Kosterlitz – Thouless transition

we have shown that for systems with a discrete symmetry and short-range interactions there is no long-range order at finite T for $D < 2$

for systems with continuous symmetry and short-range interactions, there is no long-range order at finite T for $D \leq 2$ (Hermin-Wagner theorem)

experimental findings by Chester / Young / Stephens (1972):

- use quartz crystal with a thin film of ^4He adsorbed to crystal
- at low T film is superfluid, crystal oscillates, whereas film is at rest
- for larger T superfluidity gets reduced and eventually disappears
- film is then locked to crystal and now also oscillates, oscillation frequency changes due to additional mass
- shift of resonance frequency is measure for superfluid density

$\Rightarrow$ experimental evidence of phase transition in 2d systems, how to reconcile with Hermin-Wagner theorem?

- superfluidity is characterized by a complex $U(1)$ order parameter, equivalent to a spin theory with two real components $\rightarrow$ X-Y model

X-Y model consists of two-component spins:

$$\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y) \quad \text{with} \quad \vec{\sigma}_i^{\,2} = (\sigma_i^x)^2 + (\sigma_i^y)^2 = 1$$

the Hamiltonian for a nearest-neighbor interaction takes the form

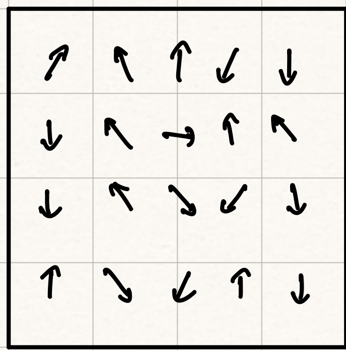
$$H = -J \sum_{\langle ij \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j$$

continuous symmetry!

$$= -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$$= -J \sum_{\langle ij \rangle} \left(1 - \frac{(\theta_i - \theta_j)^2}{2} + \dots\right)$$

$$\to \cancel{E_0} + \frac{J}{2} \int d^2\vec{r}\, \left(\nabla\theta(\vec{r})\right)^2 + \dots$$



2d rotors

continuum limit + expansion for small $\theta_i - \theta_j$

E_0: energy of completely aligned ground state of N rotors (not interesting, set $E_0 = 0$ in the following)

$$Z = e^{-\beta E_0} \int \mathcal{D}\theta \, \exp\left[-\frac{\beta J}{2} \int d^2\vec{r}\, \left(\nabla\theta(\vec{r})\right)^2\right]$$

find local minimum: $\dfrac{\delta H}{\delta\theta(\vec{r})} = 0 \quad \Rightarrow \quad \nabla^2\theta(\vec{r}) = 0$

↳ note equivalence to Coulomb Poisson law in 2d!

two solutions: 1.) $\theta_1(\vec{r}) = \text{const}$ (unstable wrt. thermal fluctuations)

2.) $\theta_2(\vec{r}) = n \tan^{-1}\left(\dfrac{x - x_0}{y - y_0}\right)$ (stable due to topological structure!)

note that

$$\oint d\vec{\ell} \cdot \nabla \Theta(\vec{r}) = 2\pi n$$

"vortex", topological excitation

closed curve that contains $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

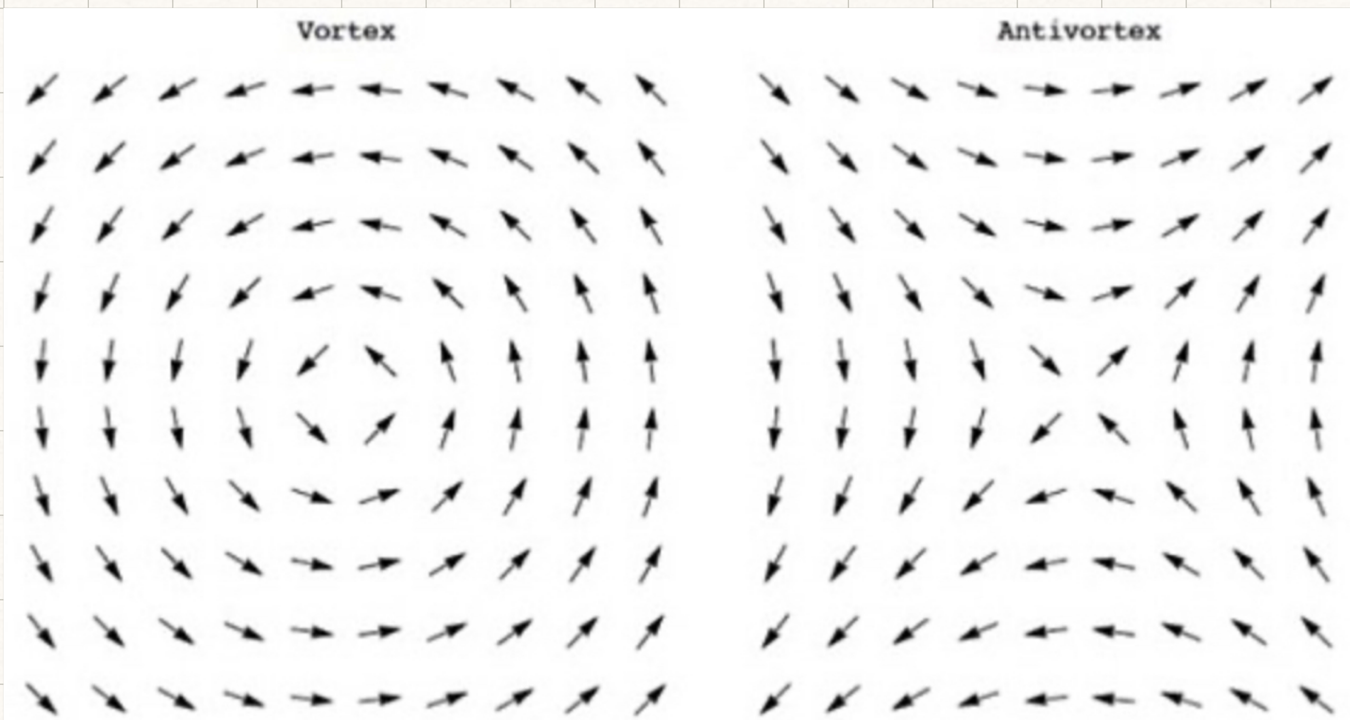
topological charge, $n > 0$ vortex

$n < 0$ antivortex

else

$$\oint d\vec{\ell} \cdot \nabla \Theta(\vec{r}) = 0$$

does not contain $\vec{r}_0$



$\Rightarrow$ general solution:

$$\Theta(\vec{r}) = \sum_{i=1}^{N} n_i \tan^{-1}\left(\frac{x - x_0^i}{y - y_0^i}\right)$$

What is the energy of a single vortex?

$$\oint d\vec{\ell} \cdot \nabla\theta(\vec{r}) = 2\pi r \, |\nabla\theta(\vec{r})| = 2\pi n \quad \Rightarrow |\nabla\theta(\vec{r})| = \frac{n}{r}$$

for a circular path (symmetry)

$$\Rightarrow \quad E = \frac{J}{2} \int d^2\vec{r} \; |\nabla\theta(\vec{r})|^2$$

$$= \frac{J}{2} \int_0^{2\pi} \int_a^L r \cdot dr \, \frac{n^2}{r^2} = \frac{2\pi J n^2}{2} \log \frac{L}{a}$$

L: linear size of system

a: short-distance cutoff (lattice spacing)

→ energy of single vortex diverges, i.e. is macroscopically large for $L \to \infty$

→ excitation of single vortex energetically not favored

How about a vortex-antivortex pair?

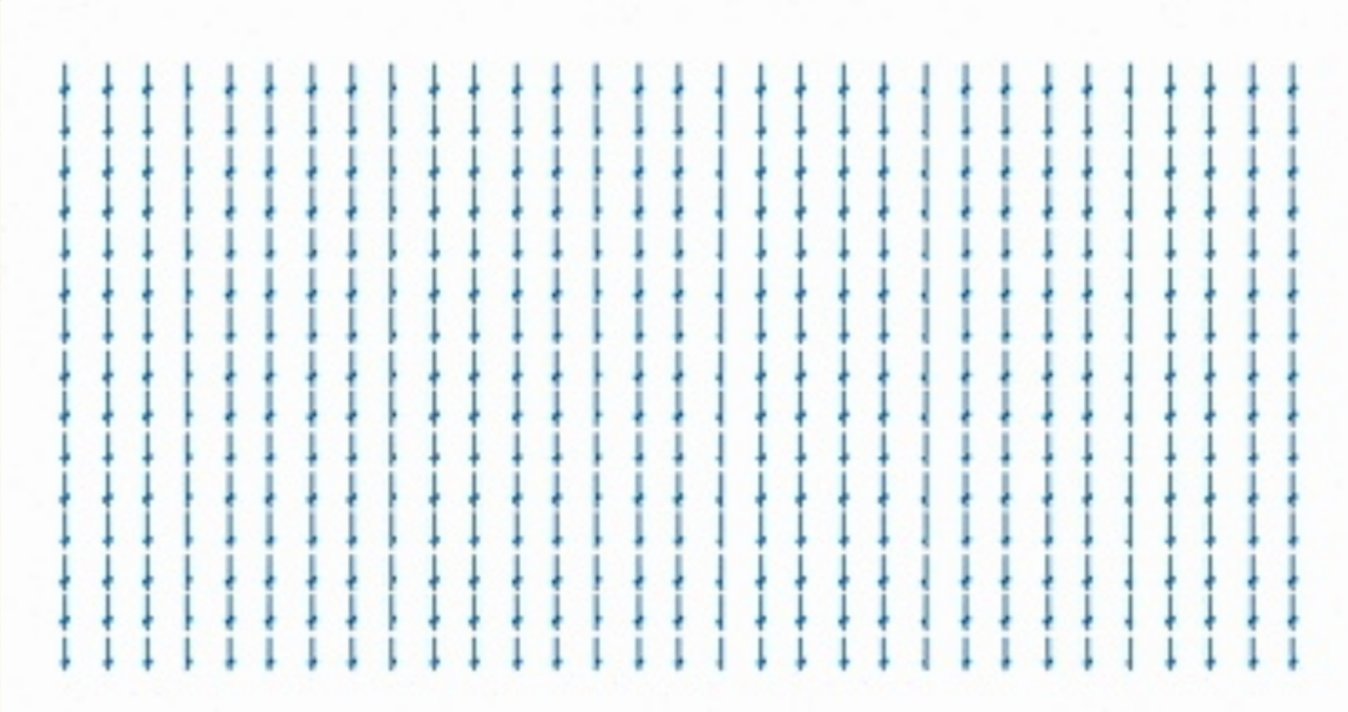
$$E_{pair} \sim 2\pi J \log \frac{R}{a} \qquad R: \text{distance of pair}$$

can be shown by noting that there is a one-to-one correspondence to 2d Coulomb system (note $\nabla^2\theta = 0$, Poisson law)

⇒ can be small for small R, at low T vortex-antivortex pairs can be excited!

Idea of Kosterlitz and Thouless:
at low T vortices are bound in pairs, at higher $T > T_c$ they become unbound (plasma of vortices)

⇒ KT transition



consider stability of system w.r.t. creation of single vortices ($n=1$).

$$F = E - TS \quad , \qquad S = k_B \log\left(\frac{L}{a}\right)^2$$

↳ number of possible lattice sites vortex can occupy

$$\Rightarrow \quad F = (\pi J - 2 k_B T) \log \frac{L}{a}$$

for $T > T_c = \frac{\pi J}{2 k_B}$ it is favorable to create isolated vortices $\Rightarrow$ unbinding

note that this can only happen in 2d since here E and S have the same $\log \frac{L}{a}$ dependence!

for general number of spatial dimensions D the correlation function

$$\langle \vec{\sigma}(\vec{r}) \; \vec{\sigma}(0) \rangle$$

has the following form for $r \to \infty$ and low T:

$$\langle \vec{\sigma}(\vec{r}) \; \vec{\sigma}(0) \rangle = \begin{cases} e^{-cT} = \text{finite} & D > 2 \text{, long-range order} \\ \left(\frac{r}{2}\right)^{-\eta} & D = 2 \text{ algebraic behaviour} \\ e^{-\frac{Tr}{2Ja}} & D = 1 \text{ no long-range order} \end{cases}$$

at large T system should behave like paramagnet

$$\langle \vec{\sigma}(\vec{r}) \; \vec{\sigma}(0) \rangle = e^{-r/\xi}$$

$\Rightarrow$ transition from algebraic to exponential behaviour indicates phase transition

it can be shown that $\xi(T) \sim \exp\left(\frac{c}{(T-T_{KT})^{\frac{1}{2}}}\right)$ for $T > T_{KT}$

KT transition is of infinite order

all considerations so far neglected interaction effects between vortices

→ RG calculations by Kosterlitz and Thouless showed that conclusions above don't change by interaction effects an interesting prediction of KT theory is the existance of a universal number:

$$\underbrace{m^2}_{^4He \text{ mass}} \frac{}{\hbar^2} \frac{k_B \cdot T_{KT}}{\rho_s(T_{KT}^-)} = \frac{\pi}{2}$$

jump in superfluid density $\rho_s(T)$

independent of substrate, layer thickness etc.

→ slides