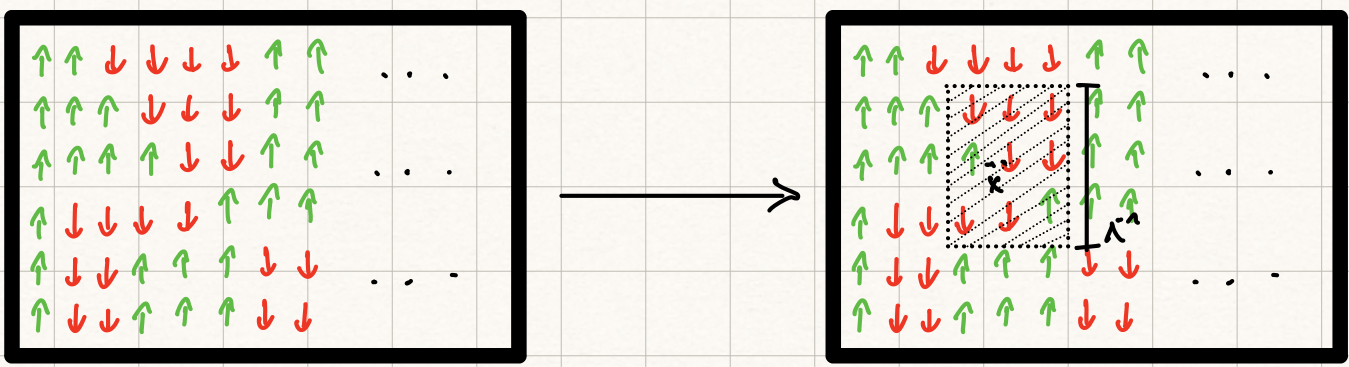
Review of previous lecture (June 03)

generalization of Landau theory to inhomogeneous configurations:

$$L = \int d^{3\ldots}\vec{x}\ \mathcal{L}\left[\eta(\vec{x})\right]$$

assign a local magnetization $m_\Lambda(\vec{x})$ to each point in space

via <u>coarse graining</u>: $m_\Lambda(\vec{x}) = \dfrac{1}{N_\Lambda(\vec{x})} \sum_{i \in \vec{x}_\Lambda} \langle \sigma_i \rangle$



$a \ll \Lambda^{-1} \leq \xi(T)$

$\Rightarrow$ $m_\Lambda(\vec{x})$ is a smooth and slowly varying function, does contain Fourier components of wave numbers $k \leq \Lambda$

$$Z_c = \sum_{\{\sigma_i^*\}} \underbrace{\sum_{\{\bar{\sigma}_i\}}}_{\substack{\text{all microscopic configurations} \\ \text{consistent with magnetization } m_\Lambda(\vec{x})}} e^{-\beta H} = \int \mathcal{D}m_\Lambda(\vec{x})\ e^{-\beta L[m_\Lambda(x)]}$$

remaining configurations not consistent with $m_\Lambda(\vec{x})$

discretized version: $\int \mathcal{D}m_\Lambda(\vec{x}) \to \prod_{i=1}^{N} \int dm(\vec{x}_i)$

close to the critical point $L[m_\Lambda(\vec{x})]$ takes the form

$$L[m_\Lambda(\vec{x})] = \int d^3\vec{x} \left[a_0 - B\, m_\Lambda(\vec{x}) + a_2\, m_\Lambda^2(\vec{x}) + \frac{\gamma}{2}\left(\nabla m_\Lambda(\vec{x})\right)^2 + a_4\, m_\Lambda^4(\vec{x}) \right]$$

↳ exercise

or in momentum space

$$L[m_\Lambda(\vec{k})] = V \cdot a_0 - B\, \underset{\substack{\uparrow \\ \text{for constant } B}}{m_\Lambda(0)} + \int \frac{d^3\vec{k}}{(2\pi)^3} \underbrace{\left(a_2 + \frac{\gamma}{2} k^2\right)}_{\text{from expansion of } J(\vec{k}) \sim c_1 + c_2 k^2 \text{ for small } k} m_\Lambda(\vec{k})\, m_\Lambda(-\vec{k})$$

$$+ a_4 \prod_{i=1}^{4} \int^{\Lambda} d\vec{k}_i\, m_\Lambda(\vec{k}_i) \cdot \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$

<u>Ginzburg–Landau–Wilson action</u>

- describes long-wavelength order-parameter fluctuations, integration over $m_\Lambda(\vec{k})$ yields contributions from long-wavelength fluctuations to partition function
- effect of short-wavelength contributions encoded in the parameters a_i, either treat them as free parameters or microscopic calculation from Hamiltonian via coarse graining
- quantity $e^{-\beta L[m_\Lambda]}$ proportional to probability density for observing an order-parameter distribution specified by the field configuration $m_\Lambda(\vec{x})$ resp $m_\Lambda(\vec{k})$
- partition function obtained by summing $e^{-\beta L[m_\Lambda]}$ over all possible configurations $m_\Lambda(\vec{x})$
- term $m_\Lambda(\vec{k})\, m_\Lambda(-\vec{k})$ describes kinetic term, term $m_\Lambda(\vec{k}_1)\, m_\Lambda(\vec{k}_2)\, m_\Lambda(\vec{k}_3)\, m_\Lambda(\vec{k}_4)$ interaction of modes

- in case of constant configurations $m_\Lambda(\vec{x}) = \bar{m}$, $\nabla m_\Lambda(\vec{x}) = 0$ we obtain

$$Z_c \sim \int_{-\infty}^{\infty} d\bar{m} \; e^{-\beta L(\bar{m})}$$

approximating the integral further by the most likely configuration (saddle point approximation) leads to:

$$\left.\frac{\partial L(\bar{m})}{\partial \bar{m}}\right|_{\bar{m}=\bar{m}^*} = 0 \quad \Rightarrow \quad \text{mean-field result, Landau theory}$$

<u>analogous:</u> classical physics $\Leftrightarrow$ quantum mechanics

$$U(x_b, t_b, x_a, t_a) = \langle x_b | e^{-i(t_b - t_a)\hat{H}/\hbar} | x_a \rangle$$

$$= \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t) \; e^{iS[x(t)]/\hbar}$$

action

classical mechanics results from saddle point approximation:

$$\delta S[x(t)] = 0 \quad \Rightarrow \quad \text{Euler-Lagrange equations}$$

"mean-field result"

Validity and breakdown of mean-field theory

- mean-field theory amounts to evaluating the partition function in saddle point approximation for a constant magnetization

- approximation can be improved by including fluctuations to quadratic order (Gaussian approximation)

- consider Ginzburg-Landau-Wilson functional for $B=0$ and after dropping the quartic terms:

$$L[m_\lambda] = \int d^3\vec{x} \left(\frac{\gamma}{2} \left(\nabla m_\lambda(\vec{x})\right)^2 + a_2 m_\lambda^2(\vec{x}) \right) + a_0 \cdot V$$

we use the Fourier expansion for $m_\lambda(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} m_\lambda(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$

$$\Rightarrow L[m_\lambda] = \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\frac{\gamma}{2} \vec{k}^2 + a_2 \right) m_\lambda(\vec{k}) m_\lambda(-\vec{k}) + a_0 V$$

$*$

- mean field theory takes into account contributions at $\vec{k}=0$ (uniform state)

- relation $*$ is called Gaussian approximation, takes fluctuations at finite $\vec{k}$ into account, contributions at different $\vec{k}$ independent (non-interacting fluctuations)
- Gaussian approximation can be solved exactly
- mean field + Gaussian approximation leads to the <u>same critical exponents</u> like mean-field theory!

study if Gaussian approximation can be systematically improved by perturbatively including effects from interaction terms $\sim a_4$:

the functional for $B=0$ in D dimensions reads:

$$L[m_\Lambda] = \int d^D\vec{x}\, \left[a_2\, m_\Lambda^2(\vec{x}) + \frac{\gamma}{2} (\nabla m_\Lambda(\vec{x}))^2 + a_4\, m_\Lambda^4(\vec{x}) \right]$$

close to T_c we know from Landau theory that: $a_2(T) \sim a_2^1 \cdot t$

we perform a dimensional analysis of the different terms in $L[m_\Lambda]$:

introduce $\bar{m}_\Lambda = \sqrt{\beta\gamma}\, m_\Lambda$, $\frac{a_2^1 t}{\gamma} \equiv \frac{r_0}{2} \sim t$, $\frac{a_4}{\beta\gamma^2} = \frac{u_0}{4}$

$$\Rightarrow \beta L[\bar{m}_\Lambda] = \int d^D\vec{x} \left(\frac{r_0}{2} \bar{m}_\Lambda^2(\vec{x}) + \frac{(\nabla \bar{m}_\Lambda(\vec{x}))^2}{2} + \frac{u_0}{4} \bar{m}_\Lambda^4(\vec{x}) \right)$$

analyse dimensions: $\left[\beta L[\bar{m}_\Lambda] \right] = 1$

$$\Rightarrow \left[\int d^D\vec{x}\, (\nabla \bar{m}_\Lambda(\vec{x}))^2 \right] = 1 = L^D L^{-2} [\bar{m}_\Lambda]^2 \Rightarrow [\bar{m}_\Lambda] = L^{1-\frac{D}{2}}$$

$$[r_0] = L^{-(2-D)} L^{-D} = L^{-2}$$

$$[u_0] = L^{-(4-2D)} L^{-D} = L^{D-4}$$

define dimensionless fields and couplings:

$$\phi_\Lambda = \frac{\bar{m}_\Lambda}{L^{1-\frac{D}{2}}} , \quad \bar{x} = \frac{\vec{x}}{L} , \quad \bar{u}_0 = \frac{u_0}{L^{D-4}} , \quad L = r_0^{-\frac{1}{2}}$$

$$\Rightarrow \quad Z = \int \mathcal{D}\phi_\Lambda \, e^{-H_0 - H_1} \approx \int \mathcal{D}\phi_\Lambda \, e^{-H_0} \left[1 - H_1 + \frac{1}{2} H_1^2 - \dots \right]$$

with
$$H_0 = \int d^D\tilde{x} \left(\frac{\phi_\Lambda(\tilde{x})}{2} + \frac{\left(\nabla\phi_\Lambda(\tilde{x})\right)^2}{2} \right)$$

$$H_1 = \int d^D\tilde{x} \, \frac{1}{4} \bar{u}_0 \, \phi_\Lambda^4(\tilde{x}) \quad , \quad \bar{u}_0 = u_0 r_0^{\frac{D-4}{2}} \sim t^{\frac{D-4}{2}}$$

$\Rightarrow$ interaction term becomes arbitrarily large as we approach the critical point ($t \to 0$, $\xi \to \infty$) for $D < 4$, no matter how small the original coupling u_0!

$\Rightarrow$ it is not possible to systematically improve the Gaussian approximation by adding interaction effects perturbatively, non-perturbative methods necessary!

<u>Conclusions:</u> (for Ising universality class, i.e. the form of the Landau functional as above)

(a) for $D > 4$ Landau theory is reliable and gives correct critical exponents

(b) for $D < 4$ Landau theory is not valid

(c) for $D = 4$ fluctuations give logarithmic corrections to mean-field results

$D_{crit} = 4$ is called the upper critical dimension and is in general given by the relation

$$D_{crit} = \frac{2\beta + \gamma}{\nu}$$

β, γ, ν critical exponents

$\left(\beta = \frac{1}{2}, \gamma = 1, \nu = \frac{1}{2} \text{ for Ising} \right)$

not shown in this lecture → Goldenfeld (p. 110)

↓

value agrees with result from naive dimensional analysis (see below)

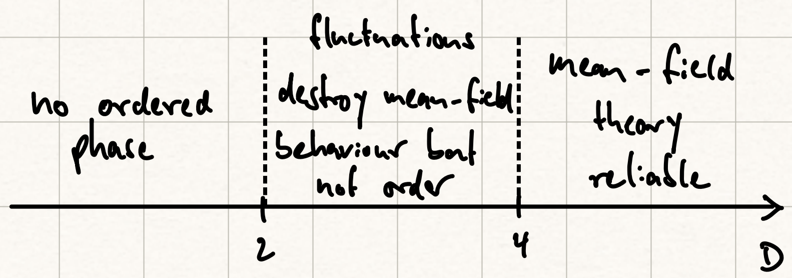


Table 3.1 CRITICAL EXPONENTS FOR THE ISING UNIVERSALITY CLASS

Exponent	Mean Field	Experiment	Ising ($d = 2$)	Ising ($d = 3$)
α	0 (disc.)	0.110 – 0.116	0 (log)	0.110(5)
β	1/2	0.316 – 0.327	1/8	0.325±0.0015
γ	1	1.23 – 1.25	7/4	1.2405±0.0015
δ	3	4.6 – 4.9	15	4.82(4)
ν	1/2	0.625±0.010	1	0.630(2)
η	0	0.016 – 0.06	1/4	0.032±0.003

Goldenfeld p. 111

anomalous dimension

Study implications of the value of critical exponent ν :

$$\xi \sim |T-T_c|^{-\nu} \qquad (\text{mean-field result: } \nu = \tfrac{1}{2})$$

define anomalous dimension θ : $\nu = \frac{1}{2} - \theta$

use dimensional analysis : $[\xi] = L$ and $[r_0] = L^{-2}$

$r_0 \sim t$

see above

$$\Rightarrow \xi \sim r_0^{-\frac{1}{2}} \sim t^{-\frac{1}{2}} \quad \rightarrow \nu = \tfrac{1}{2},\ \theta = 0$$

that means based on these general arguments ν should always take the mean-field value $\nu = \frac{1}{2}$, $\theta = 0$
however, based on table above, that is not the case

How is this possible?
→ violation of naive dimensional analysis
→ there must be another length scale in the system!

despite the fact that long wavelength phenomena dominate physics close to the critical point, also microscopic length scales show up in critical exponents!

consider as an example the lattice spacing as an additional scale:

$$[\xi] = L \,, \quad [a] = L \,, \quad [r_0] = L^{-2}$$

$$L_s \sim t$$

generalizing dimensional analysis leads to:

$$\xi = r_0^{-\frac{1}{2}} f(r_0 a^2)$$

→ see also exercise sheet 3

↑ function to be determined

What happens close to the critical point $t \to 0$?

assume $f(x) \sim x^\theta$ for $x \to 0$ (non-analytic behaviour)

$$\Rightarrow \xi \sim t^{-\frac{1}{2}+\theta} a^{2\theta}$$

this result is remarkable, even though $\frac{a}{\xi} \ll 1$, we cannot in general replace a function $\Phi\left(\frac{a}{\xi}\right)$ by $\Phi(0)$, otherwise a would not appear in any quantity

⇒ effects at very different length scales contribute to the physics of critical phenomena !