Review of previous lecture (June 30)

RG transformations form a semi-group:

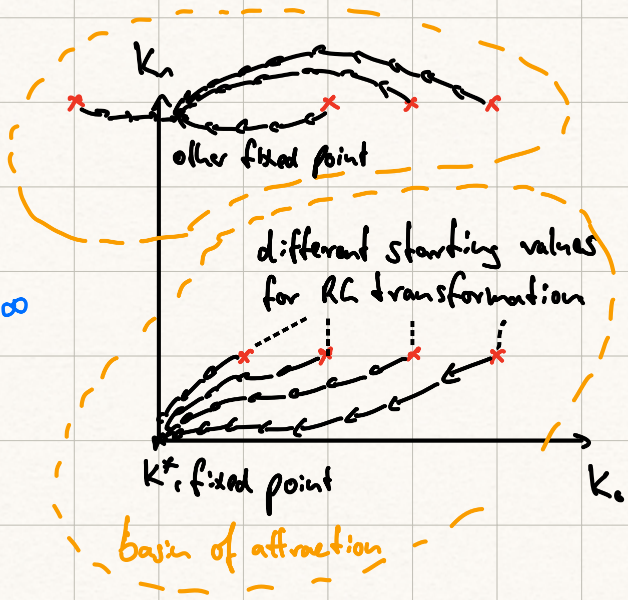
$$[k''] = R_{\ell_2}[k'] = R_{\ell_2}\cdot R_{\ell_1}[k] = R_{\ell_1\cdot\ell_2}[k]$$

$$R_{\ell=1}[k] = k$$

in general no inverse $\quad R_\ell^{-1}[k]$

fixed points

- RG transformations have to be analytic since only a finite number of degrees involved
- study RG flow for $n\to\infty$ RG transformations
- at a fixed point we have



$$\xi[k^*] = \frac{\xi[k^*]}{\ell}$$

$\Rightarrow \xi[k^*] = 0$ or $\xi[k^*] = \infty$

$\downarrow$ trivial fixed point $\qquad \downarrow$ critical fixed point

- RG flow near a fixed point

$$K_n' = K_n^* + \delta K_n' = K_n^* + \sum_m M^\ell_{nm}\,\delta K_m$$

with $M^\ell_{nm} = \left.\dfrac{\partial K_n'}{\partial K_m}\right|_{K_m = K_m^*}$

for the sake of simplicity we will assume for the following discussion that the eigenvectors $\vec{e}^{(i)}$ are orthonormal, that is generally not true (see 2d Ising model), but does not change the main arguments

$$\Rightarrow \delta k_n' = \sum_m M^l_{nm} \delta k_m$$

$$= \sum_m M^l_{nm} \sum_i a^{(i)} e_m^{(i)}$$

$$= \sum_i a^{(i)} \lambda_l^{(i)} e_n^{(i)} = \sum_i \underbrace{a'^{(i)}}_{\text{projection of } \delta \vec{k}' \text{ on eigenvectors } \vec{e}^{(i)}} e_n^{(i)}$$

depending on $\lambda_l^{(i)}$, some components of δk grow under M^l while others shrink.

a) $|\lambda_l^{(i)}| > 1$, i.e. $y_i > 0$: $a^{(i)}$ grows during RG flow

$\longrightarrow$ relevant eigenvectors / eigenvalues / directions

b) $|\lambda_l^{(i)}| < 1$, i.e $y_i < 0$: $a^{(i)}$ shrinks during RG flow

$\longrightarrow$ irrelevant eigenvectors

c) $|\lambda_l^{(i)}| = 1$, i.e. $y_i = 0$: $a^{(i)}$ invariant

$\longrightarrow$ marginal eigenvectors

$\Rightarrow$ for $\vec{k}$ near $\vec{k}^*$ (not on the critical manifold), the RG flows away from $\vec{k}^*$ are associated with relevant eigenvectors, irrelevant eigenvectors correspond to directions of flow into the fixed point

$\Rightarrow$ eigenvectors corresponding to irrelevant eigenvalues span critical manifold

global and local properties of the RG flow

global behaviour of RG flow determines phase diagram:

a) start at a given point $\vec{K}^{(0)} = (K_0^{(0)}, K_1^{(0)}, K_2^{(0)}, \dots)$

b) iterate RG transformations: $\vec{K}^{(0)} \to \vec{K}^{(1)} \to \vec{K}^{(2)} \to \dots \to \vec{K}^{(n)}$

c) identify fix points to which the system flows,
state of system described by this fixed point corresponds to the phase at the original point $\vec{K}_0$ (note that partition function is preserved along RG trajectory if RG transformations are performed exactly)

classification of fixed points:

a) sinks: all trajectories flow into fix point (no relevant directions), sinks correspond to bulk phases

example: Ising model in 2d
sink at $B = \pm\infty$, $T=0$: at finite B there is a finite magnetization for all T

b) discontinuity / continuity fixed points (1 relevant direction)

$\downarrow$

discontinuity	continuity
phase boundary	phase of system
for Ising model:	no transition in vicinity
all points on line $B=0$ for $T<T_c$	all points on line $B=0$, $T>T_c$
$\to$ flow to $B=0$, $T=0$	$\to$ flow to $B=0$, $T=\infty$
first order phase transition when crossing $B=0$ from $B>0$ or $B<0$	

both fixed points unstable with respect to $B=0 \to B=0^{\pm}$ (relevant direction)

$\downarrow$ flows to sinks (a)

c) critical points, multiphase coexistence (2 relevant directions)

2 couplings must be tuned to place system at the critical point ($B=0$, $T=T_c$)

$\downarrow$

flow into critical fixed point

Table 9.1 CLASSIFICATION OF FIXED POINTS

Codimension	Value of ξ	Type of Fixed Point	Physical Domain
0	0	Sink	Bulk phase
1	0	Discontinuity FP	Plane of coexistence
1	0	Continuity FP	Bulk phase
2	0	Triple point	Triple Point
2	∞	Critical FP	Critical manifold
Greater than 2	∞	Multicritical point	Multicritical point
Greater than 2	0	Multiple coexistence FP	Multiple coexistence

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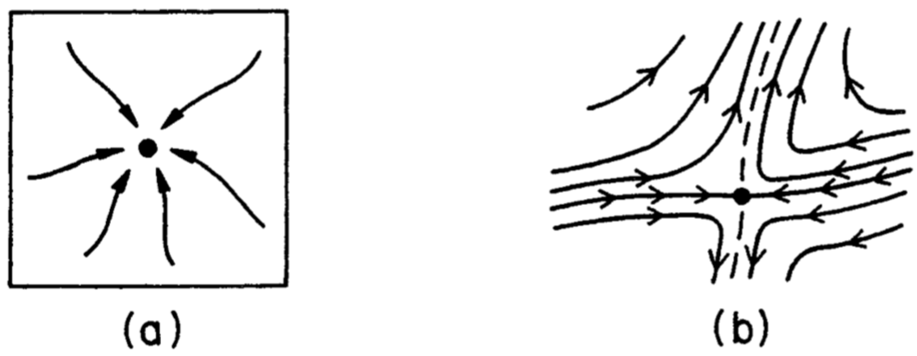

Figure 9.2 Renormalisation group flows near a critical fixed point: (a) View of flows on the critical manifold. (b) View of flows off the critical manifold.

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$$\mathcal{H} = K_1 \sum_{<ij>} S_i S_j + K_2 \sum_{ij=n.n.n.} S_i S_j \quad K_1 = J_1/k_B T \quad K_2 = J_2/k_B T$$

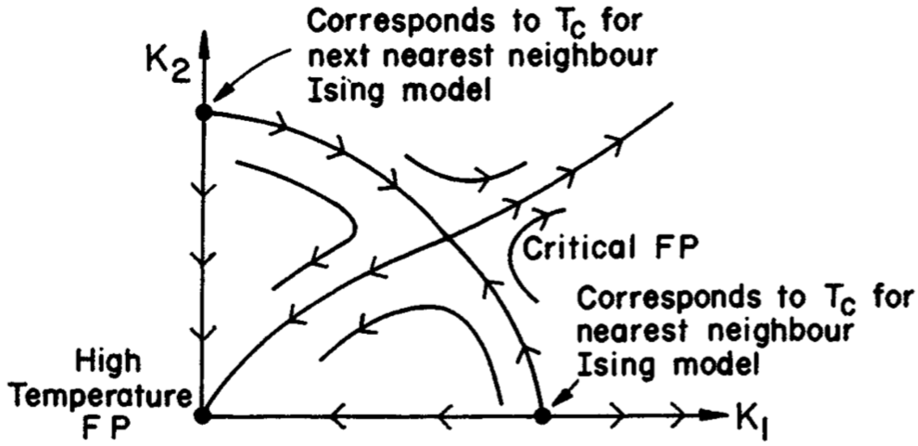

Figure 9.3 Flow diagram for an Ising model with nearest and next nearest neighbour interactions.

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local properties of RG flow around critical fixed point determines critical behaviour

- trajectories on the critical manifold remain on the manifold and flow to critical fixed point
- trajectories that start slightly off the critical manifold initially flow towards critical fixed point, but are ultimatelly repelled due to relevant couplings
- the same relevant eigenvalues (see below) drive all slightly off-oriented systems away from critical manifold (for a given universality class), independent of original values of coupling constants
  ↳ universality

critical exponents from the RG flow

- when iterating RG transformation for $T = T_c + \varepsilon$ $(B=0)$, trajectory approaches fixed point and is then repelled towards $T=\infty$ fixed point (complete disorder, $\xi \to 0$) while close to critical fixed point $\xi \gg a$ (lattice spacing)
- "turning point" happens roughly when $\xi \sim a$, i.e.

$$\frac{\xi^{(n)}}{a} = \frac{\xi}{a\ell^n} \equiv x = O(1) \qquad (*)$$

- represent a point close to fixed point in terms of the eigenvectors of R_ℓ:

$$\vec{K} = \vec{K}^* + u_1 \vec{e}_1 + u_2 \vec{e}_2 + \dots$$

after n iterations:

$$\vec{K}^{(n)} = \vec{K}^* + u_1 \underbrace{\lambda_1^n}_{\substack{\text{if } |\lambda_1|>1, \\ \text{relevant}}} \vec{e}_1 + u_2 \underbrace{\lambda_2^n}_{\substack{\text{if } |\lambda_2|<1, \\ \text{irrelevant}}} \vec{e}_2 + \dots$$

- as T approaches T_c, $u_1(T) \to 0$ (why?), whereas $u_2, u_3, \dots \to \text{const}$ (typically)

$\longrightarrow$ expand u_1 around T_c: $u_1 = \tilde{u}_1 (T - T_c) + O\left((T-T_c)^2\right)$

$$\Rightarrow \quad \vec{K}^{(n)} = \vec{K}^* + \tilde{u}_1 (T - T_c) \lambda_1^n \vec{e}_1 + u_2 \lambda_2^n \vec{e}_2 + \dots$$

- "turning point" is given by the condition

$$\tilde{u}_n (T - T_c)\, \lambda_n^n \equiv y = O(1) \quad (**)$$

combining with * leads to

$$\xi = a \cdot x \cdot \ell^n = a\, x\, \ell^{\left[\frac{\log \frac{y}{\tilde{u}_n (T-T_c)}}{\log \lambda_n}\right]}$$

$$= a\, x \exp\left[\log \ell \; \frac{\log \frac{y}{\tilde{u}_n (T-T_c)}}{\log \lambda_n}\right]$$

$$= a\, x \left(\frac{y}{\tilde{u}_n (T-T_c)}\right)^{\nu}$$

$$\boxed{\text{with } \nu = \frac{\log \ell}{\log \lambda_n}}$$

$$\Rightarrow \xi = a\, x \left(\frac{y}{\tilde{u}_n}\right)^{\nu} (T - T_c)^{-\nu}$$

Critical exponent of correlation length $\left(\xi \sim (T-T_c)^{-\nu}\right)$ is independent of all system specific constants $x, y, \tilde{u}_n, \ldots$

- argument above can be repeated for $T = T_c - \varepsilon$, results in same exponent $\Rightarrow \xi \sim |T - T_c|^{-\nu}$
- the calculation of other critical exponents depend on details how block variables are defined