doesn't vanish but $G_0 \sim 1/q^2$:

$$G^{\mu\nu}(q) \sim \frac{1}{q^2} + \delta\Pi \frac{1}{q^4}$$

more IR singular. Potential for log IR singularity.

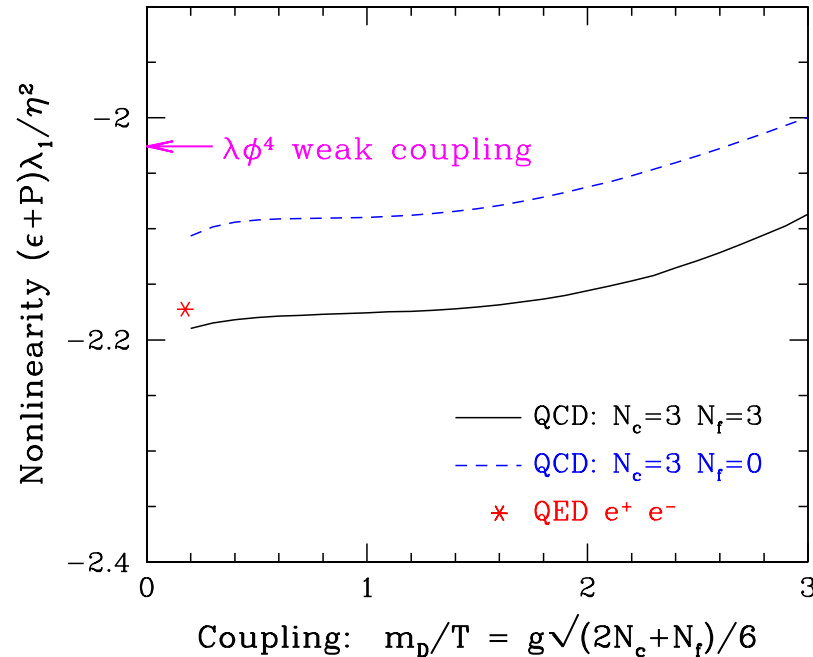
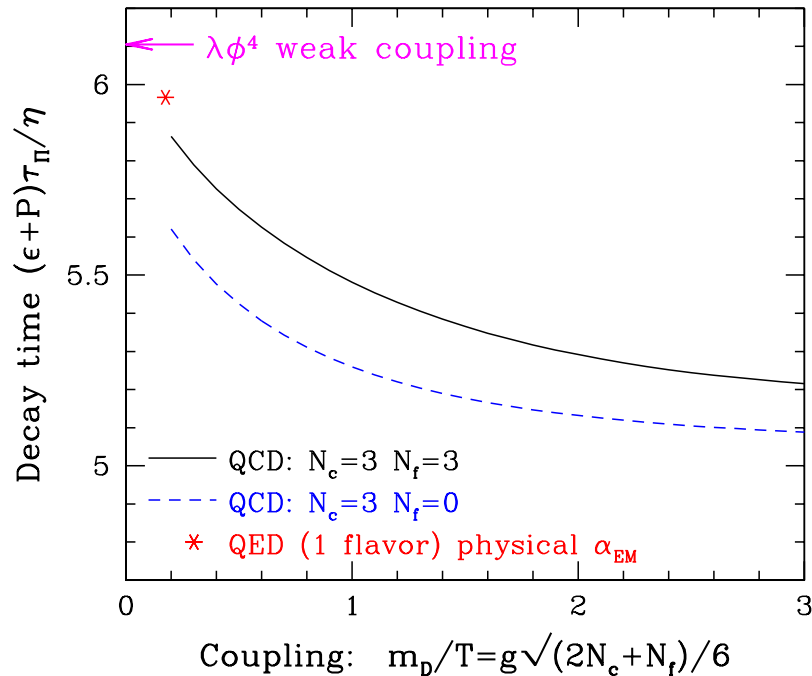
Turns out to cancel in angular stuff for $2 \leftrightarrow 2$,

but not for $1 \leftrightarrow 2$ (splitting).

Splitting rate is IR log singular. I don't have exact results!

Results

$\lambda_3 = \kappa = 0$. $\lambda_2 = -2\eta\tau_{\Pi}$. τ_{Π} , λ_1 nontrivial:



Size of uncertainty is thinner than lines in plots!

Ratios are very stable with value of coupling.

QCD vs SYM comparison

Ratio	QCD value	SYM value
$\frac{\tau_{\text{II}}(\epsilon+P)}{\eta}$	5 to 5.9	2.6137
$\frac{\lambda_1(\epsilon+P)}{\eta^2}$	-2 to -2.2	2
$\frac{\lambda_2(\epsilon+P)}{\eta^2}$	-10 to -11.8	-2.77
$\frac{\kappa(\epsilon+P)}{\eta^2}$	0	4
$\frac{\lambda_3(\epsilon+P)}{\eta^2}$	0	0

Good news: Not qualitatively different.

Bad news: “exact” kinetic theory relation $\lambda_2 = -2\eta\tau_{\text{II}}$ not actually general.

Conclusions

- Hydro seems sensible framework in heavy ion coll.
- Shear viscosity should be quantified!
- Requires expansion to 2'nd order in gradients
- Calculation in pert. QCD is intricate.
- Ratios are relatively robust. But Pert Thy and SYM give rather different predictions in detail.

Limitation of kinetic theory method?

How does one compute non-kinetic corrections?