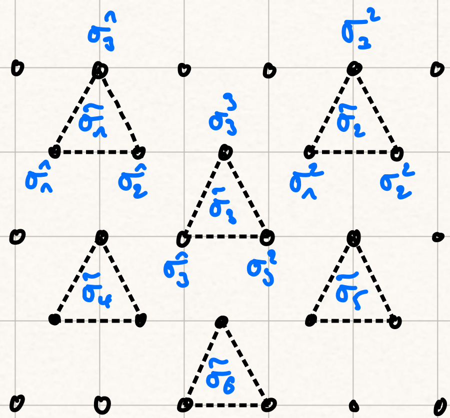
RG analysis of 2d Ising model on triangular lattice

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i$$

construct block spins by grouping together 3 spins each and assign spin according to majority rule:

$$\tilde{\sigma}_I = \operatorname{sign}\left(\sigma_1^I + \sigma_2^I + \sigma_3^I \right)$$



the coarse-grained Hamiltonian H' is then given by:

$$e^{-\beta H'(\{\tilde{\sigma}_I\})} = \sum_{\{\sigma_i^I\}} e^{-\beta H(\{\sigma_i\})}$$

$\hookrightarrow$ sum over all configurations in block spin $\tilde{\sigma}_I$

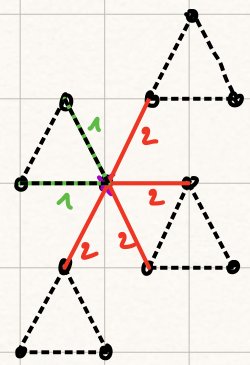
$\tilde{\sigma}_I$	σ_1^I	σ_2^I	σ_3^I	$e^{-\beta H_0}$
+	+	+	+	e^{3K}
+	−	+	+	e^{-K}
+	+	−	+	e^{-K}
+	+	+	−	e^{-K}
−	−	−	−	e^{3K}
−	+	−	−	e^{-K}
−	−	+	−	e^{-K}
−	−	−	+	e^{-K}

$*$

there are two different types of interaction contributions in a RG transformation:

1. interactions within one block spin (intra-cell interactions)

2. interactions with spins of adjacent block spins (inter-cell interactions)



split Hamiltonian into two parts (set $B=0$ for the moment):

$$H_0 = -J \sum_I \sum_{i,j\in I} \sigma_i^I \sigma_j^I = -J \sum_I \left(\sigma_1^I \sigma_2^I + \sigma_2^I \sigma_3^I + \sigma_3^I \sigma_1^I \right)$$

$$V = -J \sum_{I\neq J} \sum_{\substack{i\in I \\ j\in J}} \sigma_i^I \sigma_j^J$$

in order to truncate the number of induced interactions by the RG transformations, we will treat V as a perturbation using the cumulant expansion (strictly not justified!)

$$\Rightarrow e^{-\beta H'(\{\bar{\sigma}_I\})} = \sum_{\{\sigma_i^I\}} e^{-\beta H_0(\{\sigma_i\}) - \beta V(\{\sigma_i\})}$$

$$= \left\langle e^{-\beta V} \right\rangle_0 \sum_{\{\sigma_i^I\}} e^{-\beta H_0(\{\sigma_i\})}$$

$$= \left\langle e^{-\beta V} \right\rangle_0 \; Z_0^{\boxed{\frac{N}{3}}}$$

number of block spins

with

$$\langle A \rangle_0 = \frac{\sum_{\{\sigma_i I\}} e^{-\beta H_0(\{\sigma_i\})} A(\{\sigma_i\})}{\sum_{\{\sigma_i I\}} e^{-\beta H_0(\{\sigma_i\})}}$$

and $$Z_0(K) = \sum'_{\sigma_1,\sigma_2,\sigma_3} e^{K(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)} = 3e^{-K} + e^{3K}$$

$\downarrow$ for a given value of $\tilde{\sigma}_I$

$\downarrow$ result independent of value of $\tilde{\sigma}_I$! (see table above)

for evaluation of $\langle e^{-\beta V} \rangle_0$ we use the cumulant expansion:

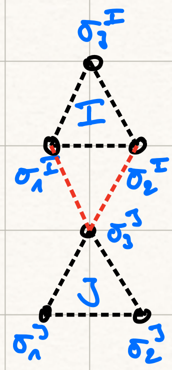
$$\langle e^{-\beta V} \rangle_0 = \exp\left[-\langle \beta V \rangle_0 + \frac{1}{2} \langle (\beta V_0)^2 \rangle - \frac{1}{2} \langle \beta V \rangle_0^2 + \dots \right]$$

$$\Rightarrow \beta H'(\{\tilde{\sigma}_I\}) = -\frac{N}{3} \log Z_0(K) + \langle \beta V \rangle_0 - \frac{1}{2}\left(\langle (\beta V)^2 \rangle - \langle \beta V \rangle^2 \right)$$

$\downarrow$ here we discuss only this term explicitly consider two adjacent block spins I and J:

interaction V_{IJ} between two blocks

is mediated by spin σ_3^J:



$$V = \sum_{I \neq J} V_{IJ} \,, \quad V_{IJ} = -J\, \sigma_3^J \left(\sigma_1^I + \sigma_2^I \right)$$

$$\Rightarrow \langle V_{IJ} \rangle_0 = -J \left(\langle \sigma_3^J \sigma_1^I \rangle_0 + \langle \sigma_3^J \sigma_2^I \rangle_0 \right)$$

$$= -2J \langle \sigma_3^J \sigma_1^I \rangle_0$$

$$= -2J \langle \sigma_3^J \rangle_0 \langle \sigma_1^I \rangle_0$$

$\uparrow$

H_0 does not couple

different blocks!

$\Rightarrow$ need to evaluate

$$\langle \sigma_3^J \rangle_0 = Z_0(K)^{-1} \sum_{\{\sigma_i^J\}} \sigma_3^J \, e^{K(\sigma_1^J \sigma_2^J + \sigma_2^J \sigma_3^J + \sigma_3^J \sigma_1^J)}$$

$$= \tilde{\sigma}_J \underbrace{\frac{e^{-K} + e^{3K}}{e^{3K} + 3e^{-K}}}_{\Phi(K)} \longrightarrow \text{verify (see table ✱)}$$

$$\Rightarrow \langle V \rangle_0 = -2J \left(\Phi(K) \right)^2 \sum_{\langle IJ \rangle} \tilde{\sigma}_I \tilde{\sigma}_J$$

$$\beta H'(\{\tilde{\sigma}_I\}) = -\frac{N}{3} \log Z_0(K) - 2K \left(\Phi(K) \right)^2 \sum_{\langle IJ \rangle} \tilde{\sigma}_I \tilde{\sigma}_J$$

$\downarrow$

again nearest neighbor Hamiltonian

$$\Rightarrow \quad K' = 2K\left(\Phi(K)\right)^2 \qquad \text{RG equation}$$

fixed points: $K^* = 2K^*\left(\Phi(K^*)\right)^2$

$\Rightarrow$ a) $K^* = 0$

b) $K^* = \infty$

c) $K^* = \frac{1}{4}\log\left(1 + 2\sqrt{2}\right) \approx 0{,}34$ $\rightarrow$ verify!

comparison to exact solution:

- critical fix point: $K^*_{exact} = \frac{1}{4}\log 3 \approx 0{,}27$
- relevant RG eigenvalue $\lambda_1^{(1)} = \left.\frac{\partial K'}{\partial K}\right|_{K^*} \approx 1{,}62$ (exact: $\lambda \approx 1{,}73$)

calculations can be improved by taking into account more terms in cumulant expansion

incorporation of second-order terms lead to induction of more couplings (second- and third-nearest neighbor interactions)

$\Rightarrow$ $\lambda_1^{(2)} \approx 1{,}77$ (closer to exact solution...)

however, cumulant expansion is not rigorous, V can in general not be treated as perturbation

$\Rightarrow$ convergence pattern not systematic or uniform

- inclusion of external magnetic field B leads to second relevant eigenvalue $\lambda_2^{(0)} = \frac{3}{\sqrt{2}} \approx 2{,}12$

$\lambda_2^{(1)} \approx 3{,}06$

$\lambda_2^{(4)} \approx 2{,}76$ exact: $\lambda_2^{exact} \approx 2{,}8$

for general number of spatial dimensions D the correlation function

$$\langle \vec{\sigma}(\vec{r}) \; \vec{\sigma}(0) \rangle$$

has the following form for $r \to \infty$ and low T:

$$\langle \vec{\sigma}(\vec{r}) \; \vec{\sigma}(0) \rangle = \begin{cases} e^{-cT} = \text{finite} & D>2, \text{ long-range order} \\ \left(\frac{r}{L}\right)^{-\eta} & D=2 \text{ algebraic behaviour} \\ e^{-\frac{Tr}{2Ja}} & D=1 \text{ no long-range order} \end{cases}$$

at large T system should behave like paramagnet

$$\langle \vec{\sigma}(\vec{r}) \; \vec{\sigma}(0) \rangle = e^{-r/\xi}$$

$\Rightarrow$ transition from algebraic to exponential behaviour indicates phase transition

it can be shown that $\xi(T) \sim \exp\left(\frac{c}{(T-T_{KT})^{\frac{1}{2}}}\right)$ for $T > T_{KT}$

KT transition is of infinite order

all considerations so far neglected interaction effects between vortices

→ RG calculations by Kosterlitz and Thouless showed that conclusions above don't change by interaction effects