Spectral sums for Dirac operator and Polyakov loops

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related results by:

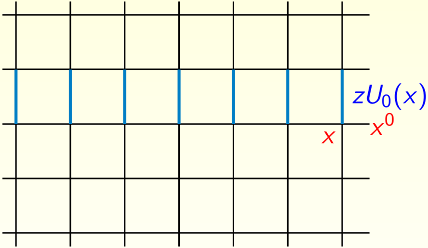
Bilgici, Bruckmann, Gattringer, Hagen, Soldner, . . .

cp. Christof Gattringer talk
cp. poster Synatschke and Wozar

Center transformations, twisting

multiply all $U_0(x^0, \boldsymbol{x})$ in one time-slice with center elements z

$$\{U_\mu(x)\} \longrightarrow \{{}^zU_\mu(x)\} \text{ twisted configuration on } V = N_\tau \cdot N_s^3$$



$\mathcal{C}_{xx}$ winds n-times around periodic time direction:

$$\mathcal{W}_{\mathcal{C}_{xx}} \longrightarrow z^n \mathcal{W}_{\mathcal{C}_{xx}}$$

Polyakov loop:

$$cP_{\boldsymbol{x}} \longrightarrow z\mathcal{P}_{\boldsymbol{x}}$$

Yang-Mills action invariant

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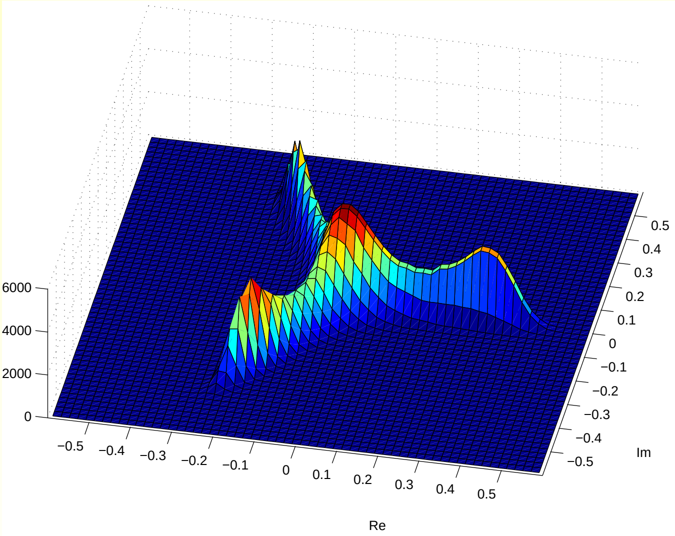

histogram of $P = \mathrm{tr}\,\mathcal{P}$

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Spectral sums for Dirac operator

- $\mathcal{D}$ with nearest neighbour interaction: Wilson, staggered, ...
  γ_5-hermiticity $\Rightarrow$ non-real eigenvalues come in pairs λ_p, λ_p^*
- spectral problem

$$\mathcal{D}\psi_{p,\ell} = \lambda_p \psi_{p,\ell}, \quad \psi_{p,\ell} \text{ normalized}$$

- matrix functions $f(\mathcal{D})$, spectral resolution in position basis

$$\langle x|\mathrm{tr}_{c,s} f(\mathcal{D})|x\rangle = \sum_{p=1}^{n_D} f(\lambda_p)\varrho_p(x), \quad \varrho_p(x) = \sum_\ell |\psi_{p,\ell}(x)|^2$$

- $\varrho_p(x)$ spectral space-time density

$$\varrho_p(\boldsymbol{x}) = \sum_{x^0} \varrho_p(x^0, \boldsymbol{x}), \quad \varrho_p \equiv \sum_x \varrho_p(x) = \mathrm{deg}(\lambda_p)$$

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- twisting: $U \to {}^zU \Longrightarrow \mathcal{D} \to {}^z\mathcal{D},\ \lambda_p \to {}^z\lambda_p,\ \varrho_p(x) \to {}^z\varrho_p(x)$
- spectral problems for twisted ${}^z\mathcal{D} \to$ center averaged sums

$$\mathcal{S}_f(x) = \sum_k z_k^* \sum_p {}^{z_k}\varrho_p(x)\, f({}^{z_k}\lambda_p)$$
$$\mathcal{S}_f \ = \sum_x \mathcal{S}_f(x) = \sum_k z_k^* \,\mathrm{Tr}\,({}^{z_k}\mathcal{D})$$

- Gattringer formula for choice $f(\lambda) = \lambda^{N_\tau}$:

$$\mathcal{S}(\boldsymbol{x}) = \sum_k z_k^* \sum_p {}^{z_k}\varrho_p(\boldsymbol{x}) \left({}^{z_k}\lambda_p\right)^{N_\tau} = \kappa'\, P(\boldsymbol{x})$$
$$\Rightarrow \mathcal{S} = \sum_k z_k^* \,\mathrm{Tr}\,\left({}^{z_k}\mathcal{D}\right)^{N_\tau} = \kappa\, L \quad \text{any} \ \ \{U_\mu(x)\}$$

L spatial average of $P(\boldsymbol{x}) = \mathrm{tr}\,\mathcal{P}(\boldsymbol{x})$

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Polyakov loop $\leftrightarrow$ spectral data of $\mathcal{D}$
Confinement $\leftrightarrow$ chiral symmetry breaking

- but: $\mathcal{S}(x)$ dominated by large eigenvalues: partial sum

$$\mathcal{S}_n = \sum_k z_k^* \sum_p^{n} \left({}^{z_k}\lambda_p\right)^{N_\tau} \quad \text{(assume } \varrho_p = 1\text{)}$$

- related problem: sick continuum limit $N_\tau \to \infty$
- generalized partial sums

$$\mathcal{S}_{f,n}(x) = \sum_k z_k^* \sum_{p=1}^{n} {}^{z_k}\varrho_p(x) f\left({}^{z_k}\lambda_p\right) \stackrel{n \to n_D}{\longrightarrow} \mathcal{S}_f(x)$$

space-averages $\mathcal{S}_{f,n}$

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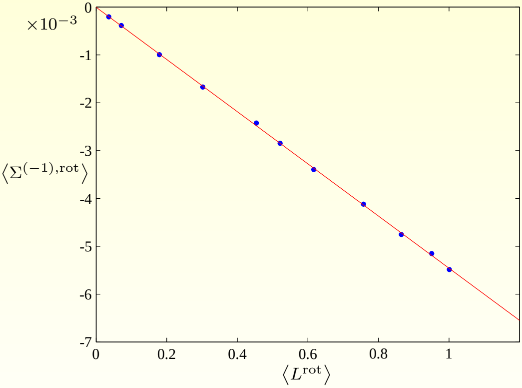
Propagator spectral sums

$$f(\lambda) = \lambda^{-s} \Longrightarrow \Sigma^{(-s)} \quad \text{Zeta-function,} \ \Sigma^{-N_t} = S$$



The expectation value of $\Sigma^{-1,\mathrm{rot}}$ as function of L^{rot}

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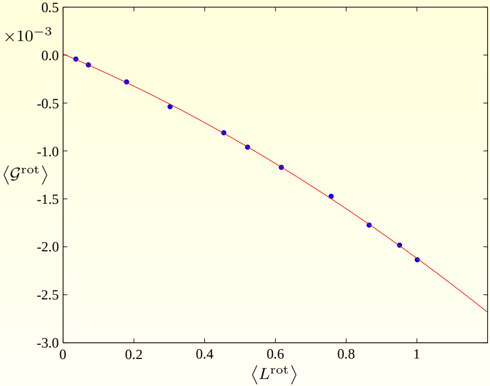
Gaussian spectral sums

$$f(\lambda) = e^{-\lambda^2} \Longrightarrow \mathcal{G} \quad \text{heat kernel}$$



The expectation value of $\mathcal{G}^{\mathrm{rot}}$ as function of L^{rot}

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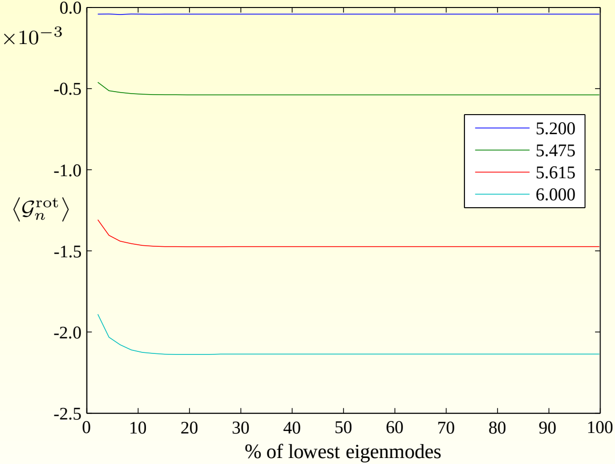
partial Gaussian sums $\mathcal{G}_n$ converge quickly $\to \mathcal{G}$, lowest 3% sufficient



small $4^3 \times 3$ lattice, $n_D = 2304$, $\beta_c = 5.49$

Spectral sums for the continuum theory

- Euclidean theory on $4d$ torus T^4, volume $V = \beta \cdot L^3$
  (anti)periodic fields in time direction with period $\beta = 1/k_B T$
  periodic (mod gauge transformations) in spatial directions
- Dirac operator

$$\mathcal{D} = i\gamma^\mu D_\mu + im, \quad D_\mu = \partial_\mu - iA_\mu, \;\; A_\mu = A_\mu^a \lambda_a$$

- spectral problem

$$\mathcal{D}\psi_{p,\ell} = \lambda_p \psi_{p,\ell}, \quad \psi_{p,\ell}(x) \;\; \text{normalized on torus}$$

- gauge transformation

$$^gA_\mu = g(A_\mu + i\partial_\mu)g^{-1}, \quad {}^g\psi(x) = g(x)\psi(x)$$

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- g periodic $\Rightarrow$ invariant λ_p and spectral density

$$\varrho_p(x) = \sum_\ell |\psi_{p,\ell}(x)|^2, \quad \varrho_p(\boldsymbol{x}) = \int_0^\beta dx^0 \varrho(x^0, \boldsymbol{x})$$

- center transformation = transformation with non-periodic $g(x)$: $g(x_0+\beta,\boldsymbol{x}) = z\, g(x_0,\boldsymbol{x}),\ z \in \mathcal{Z} \Rightarrow {}^gA_\mu$ still periodic, but

$${}^g\psi(x_0+\beta,\boldsymbol{x}) = -z\, {}^g\psi(x_0,\boldsymbol{x})$$

- twisted Diracoperator ${}^z\mathcal{D}_A \equiv \mathcal{D}_{{}^gA} = g\mathcal{D}_A g^{-1}$
  ${}^g\psi$ not eigenfunctions of ${}^z\mathcal{D}$ for center transformation
  spectrum changes if $z \neq \mathbb{1}$
- twisting: $A \to {}^gA \Longrightarrow \mathcal{D} \to {}^z\mathcal{D},\ \lambda_p \to {}^z\lambda_p,\ \varrho_p(x) \to {}^z\varrho_p(x)$

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- spectral sums in continuum: weighted center average of $\langle x|f(\mathcal{D})|x\rangle$

$$\mathcal{S}_f(x) = \lim_{n\to\infty}\mathcal{S}_{f,n}(x),\quad \mathcal{S}_{f,n}(x)=\sum_{p=1}^{n}\sum_k z_k^{*}\,{}^{z_k}\varrho_p(x)\,f({}^{z_k}\lambda_p)$$

$$\mathcal{S}_f = \int d^dx\,\mathcal{S}_f(x) = \sum_k z_k^*\,\mathrm{Tr}\,f\left({}^{z_k}\mathcal{D}\right)$$

- all spectral sums are order parameters for center symmetry

$$\mathcal{S}_f = \sum_k z_k^*\,\mathrm{Tr}\,f\left({}^{z_k}\mathcal{D}\right)$$

$$\stackrel{A\to{}^gA}{\longrightarrow} \sum_k z_k^*\,\mathrm{Tr}\,f\left({}^{z_k z}\mathcal{D}\right) \stackrel{z_k z = z_k'}{=} z\sum_k z_k'^*\,\mathrm{Tr}\,f\left({}^{z_k'}\mathcal{D}\right) = z\mathcal{S}_f$$

- similarly $\mathcal{S}_f(x)\to z\,\mathcal{S}_f(x)$

Spectral sums and Polyakov loops

- $S_f(x)$ gauge invariant $\Rightarrow$ function of $W_{\mathcal{C}_x}$
  $L \gg \beta \Rightarrow$ neglect loops winding around the spatial directions
  $\mathcal{S}_f(x)$ transforms under twists as dressed Polyakov loops $P_{\mathcal{C}_{\boldsymbol{x}}} \Rightarrow$

$$\mathcal{S}_f(\boldsymbol{x}) = \sum_{\mathcal{C}_{\boldsymbol{x}}} P_{\mathcal{C}_{\boldsymbol{x}}} \cdot F_{\mathcal{C}_{\boldsymbol{x}}}(W_{\mathcal{C}_{\boldsymbol{x}}})$$

  $F_{\mathcal{C}_{\boldsymbol{x}}}$ gauge- and center invariant
  center $U(1) \Rightarrow$ contractable $W_{\mathcal{C}_{\boldsymbol{x}}}$ and combinations $P^*_{\mathcal{C}_{\boldsymbol{x}}} P_{\mathcal{C}'_{\boldsymbol{x}}}$.
- on lattice or 't Hooft's constant field strength configurations on torus:

$$\mathcal{S}_f(\boldsymbol{x}) \xrightarrow{L \gg \beta} \text{const} \cdot P(\boldsymbol{x})$$

  lattice: hopping parameter expansion; continuum: explicit calulation

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Constant field strength configurations

- Here: Schwinger model
  for SU(2) on T^4 see arXiv:0803.0271 [hep-lat]
- 'boundary conditions'

$$\psi(x_0+\beta,x_1)=-\psi(x_0,x_1)\quad,\quad \psi(x_0,x_1+L)=e^{i\gamma(x)}\psi(x_0,x_1)$$
$$\gamma=-2\pi q\, x_0/\beta\quad,\quad q\ \text{instanton number}$$

- minimal action instanton configuration ($A_1=0$)

$$A_0=-Bx_1+\frac{2\pi}{\beta}h\Longrightarrow F_{01}\equiv B,\quad P(\boldsymbol{x})=e^{2\pi ih-i\Phi\, x_1/L}$$

- $U(1)$ twist with

$$g=e^{2\pi i\alpha x_0/\beta}\Longrightarrow z=e^{2\pi i\alpha},\quad {}^{g}\!A_0=-Bx_1+\frac{2\pi}{\beta}(h+\alpha)$$

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- spectral problem for $\mathcal{D}^2_{m=0}$ reduces to harmonic oscillator on 'circle'
  $\Rightarrow$ elliptic functions
  groundstate: θ-function; excited states $(y = x_1 + \frac{L}{q}(\ell - h - \frac{1}{2}))$

$$\psi_{p,\ell}(x) \propto e^{2\pi i \ell x_0/\beta} \sum_{s=-\infty}^{\infty} H_p\Big(\sqrt{B}\,(y+sL)\Big)\,\xi_0(y+sL)\,e^{2\pi i\, sqx_0/\beta}$$

- eigenvalues of $-\mathcal{D}^2$ twist-independent

$$\mu_p = \begin{cases} 0 & \text{degeneracy: } q \\ 2pB & \text{degeneracy: } 2q. \end{cases}$$

- center average $\Rightarrow$

$$\bar{\varrho}_p(x) = \int_0^1 d\alpha\, e^{-2\pi i\alpha}\ {}^{\alpha}\varrho_p(x) \qquad (z^* = e^{-2\pi i\alpha})$$

- $\bar{\varrho}_p(\boldsymbol{x})$ proportional to $P(\boldsymbol{x})$:

$$\bar{\varrho}_p(\boldsymbol{x}) \equiv \int dx_0\, \bar{\varrho}_p(x) = -\frac{q}{L} P(\boldsymbol{x})\, L_p(\pi q \tau)\, e^{-\pi q \tau/2}, \quad \tau = \frac{\beta}{L}$$

- general spectral sum for Schwinger-model instanton ($L_{-1} \equiv 0$)

$$\mathcal{S}_f(\boldsymbol{x}) = -\frac{q}{L} P(\boldsymbol{x}) \sum_{p=0}^{\infty} f(\mu_p) \left\{ L_p(\pi q \tau) + L_{p-1}(\pi q \tau) \right\} e^{-\pi q \tau/2}$$

- all spectral sums are proportional to $P(\boldsymbol{x})$
  convergence for which f? (cp. Banks-Casher $f(\mu) \sim \mu^{-1/2}$)
  speed of convercence depends on f

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- Gaussian spectral sum $\mathcal{G}$ for $f(\mathcal{D}) = \exp(t\mathcal{D}^2)$:

$$\mathcal{G}_t(\boldsymbol{x}) = -\frac{q}{L}\coth(tB)\,\exp\left(-\frac{\pi q\tau}{2}\coth(tB)\right)P(\boldsymbol{x})$$

- large volumes: $2\pi q = BV = \Phi$ fixed $\Rightarrow$

$$\mathcal{G}_t(\boldsymbol{x}) \longrightarrow -\frac{\beta}{2\pi t}\,e^{-\beta^2/4t}\,P(\boldsymbol{x}) \quad \beta L \gg tq$$

- small-t and large-t asymptotics

$$\mathcal{G}_t(\boldsymbol{x}) \quad \longrightarrow \quad \begin{array}{ll} -\frac{1}{L}\frac{q}{tB}e^{-\pi q\tau/(2tB)}\,P(\boldsymbol{x}) & \text{for } t \to 0 \\ -\frac{1}{L}(q - 2qe^{-2tB})e^{-\pi q\tau/2}\,P(\boldsymbol{x}) & \text{for } t \to \infty \end{array}$$

- asymptotic expansion for small t: all $a_n = 0$
  large t: zero-mode contribution $+\ O(e^{-\mu_1 t})$

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- zero-mode subtracted heat kernel

$$\mathcal{G}'_t(x) = \sum_{p:\mu_p>0} \sum_k z_k^* \, {}^{z_k}\varrho_p(x)\, e^{-t\,{}^{z_k}\mu} = \mathcal{G}_t(\boldsymbol{x}) - \bar{\varrho}_0(x)$$

exponentiell fall-off, vanishing asymptotic expansion

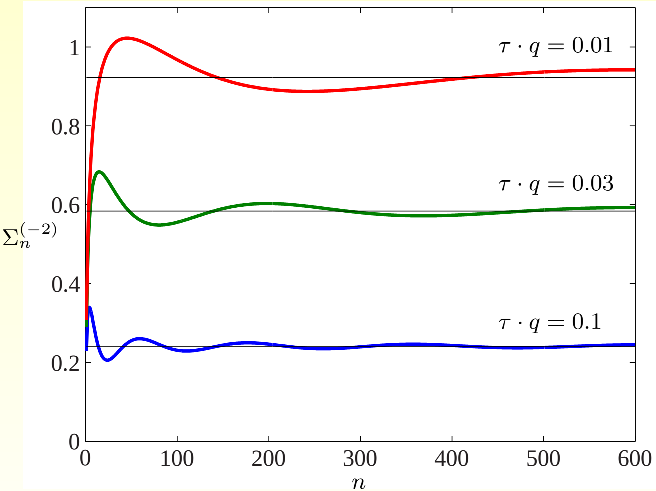
- Mellin transform exist for all s

$$\Sigma^{(-2s)}(x) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathcal{G}'_t(x)$$

- spectral sum for $f(\mathcal{D}) = 1/\mathcal{D}^2$

$$\Sigma^{(-2)}(\boldsymbol{x}) = \sum_{p,\mu_p>0} \Big(\sum_k z_k^* \frac{1}{{}^{z_k}\mu_p} {}^{z_k}\varrho_p(\boldsymbol{x}) \Big)$$

$$= \frac{\beta}{4\pi} \{\gamma + \log(\pi q \tau) - e^{\pi q \tau}\, \Gamma(0, \pi q \tau)\}\, e^{-\pi q \tau/2}\, P(\boldsymbol{x})$$



The partial sums Σ_n^{-2} as function of n

On the convergence of spectral sums

- zero-mode subtracted heat kernel

$$K'(t,x) = K(t,x) - \varrho_0(x), \quad K(t,x) = \langle x|e^{t\mathcal{D}^2}|x\rangle = \sum_p e^{-t\mu_p}\varrho_p(x)$$

- large and small-t asymptotics

$$K'(t,x) \quad \longrightarrow \quad e^{-t\mu_1}\varrho_1(x) \qquad \text{for} \quad t \to \infty$$

$$K(t,x) \quad \longrightarrow \quad t^{-d/2}\Big\{\sum_{n=0}^{N} a_n(x)t^n + \mathcal{O}(t^{N+1})\Big\} \quad \text{for} \quad t \to 0$$

- a_n gauge-invariant function of $F_{\mu\nu}$ and its covariant derivatives $\Rightarrow a_n$ center-invariant,

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- Mellin transform

$$\zeta'_{\mathcal{D}}(s,x) = \langle x|\frac{1}{(\mathcal{D}^2)^s}|x\rangle = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}K'(t,x)$$

d even: simple poles at $s=\frac{d}{2},\dots,1$,residues $a_0(x),\dots,a_{\frac{d}{2}-1}(x)$

- a_k center-invariant $\Rightarrow$

$$\Sigma^{(-2s)}(x) = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}\mathcal{G}'(t,x) = \sum_k z_k^*\,\zeta'_{z_k\mathcal{D}}(s,x)$$

no poles in $s \Rightarrow$ spectral sums Σ^{-2s} convergent $\forall s$
important: first sum over k and only afterwards over p!

- γ_5 symmetry for $m=0 \Rightarrow$

$$\mathcal{S}_{\mathcal{D}^{-2s}}(x) = \frac{1}{2}\left(1+(-1)^{2s}\right)\mathcal{S}_{(\mathcal{D}^2)^{-s}}(x)$$

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Numerical investigations for SU(2)

- static quark potential

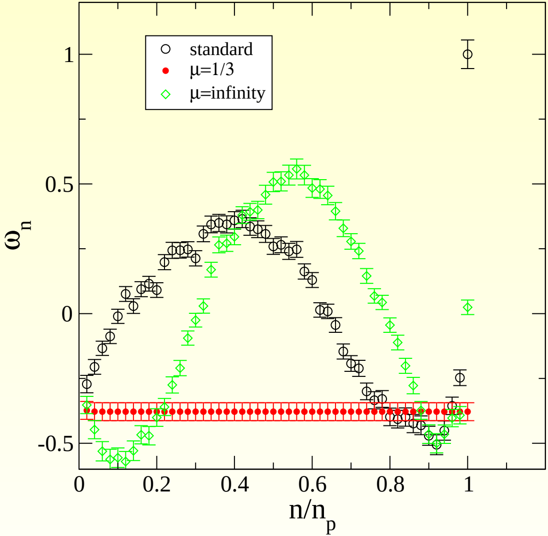
$$V(r) = -T \log C(r), \quad C(r) = \langle P(\boldsymbol{x}) P(\boldsymbol{x} + r\boldsymbol{e}_3) \rangle$$

insert Gattringers result $P(\boldsymbol{x}) = \mathcal{S}(\boldsymbol{x}) \Rightarrow$ cancellation of huge contributions (convergence proof)
$\Rightarrow$ must include all eigenfunction

- use IR-dominated spectral sum, e.g. $\mathcal{G}_n$
- correlation measure

$$\omega_n = \frac{\left\langle P(\boldsymbol{x})\, \mathcal{S}_{f,n}(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}}}{\sqrt{\left\langle P^2(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}} \left\langle \mathcal{S}_{f,n}^2(\boldsymbol{x}) \right\rangle_{\boldsymbol{x}}}}$$

average over $\boldsymbol{x}$ for fixed configuration



ω_n for $\mathcal{S}_n(\boldsymbol{x}), \mathcal{G}_n(\boldsymbol{x})$ as function of n/n_D, 6^4 lattice

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- use $\mathcal{G}_n(x)$ instead of $\mathcal{S}_n(x)$: measure ω_n almost constant
- simulations: staggered fermions, SU(2)

$$\mathcal{G}_n(x) := \frac{1}{8}\sum_{p=1}^{n}\left(\varrho_p(x)e^{-\lambda_p^2/\mu^2} - {}^z\varrho_p(x)e^{-{}^z\lambda_p^2/\mu^2}\right)$$

- improved action (rotational invariance, scaling)

$$S = \beta\sum_{\mu>\nu,x}\left[\gamma_1 P_{\mu\nu}(x) + \gamma_2 P^{(2)}_{\mu\nu}(x)\right]$$

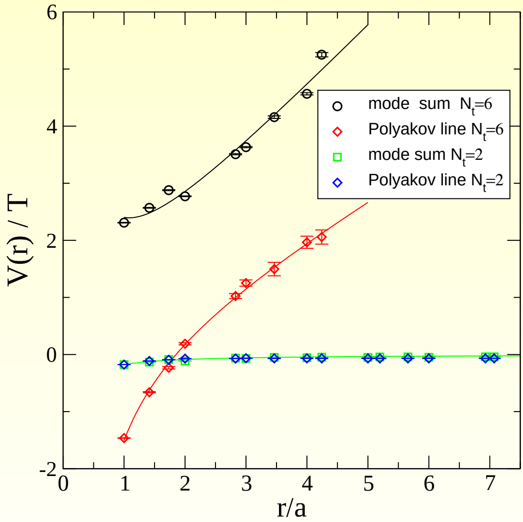
$$\beta = 1.35,\ \gamma_1 = .0348,\ \gamma_2 = -0.10121,\ \sigma a^2 = 0.1244(7)$$

- compare $V(r)$ with

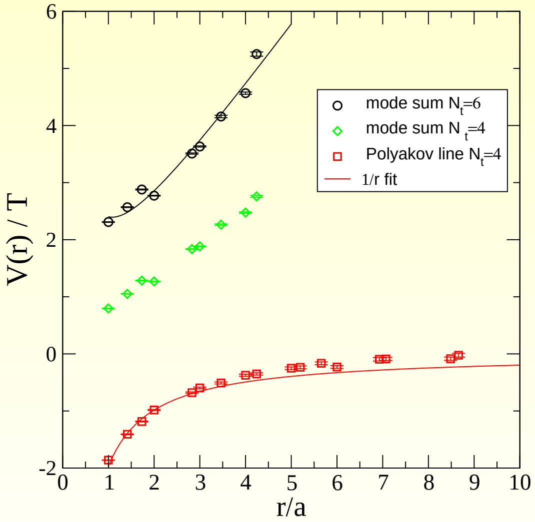
$$V_n^{\mathcal{G}}(r) = -T \log C_n^{\mathcal{G}}(r), \quad C_n^{\mathcal{G}}(r) = \langle \mathcal{G}_n(\boldsymbol{x}) \mathcal{G}_n(\boldsymbol{x} + r\boldsymbol{e}_3) \rangle$$

- simulation parameters

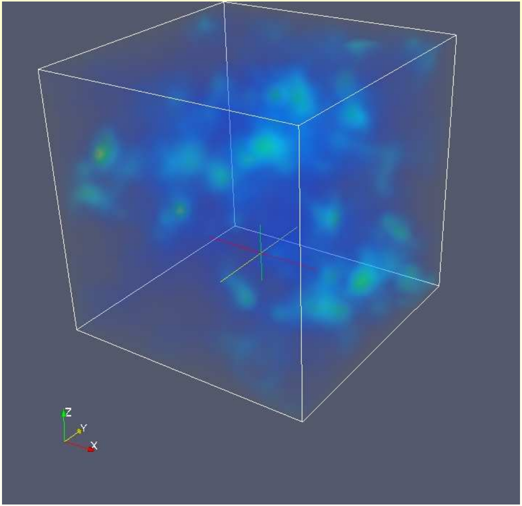
β	σa^2	lattice	T/T_c	configurations
1.35	0.1244(7)	$12^3 6$	0.7	8658
1.35	0.1244(7)	$12^3 4$	1.0	12000
1.35	0.1244(7)	$12^3 2$	2.1	12000



potential from P and $\mathcal{G}_n$, $N_t = 6$ confined, $N_t = 2$ deconfined



potential from P and $\mathcal{G}_n$ for $T \approx T_c(N_t = 4)$



mode sum $|\mathcal{G}_n(\boldsymbol{x})|$ in a 20^3 spatial hypercube. $L = 3.2$ fm
smooth texture at scale 0.3 fm

Conclusions, remarks

- all spectral sums define order parameters for center symmetry
- spectral sum approach can be formulated for continuum theories
- first sum over center elements and afterwards over the eigenvalues
  $\Rightarrow$ spectral sums exist for almost all $f(\mathcal{D})$
- reconstructed Polyakov loop locally from spectral sums on lattice
- same construction for continuum theory and constant $F_{\mu\nu}$
- beyond constant field strength?
  relation to Banks-Casher in continuum (CSB $\leftrightarrow$ confinement)

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