Second Order Hydro in QCD, SYM

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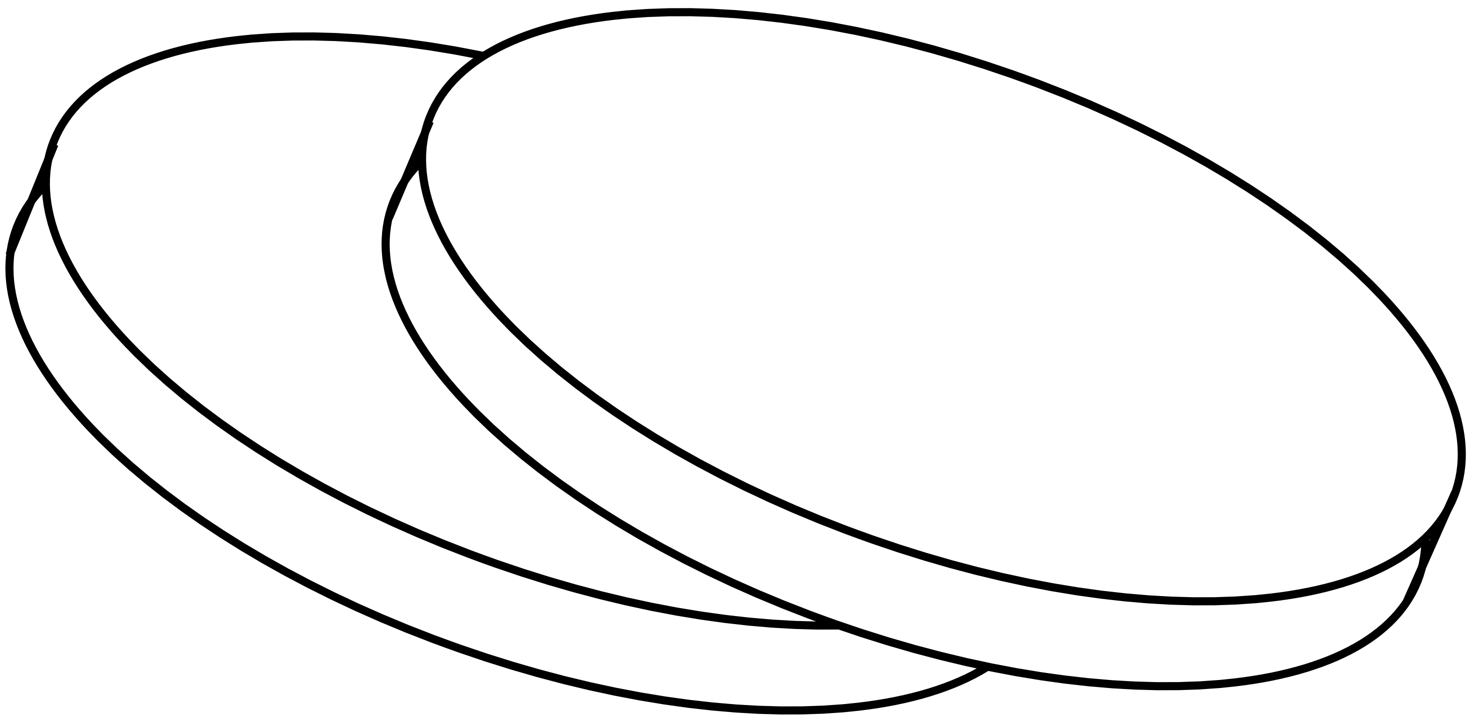
- Why do hydrodynamics in QCD?
- Why find 2'nd order coefficients and what are they?
- Kinetic theory: setup
- Kinetic theory: details
- Interesting physics along the way
- Conclusions

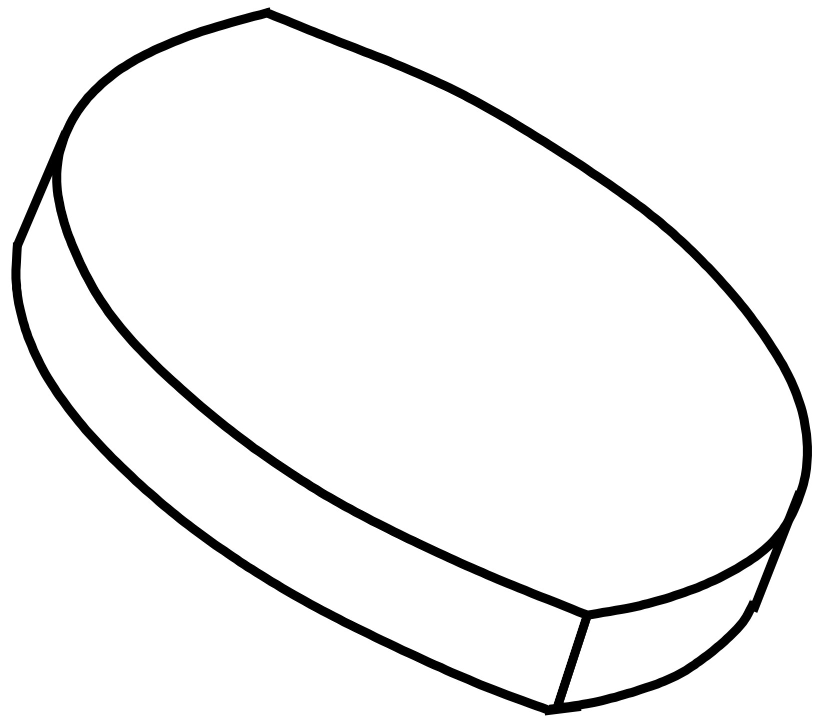
Heavy ion collisions

Accelerate two heavy nuclei to high energy, slam together.



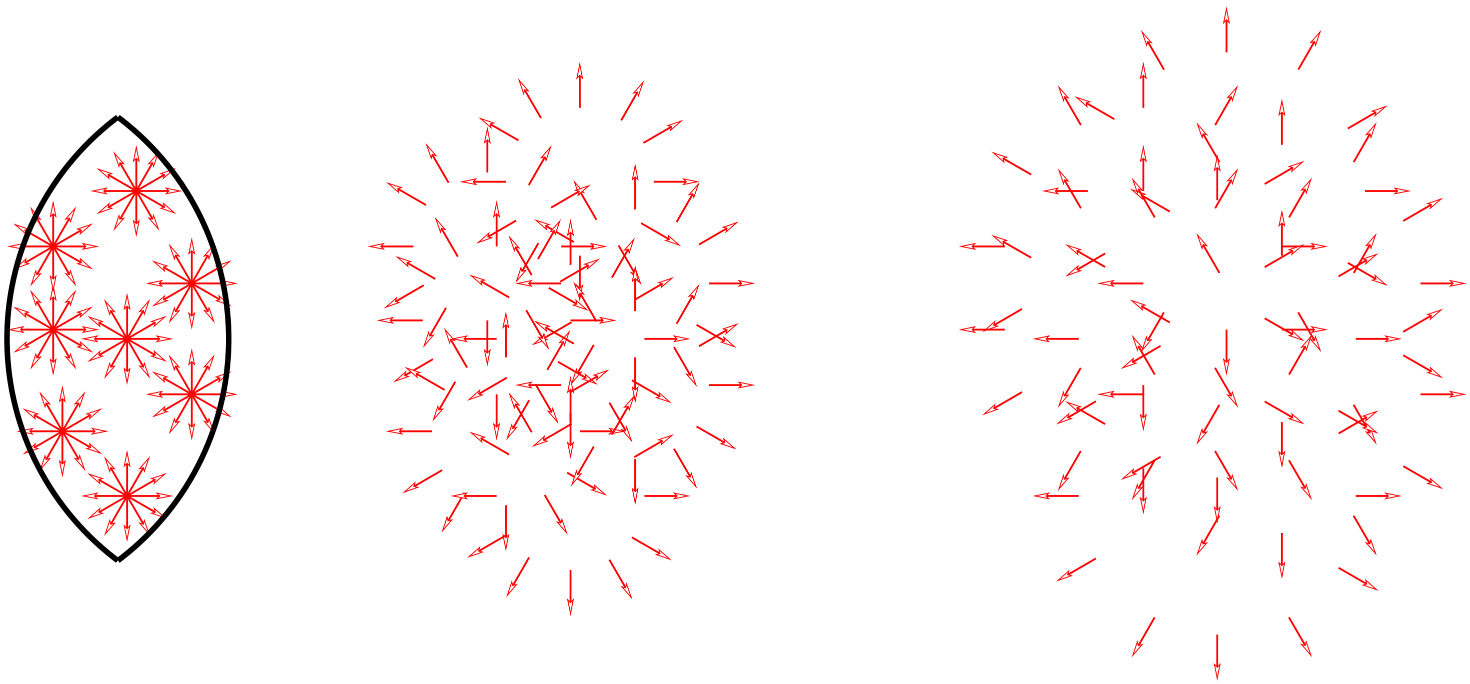
Just before: Lorentz contracted nuclei

After the scattering: region where nuclei overlapped:
"Flat almond" shaped region of $q, \bar{q}, g$ which scattered.



$\sim$2 thousand random $\mathbf{v}$ quarks+gluons: isotropic in xy plane

Behavior IF no re-interactions (transparency)

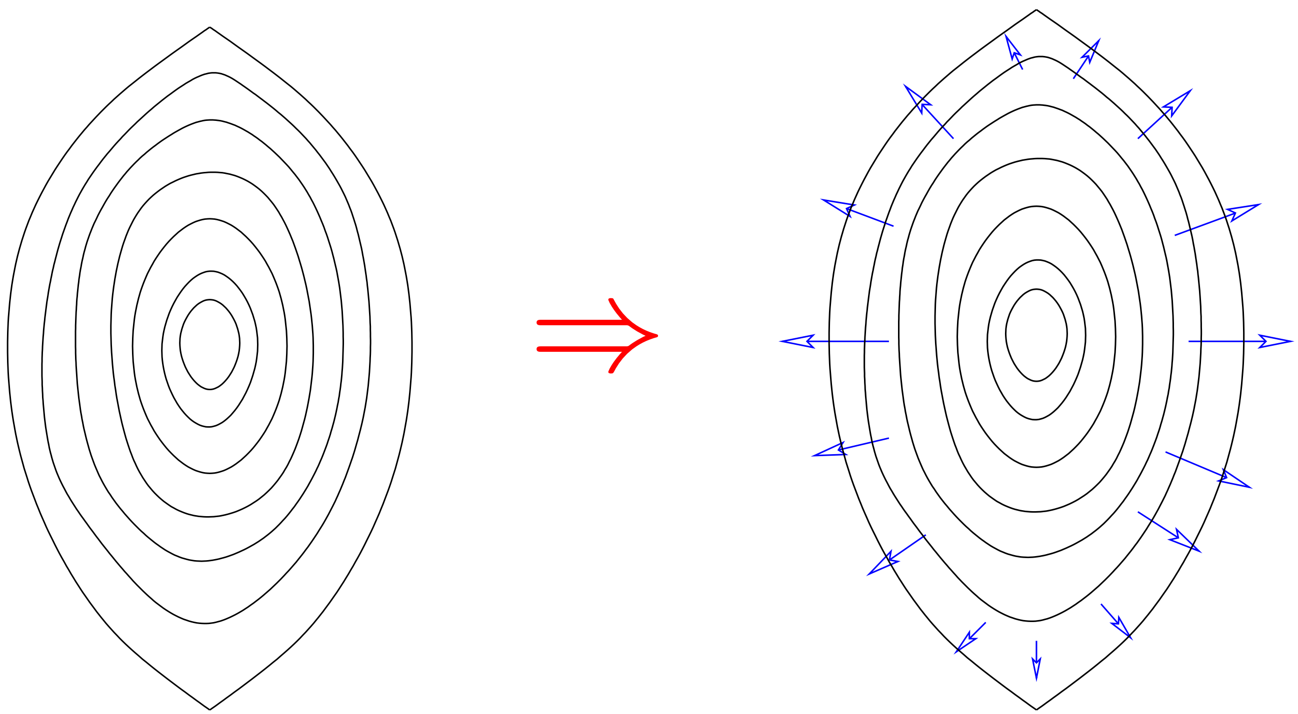


Just fly out and hit the detector.

Detector will see xy plane *isotropy*

Behavior with re-scattering: Macroscopic view

Efficient re-scattering means “stuff” acts like a *fluid*

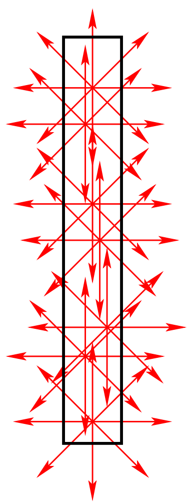


Pressure contours Expansion pattern

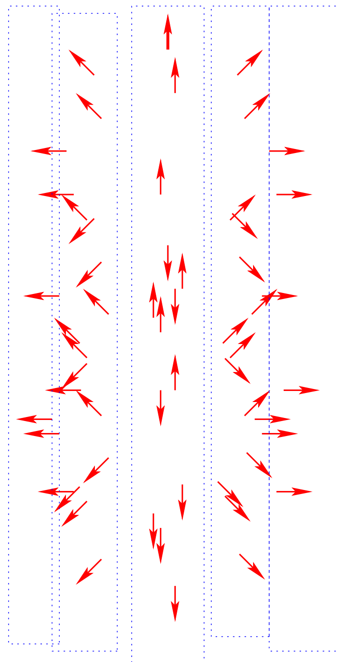
Anisotropy leads to anisotropic (local CM motion) flow.

Why? Microscopic view

Momentum selection. Consider extreme case:



becomes



Each region has CM motion and distribution of relative motions

All CM motions are in x-directions.

In local CM frame, relative motions are y-directed.

Re-scatterings equalize p_x^2, p_y^2 in relative motion only!

Free particle propagation:

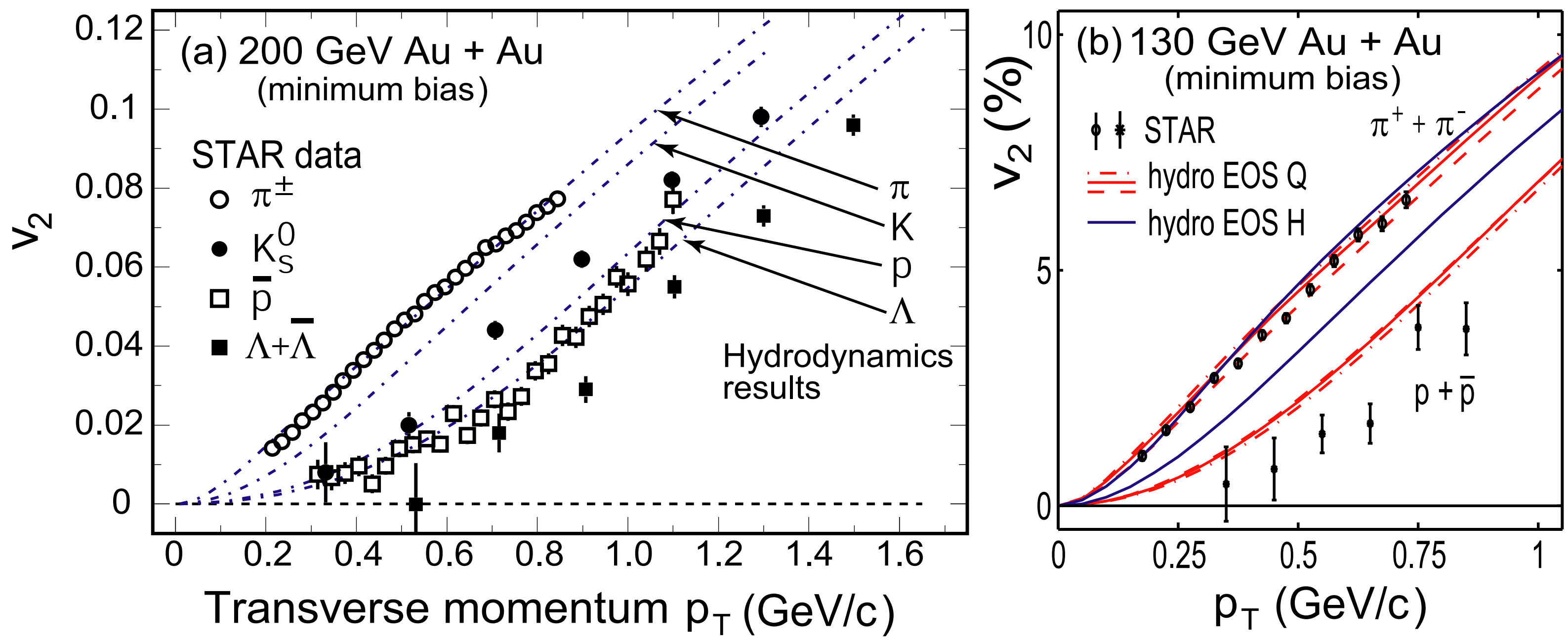
- Particle distributions locally triaxial, $\langle v_x^2 \rangle < \langle v_y^2 \rangle$, but
- System-average CM flow velocities $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- Total particle distribution $\langle v_x^2 \rangle = \langle v_y^2 \rangle$

Efficient scattering:

- Drives system *locally* towards $\langle v_{x,\mathrm{relative}}^2 \rangle = \langle v_{y,\mathrm{relative}}^2 \rangle$
- System-average CM flow still has $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- Adding these together, $\langle v_{x,\mathrm{tot}}^2 \rangle > \langle v_{y,\mathrm{tot}}^2 \rangle$

Net "Elliptic Flow" $v_2 \equiv \frac{v_x^2 - v_y^2}{v_x^2 + v_y^2}$ measures scattering

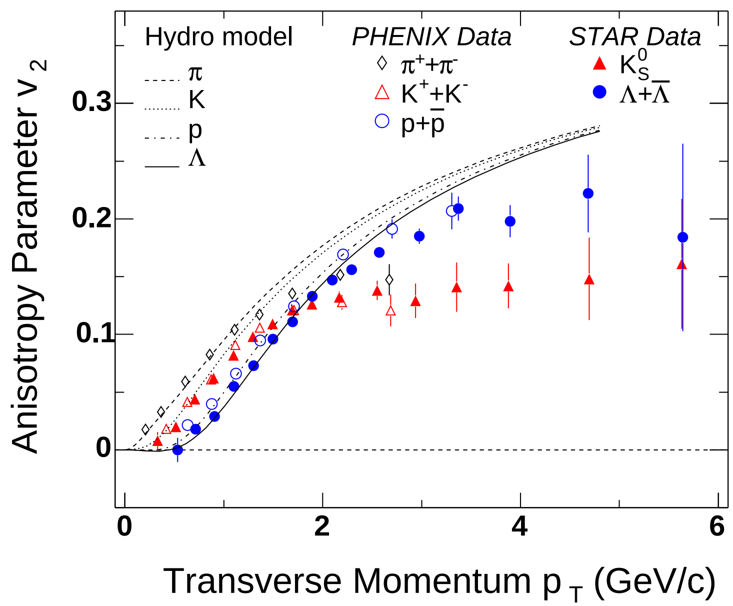
Elliptic flow is measured

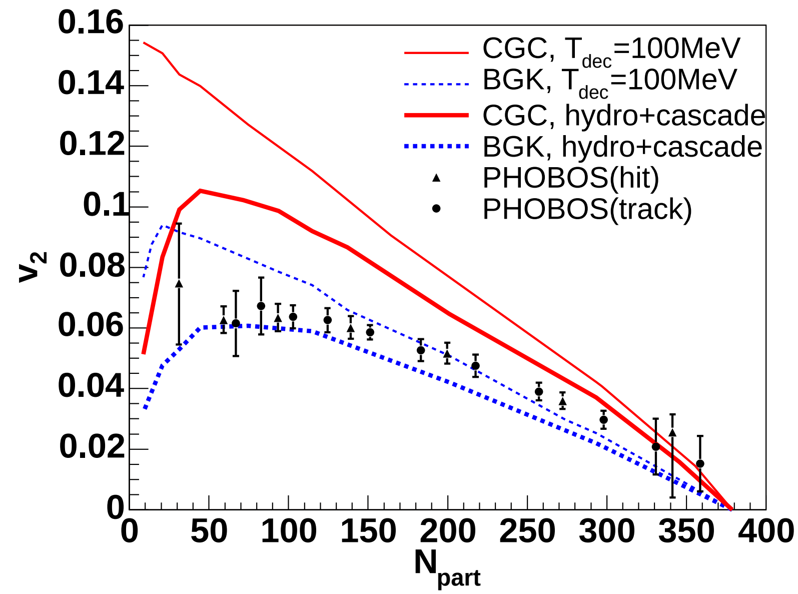


STAR experiment, minimum bias... .

We should try to understand it theoretically.

First attempt: ideal hydro





Works, DEPENDING on initial conditions.

Corrections to ideality exist, but are “small” (?)

Can we quantify that?

Ideal Hydrodynamics

Ideal hydro: stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{(4 equations, 10 unknowns)}$$

plus local equilibrium *assumption*:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} = \epsilon u^\mu u^\nu + P(\epsilon)\Delta^{\mu\nu}, \\ u^\mu u_\mu &= -1, \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \end{aligned}$$

depends on 4 parameters (ϵ, 3 comp of u^μ): closed.

Ideal hydro works well: corrections *eg, viscosity* small
Claim: "Most Perfect Liquid" exotic behavior. **Quantify!**

Nonideal Hydro

Assume that ideal hydro is "good starting point," look for small systematic corrections.

Near equilibrium iff $t_{\rm therm} \ll t_{\rm vary}, l_{\rm vary}/v$ (so ∂ small)

Allows expansion of corrections in gradients:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu}[\partial, \epsilon, u] \\ \Pi^{\mu\nu} &= \mathcal{O}(\partial\mu, \partial\epsilon) + \mathcal{O}(\partial^2\mu, (\partial\mu)^2, \ldots) + \mathcal{O}(\partial^3 \ldots) \end{aligned}$$

For Conformal theory $T^\mu_\mu = 0 = \Pi^\mu_\mu$, 1-order term unique:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}\,, \quad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial\cdot u\right)$$

Coefficient η is shear viscosity.

Viscous hydro

So why not consider (Navier-Stokes)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \quad ?$$

Because in **relativisitc** setting, it is

- Acausal: shear viscosity is transverse momentum diffusion. Diffusion $\partial_t P_\perp \sim \nabla^2 P_\perp$ has instantaneous prop. speed. Müller 1967, Israel+Stewart 1976
- Unstable: $v > c$ prop $+$ non-uniform flow velocity $\rightarrow$ propagate from future into past, exponentially growing solutions. Hiscock 1983

Problem only on short length scales where $\eta|\sigma| \sim P$. But numerics must treat these scales (or "numerical viscosity" which exceeds η is present)

Israel-Stewart approach

Add one second order term:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\Pi \, u^\alpha \partial_\alpha \sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \longrightarrow -\Pi$ in order-2 term:

$$\tau_\Pi \, u^\alpha \partial_\alpha \Pi^{\mu\nu} \equiv \tau_\Pi \, \dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

"Telegraph" relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$.

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_Π)

But why only one 2'nd order term???

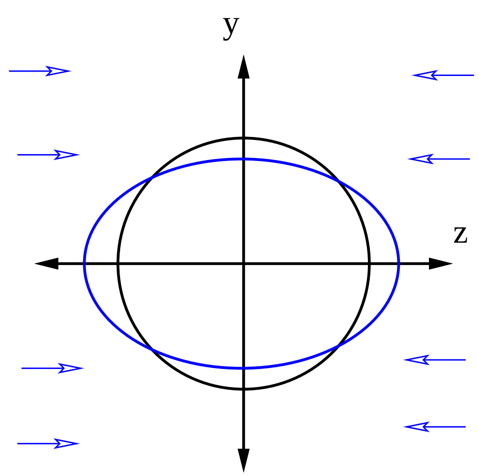
Second order hydrodynamics

It is more consistent to include all possible 2'nd order terms.
Assume *conformality* and *vanishing chem. potentials*:
5 possible terms Baier *et al*, [arXiv:0712.2451]

$$
\begin{aligned}
\Pi^{\mu\nu}_{2\text{ ord.}} = \; & \eta\tau_\Pi \left[u^\alpha \partial_\alpha \sigma^{\mu\nu} + \frac{1}{3} \sigma^{\mu\nu} \partial_\alpha u^\alpha \right] + \lambda_1 \left[\sigma^\mu_\alpha \sigma^{\nu\alpha} - (\text{trace}) \right] \\
& + \lambda_2 \left[\frac{1}{2} (\sigma^\mu_\alpha \Omega^{\nu\alpha} + \sigma^\nu_\alpha \Omega^{\mu\alpha}) - (\text{trace}) \right] \\
& + \lambda_3 \left[\Omega^\mu{}_\alpha \Omega^{\nu\alpha} - (\text{trace}) \right] + \kappa \left(R^{\mu\nu} - \ldots \right) , \\
\Omega_{\mu\nu} \equiv \; & \frac{1}{2} \Delta_{\mu\alpha} \Delta_{\nu\beta} (\partial^\alpha u^\beta - \partial^\beta u^\alpha) \;\; [\text{vorticity}] \, .
\end{aligned}
$$

Now, besides η, we have 5 more unknown coefficients.

What do these coefficients mean?

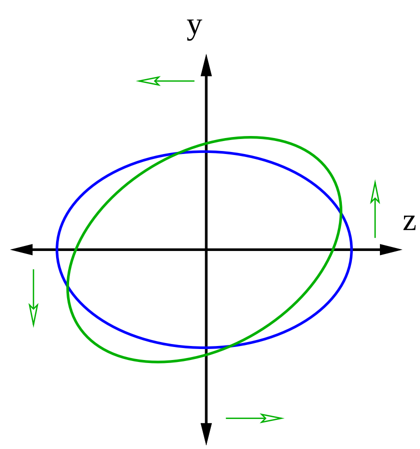


Contraction $\longrightarrow$ prolateness.

η: extent of prolateness.

τ_Π: how long prolateness lasts after contraction ends.

λ_1: induced prolateness depends on how prolate it already is!



(Vorticity) rotation

Vorticity means rotation.

λ_2: how much prolateness gets rotated in a rotating system.

Depends on how long prolateness lives.

Naively expect $\lambda_2 = -2\eta\tau_\Pi$.

Last, κ is curvature-dependence in EOS. Not really transport!

Second order: philosophy

(nonideal) hydro only consistent if η makes small corr.

These τ_Π, $\lambda_{1,2,3}$ make smaller corrections. κ irrelevant.

Make reasonable estimate for τ_Π, λ_{123}, test sensitivity

Guy argues: Ratios should be relatively robust

- Forget $\frac{\eta}{s}$. Think of $\frac{\eta}{P+\epsilon} = t_\eta$ a *timescale*. Pert: $1/g^4 T$
- Next order: $\frac{\lambda_1}{P+\epsilon} = t_\lambda^2$, $\frac{\eta\tau_\Pi}{P_\epsilon} = t_\pi^2$ Pert: $1/g^8 T^2$
- All are thermalization times. One expects $t_\lambda \sim t_\eta \sim t_\pi$.

Determine ratios where you can, use as priors in fit

Analogue model of QCD

We cannot solve QCD at physical coupling!
Consider an $Analogue\ Model$: $\mathcal{N}=4$ SYM:

- Also a gauge theory with fermionic (and scalar) matter
- Theory exists at all values of coupling, weak to strong
- Strong coupling solvable by AdS-CFT approach BUT
- Conformal–coupling does not run
- only solvable in infinite N_c limit
- Too much matter (pert: dominates screening+scattering)

How to compute these coefficients?

Hydro is response to disturbance.

Disturb system with $h_{xy}(k,\omega)$, measure T_{xy}.

Equivalent to G_R for two T_{xy}'s. Gets linear terms:

$$G_R^{T_{xy}T_{xy}}(\omega,k) = P - i\omega\eta + \omega^2\eta\tau_\Pi - \frac{\kappa}{2}(k^2+\omega^2)$$

Leading-order: pressure.

First correction: dissipative frequency dependent η

Next correction: τ_Π is higher-frequency bit

κ is k^2 bit since $k^2 h_{xy}$ gives nonzero curvature.

$\lambda_{1,2,3}$ require solution of fully nonlinear flow pattern.

SYM: coefficients computed

Baier et al [arXiv:0712.2451] and the Tata group, [arXiv:0712.2456] have computed the 2'nd order coefficients using holography:

Quantity	SYM value	Guy's ratio
τ_Π	$\frac{2-\ln 2}{2\pi T}$	$\frac{(\epsilon+P)\tau_\Pi}{\eta} = 4 - 2\ln(2)$
λ_1	$\frac{N_c^2 T^2}{16}$	$\frac{\lambda_1(\epsilon+P)}{\eta^2} = 2$
λ_2	$\frac{-\ln(2)\, N_c^2 T^2}{8}$	$\frac{\lambda_2(\epsilon+P)}{\eta^2} = -4\ln(2)$
κ	$\frac{N_c^2 T^2}{8}$	$\frac{\kappa(\epsilon+P)}{\eta^2} = 4$
λ_3	0	$\frac{\lambda_3(\epsilon+P)}{\eta^2} = 0$

Second Analogue Theory

QCD but at weak coupling using perturbation theory!

corr funcs nearly Gaussian: keep track of 2-point function.

Value of 2-pt function has interpretation as particle number:
$\phi^\dagger\phi$ is $\frac{1}{2} + \hat{N}$ number operator of free thy.

Leading-order: free propagation. Scatterings "rare".

$\Delta x \Delta p \ll 1$, classical particle propagation.

Long time dynamics: need to include scatterings, but in,out particles are uncorrelated.

Kinetic theory – q, g scattering from each other

Kinetic theory

State, all measurables described by particle distrib. $f_a(x,p)$:

$$T^{\mu\nu}(x) = \sum_a \int_p 2P^\mu P^\nu f_a(x,p)\,, \qquad \int_p \equiv \int \frac{d^3p}{(2\pi)^3 2p^0}$$

(Assumes weak coupling, slow x^μ dependence, little else)

Dynamics: Boltzmann equation (Schwinger-Dyson eq):

$$2P^\mu \partial_{\mu_x} f(x,p) \;\; = \;\; -\mathcal{C}[p, f(x,q)]$$

LHS: particle propagation. $p^0 \equiv \sqrt{\mathbf{p}^2} \equiv p$

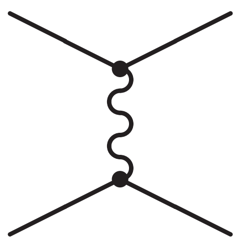
RHS: scattering (Im self-energy). Local in x but not p.

Theory dependence all contained in detailed form of $\mathcal{C}[f]$.

In our case, described in detail in AMY5: `hep-ph:0209353`

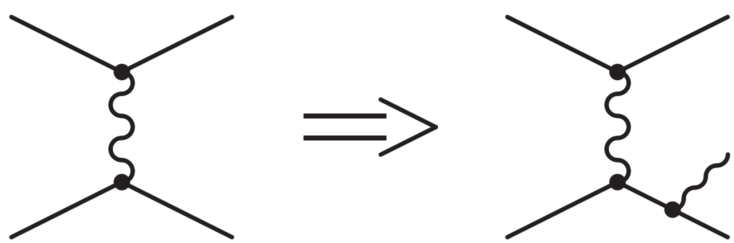
Mini-review: scattering in QCD

Dominant scattering process is Coulomb scattering



Effects Log divergent! NEED to include plasma screening effects which make effects finite. Leads to log coupling sensitivity!

Collinear splitting also very efficient:



Needed at leading order. Nasty, due to coherence effects

Expansion in (hydro) gradients

$$f(x,p) = f_0(\beta(x), u(x), p) + f_1(\partial, \beta, u, p) + f_2(\partial^2, \beta, u, p)$$

Subscript counts order in derivatives. β, u and $\epsilon, \vec{P}$ dual

LHS of Boltzmann has 1 deriv: RHS has 0.

$$\mathcal{O}\ 0: \quad \mathcal{C}[p, f_0(x,q)] = 0 \quad \rightarrow \quad f_{0,a} = \frac{1}{\exp(-\beta u^\mu P_\mu) \pm 1}$$

$$\mathcal{O}\ 1: \quad 2P^\mu \partial_{\mu_x} f_0(\beta(x), u(x), p) = -\mathcal{C}_1[p, f(q)]$$

where $\mathcal{C}_1$ is $\mathcal{C}$ expanded to lin. order in f_1.

$$\mathcal{O}\ 2: \quad 2P^\mu \partial_{\mu_x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, fq]$$

with $\mathcal{C}_{11}$ 2 order in f_1, $\mathcal{C}_2$ lin. order in f_2

First order in expansion

Organize it as

$$f_1(q) = -\mathcal{C}^{-1}_{1,qp} 2P^\mu \partial_\mu f_0(-\beta u^\nu P_\nu)$$

Gradients of free-theory distribution act as source for f_1.

$$2P^\mu \partial_\mu f_0 = -f_0' \beta P^\mu P^\nu (\partial_\mu u_\nu + u_\mu \partial_\nu \beta)$$

Organize source in spherical harmonics. $\ell = 0, 1$ determine u, β:

$$\partial_t \beta = \frac{\beta}{3} \partial_i u_i \qquad \text{and} \qquad \partial_t u_i = \frac{1}{\beta} \partial_i \beta$$

Remaining term is nontrivial:

$$f_1(q) = \mathcal{C}^{-1}_{1,qp} (p_i p_j - p^2 \delta_{ij}/3) \beta \sigma_{ij} f_0'$$

Solution always of form $f_1(p) = \frac{\beta^3}{2} \sigma_{ij} (p_i p_j - p^2 \frac{\delta_{ij}}{3}) \chi(-\beta u_\mu P^\mu)$

Second order Boltzmann Equation

$$2P^\mu \partial_{\mu_x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, fq]$$

Organize it as

$$f_2 = -\mathcal{C}_1^{-1}\Big(2P^\mu \partial_\mu f_1 + \mathcal{C}_{11}[f_1]\Big)$$

Term on right acts like a source for 2'nd order departure f_2. Two pieces: effect of inhomogeneity on 1'st order departure, nonlinearity of collision operator in departure from equilib.

Determining Π^{ij} only requires $\ell = 2$ moment of f_2, which simplifies calculation: only need $\ell = 2$ of RHS.

Term $2P^\mu \partial_\mu f_1$

Inhomogeneous flow when f already skewed. Consider

$$2P^\mu \partial_\mu \sigma_{\alpha\beta}(P^\alpha P^\beta - g^{\alpha\beta} p^2/3)\beta^3 \chi(-\beta u^\gamma P_\gamma)$$

Two types of terms: ∂_μ acts on $\sigma_{\alpha\beta}\beta^3$, or on χ

First term:

$$P^\mu P^\nu P^\alpha \left(\partial_\mu \sigma_{\nu\alpha} + 3\sigma_{\nu\alpha}\partial_\mu \ln\beta \right) \beta^3 \chi$$

Only contributions to Π^{ij} when 2 P's space, one time.

$$E p_i p_j \left(\partial_0 \sigma_{ij} + 2\partial_i \sigma_{0j} + 3\sigma_{ij}\partial_0 \ln\beta \right)$$

Contributes to τ_Π, λ_2, λ_1. Second term: contributes to λ_1.

What we get so far

One contribution to τ_Π, λ_2, and λ_1.

Automatically in ratio $1 : -2 : -1$. Therefore

$$\lambda_2 = -2\eta\tau_\Pi$$

Extra independent (positive) contribution to λ_1.

Detailed values depend on functional form of $\chi(\beta E)$.

Specific *Ansatz* (Grad 14-moment) gives specific values:

$$\frac{\eta\tau_\Pi}{\epsilon + P} = \frac{6\zeta(4)\zeta(6)}{\zeta^2(5)} \left(\frac{\eta}{\epsilon + P}\right)^2 , \quad \frac{\lambda_1}{\eta\tau_\Pi} = 1 \,.$$

But we *solve* for $\chi(\beta E)$–slightly different value

Hard part: $\mathcal{C}_{11}$

Nonlinear term in collision operator:
"extra" particles scattering from other "extra" particles

Theory dependent. Consider $2 \leftrightarrow 2$ scattering (present in most theories):

$$\mathcal{C}[p, f[q]] \;=\; \int_{kp'k'} (2\pi)^4 \delta^4(P{+}K{-}P'{-}K')|\mathcal{M}^2| \times$$
$$\Big(f(p)f(k)[1{\pm}f(p')][1{\pm}f(k')] - f(p')f(k')[1{\pm}f(p)][1{\pm}f(k)] \Big)$$

First order expansion: define $\bar{f}_1 = f_0(1{\pm}f_0)f_1$.

$$\Big(f(p)f(k)[1{\pm}f(p')][1{\pm}f(k')] - f(p')f(k')[1{\pm}f(p)][1{\pm}f(k)] \Big)$$
$$= \quad 0 + f_0(p)f_0(k)[1{\pm}f_0(p')][1{\pm}f_0(k')]\Big(\bar{f}_1(p){+}\bar{f}_1(k){-}\bar{f}_1(p'){-}\bar{f}_1(k') \Big)$$

That's what we needed in defining $\mathcal{C}_1$. Used twice already!

Next order: $f_0(p)f_0(k)[1 \pm f_0(p')][1 \pm f_0(k')]$ times

$$\bar{f}_1(p)\bar{f}_1(k)f_0(p)f_0(k)(e^{\frac{p+k}{T}}-1)+\bar{f}_1(p')\bar{f}_1(k')f_0(p')f_0(k')(1-e^{\frac{p+k}{T}})$$
$$+\Big[\bar{f}_1(p)\bar{f}_1(p')f_0(p)f_0(p')\left(e^{\frac{p}{T}}-e^{\frac{p'}{T}}\right)+(p'\to k')$$
$$+(p\to k)+(p,p'\to k,k')\Big]$$

Note that $\bar{f}_1(p) \propto \sigma_{ij}(p_i p_j - \delta_{ij} p^2/3)$.
In evaluating $\langle S_{ij} | \mathcal{C}_{11}[f_1] \rangle$ we meet angular integrations:

$$\text{defining } p_{\langle i} q_{j\rangle} = \frac{3p_i q_j + 3q_i p_j - 2p\cdot q\delta_{ij}}{6}\,, x_{pq} \equiv p\cdot q\,,$$
$$\sigma_{lm}\sigma_{rs}\int d\Omega_{\rm global} p_{\langle i}p_{j\rangle} q_{\langle l}q_{m\rangle} r_{\langle r}r_{s\rangle}$$
$$= \frac{4}{35}\left(\sigma_{il}\sigma_{jl}-\frac{\delta_{ij}}{3}\sigma_{lm}\sigma_{lm}\right) \times \Big(3x_{pq}x_{pr}x_{qr}-x_{pp}x_{qr}^2-x_{qq}x_{pr}^2-x_{rr}x_{pq}^2+2x_{pp}x_{qq}x_{rr}/3\Big)$$

Using these, one can bludgeon $\mathcal{C}_{11}$ term to death. Contributes only to λ_1.

Subtlety!

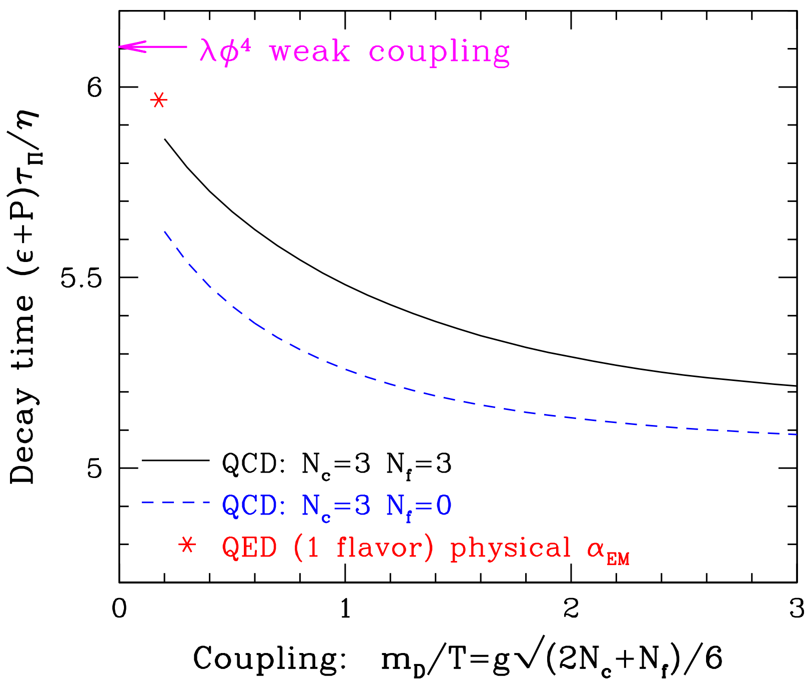
Preceding assumed that matrix element $|\mathcal{M}^2|$ is f independent.
In gauge and Yukawa theories, f enters $\mathcal{M}$ through screening!
Change in $f_0 \to f_0 + f_1$ changes screening, leading to correction to $|\mathcal{M}|^2$ linear in f_1.

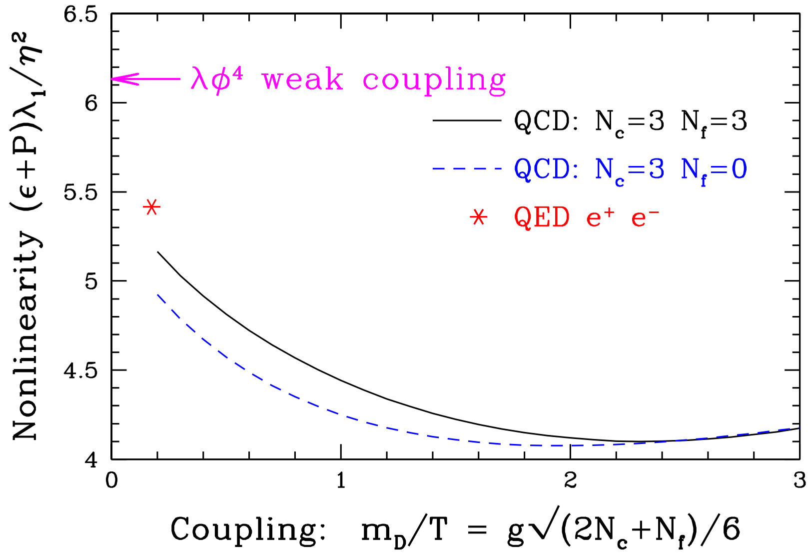
This is where things get hard.

So I won't tell you about it.

Results

$\lambda_3 = \kappa = 0$. $\lambda_2 = -2\eta\tau_\Pi$. τ_Π, λ_1 nontrivial:





Ratios vary with coupling, but only weakly!

QCD vs SYM comparison

Ratio	QCD value	SYM value
$\frac{\tau_{\Pi}(\epsilon+P)}{\eta}$	5 to 5.9	2.6137
$\frac{\lambda_1(\epsilon+P)}{\eta^2}$	4.1 to 5.2	2
$\frac{\lambda_2(\epsilon+P)}{\eta^2}$	-10 to -11.8	-2.77
$\frac{\kappa(\epsilon+P)}{\eta^2}$	$\mathcal{O}(g^8)$	4
$\frac{\lambda_3(\epsilon+P)}{\eta^2}$	0	0

Weakly coupled $\mathcal{N}=4$ SYM will be almost same as QCD results presented here. Difference is with coupling, not with theory! NLO order in expansion not known on either side!

Conclusions

- Hydro seems sensible framework in heavy ion coll.
- Shear viscosity should be quantified!
- Requires expansion to 2'nd order in gradients
- Calculation in pert. QCD is intricate.
- Ratios are relatively robust. But strong-coupling results are incompatible with kinetic theory framework.

Strong coupling DOES NOT have quasiparticles.
How does one compute non-kinetic corrections?