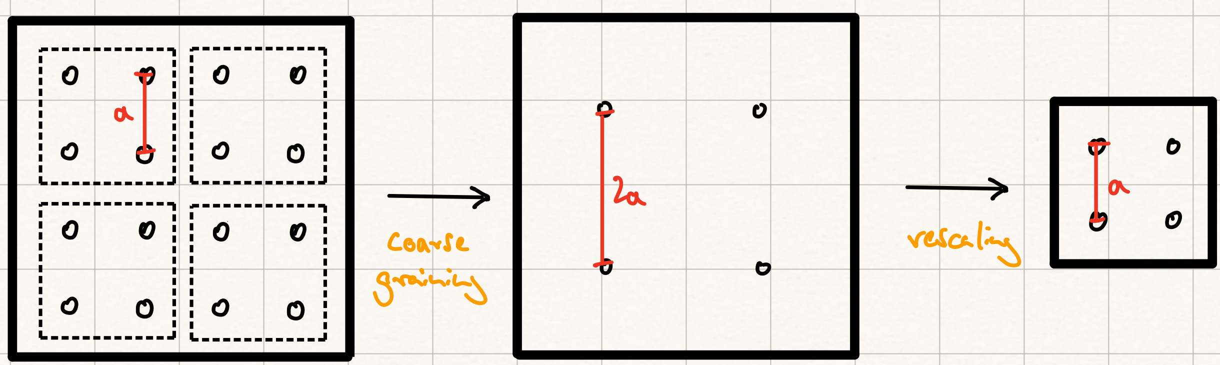
Review of previous lecture (June 13)

The Renormalization Group (RG)

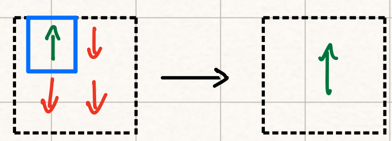
systematic framework to progressively coarse grain a microscopic description by means of a series of transformations that typically involve two steps each (in real space)

1.) coarse grain the degrees of freedom of a microscopic system (e.g. by introducing "block spins")

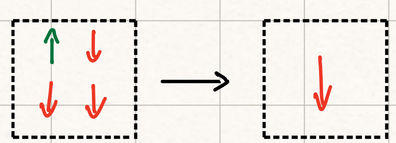
2.) Rescale basic variables such as lengths



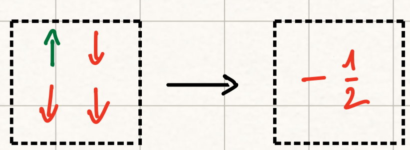
coarse graining NOT unique:



(decimation)



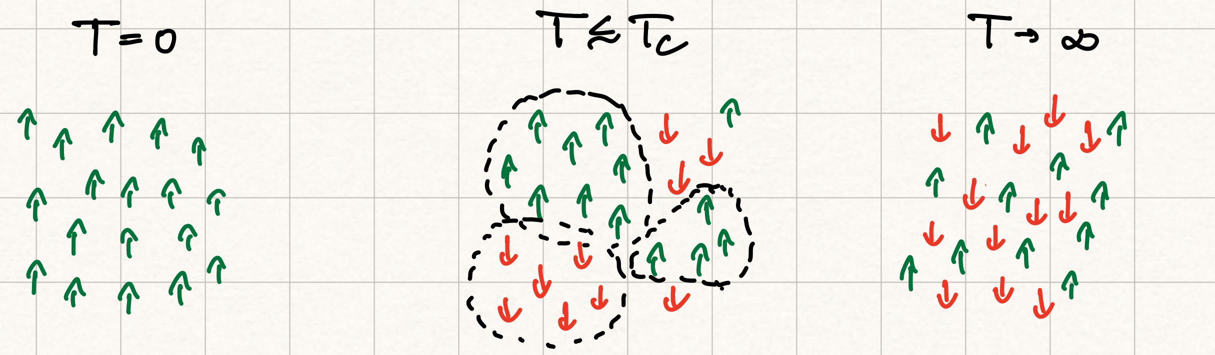
(majority)



(average)

What happens after $N \to \infty$ RG transformations?

Consider the 2d Ising model at 3 different temperatures:



perfect long-range order ①

correlation length of size $\xi(T)$ ②

complete disorder ③

① all spins are aligned, system will not change, no matter how much we coarse grain, system invariant under RG transformations
$\Rightarrow$ "fixed point" at $T = 0$

③ all spins randomly oriented, again system will look identical at every coarse graining scale
$\Rightarrow$ fixed point at $T \to \infty$

② spins are a random sea with islands of correlated spins, each only exists for a brief amount of time, correlation length: $\xi(T)$

Question: are there RG fixed points around T_c? *
for this we need to discuss the physical interpretation of ξ in more detail.

reminder : $G(i,j) = \langle (\sigma_i - \langle\sigma_i\rangle)(\sigma_j - \langle\sigma_j\rangle)\rangle$

$$= \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$$

$$\equiv \exp\left(-\frac{|i-j|}{\xi}\right)$$

Consider the 3 different temperatures above:

① all spin perfectly aligned : $\sigma_i = \langle\sigma_i\rangle$

$\Rightarrow G(i,j) = 0 \;,\; \xi(T=0) = 0$

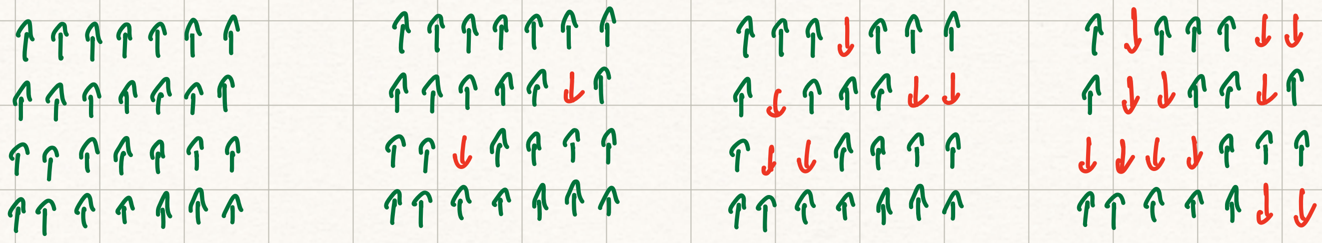
③ all spins randomly oriented : $\langle\sigma_i\rangle = 0$

no correlation between two spins : $\langle\sigma_i\sigma_j\rangle = 0$

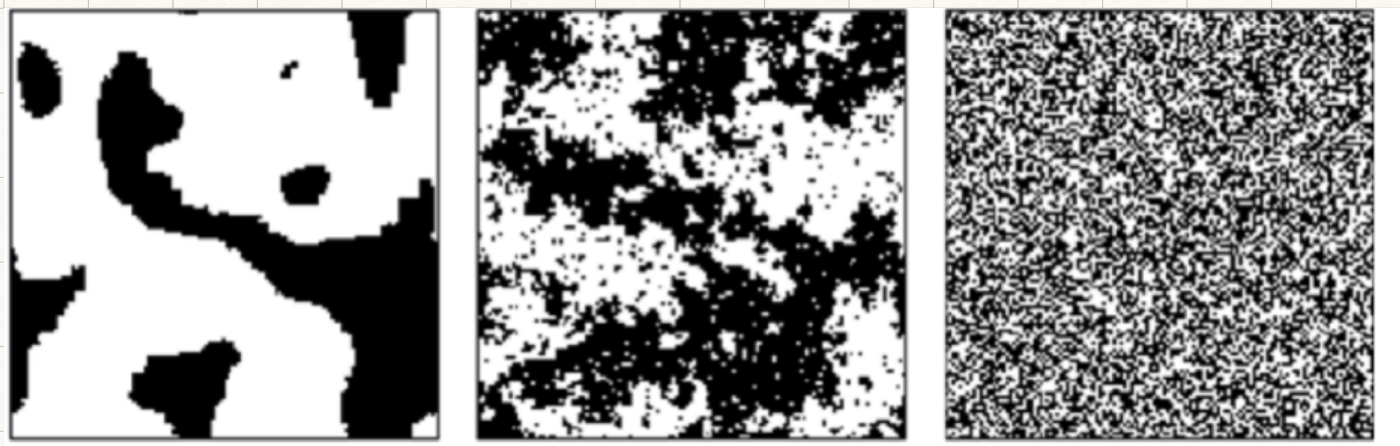
$\Rightarrow G(i,j) = 0 \;,\; \xi(T\to\infty) = 0$

ξ measures length scales over which fluctuations are correlated, not the degrees of freedom themselves

② as T is increased from T=0, individual spins will start to flip in a sea of spins pointing in the opposite direction, i.e $\xi(T)$ very small.
as we approach T_c the size of spin clusters in the sea of spins can increase and will have a certain distribution of sizes characterized by ξ



$T=0$	$T_1>0$	$T_2>T_1$	$T_3<T_c$
$\xi=0$	$\xi_1>0$	$\xi_2>\xi_1$	$\xi_3>\xi_2$



$T<T_c$	$T\sim T_c$	$T>T_c$

a great git repo for 2d Ising model:

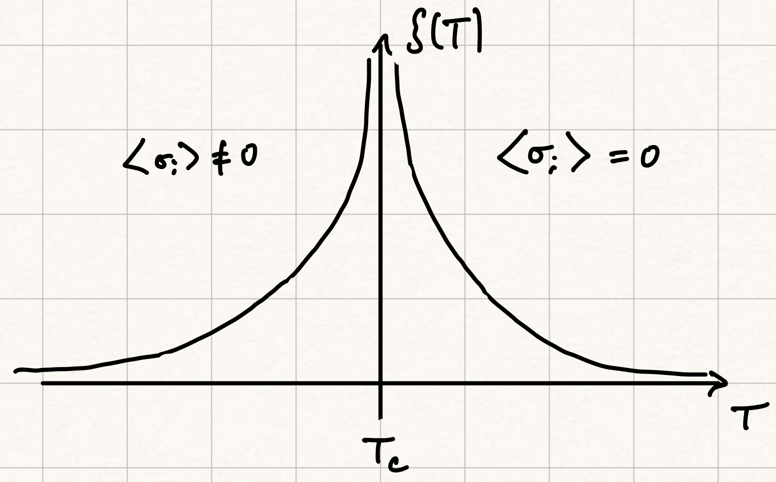
https://github.com/mattbierbaum/ising.js

including a web demo:

https://mattbierbaum.github.io/ising.js/

if $\xi \to \infty$, $G(i,j) \to$ const., that means σ_i and σ_j are correlated by the same amount, independent of the distance $|i-j|$. Physically that means there are spin clusters of all sizes in the system! $\to$ critical opacity

for $T=T_c^+$ the situation is the same with the only difference that $\langle\sigma_i\rangle = 0$ here:



Now coming back to question * above:

if ξ is finite, ordering effects will be hidden after a finite number of RG steps, as $\xi(T) \to \infty$ for $T \to T_c$ no finite number of RG transformations will hide correlations $\Rightarrow$ RG fixed point at $T = T_c$!

for critical phenomena the fixed points at $T=0$ and $T \to \infty$ are called trivial fixed points,

the case $T=T_c$ is called non-trivial fixed point (or critical fixed point)

Study the behaviour of the partition function Z_c and the Hamiltonian under an RG transformation R_ℓ

R_b: decompose lattice into blocks of length $\ell \cdot a$ and perform block spin operations

$$H = H(\{\sigma_i\}, \underbrace{K_0, K_1, K_2, \dots}_{B_i,\ J_{ij},\ K_{ijk}, \dots})$$

$$\underbrace{R_\ell [H(\{\sigma_i\}, [K_i])]}_{N \text{ spins}} = \underbrace{H'(\{\tilde{\sigma}_i\}}_{\frac{N}{\ell^D} \text{ block spins}}, \underbrace{[K_i'])}_{\text{modified couplings}}$$

ideally, if the RG transformation can be computed exactly, i.e. all the new coupling constants, the partition function will be invariant under the RG transformation:

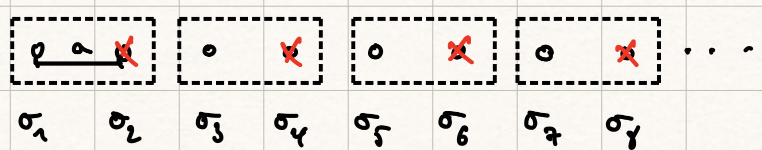
$$R_\ell [Z_c] = Z_c$$

application of the RG to the 1d Ising model ($B=0$)

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

$$\Rightarrow \quad Z_N = \mathrm{Tr}\, e^{-\beta H} = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} e^{\beta J \sum_i \sigma_i \sigma_{i+1}}$$

carry out spin sum for all even spin indices (decimation):



$\Rightarrow$ creates a system of block spins with lattice spacing $2a$

consider typical term in partition function involving spin σ_i:

$$\sum_{\sigma_i = \pm 1} e^{\beta J \sigma_i (\sigma_{i+1} + \sigma_{i-1})} = 2 \cosh\left[\beta J (\sigma_{i+1} + \sigma_{i-1})\right]$$

since this term can only depend on if σ_{i+1} and σ_{i-1} are parallel or antiparallel we can express the term above in the form

$$2\cosh\left[\underbrace{\beta J}_{\equiv k_1}(\sigma_{i+1}+\sigma_{i-1})\right] = e^{k_0' + k_1' \sigma_{i-1}\sigma_{i+1}}$$

with unknown constants k_0' and k_1'. These can be determined by considering two cases:

$$\sigma_{i+1} = -\sigma_{i-1} \quad \Rightarrow \quad 2 = e^{k_0' - k_1'}$$
$$\sigma_{i+1} = \sigma_{i-1} \quad \Rightarrow \quad 2\cosh(2k_1) = e^{k_0' + k_1'}$$

$$\Rightarrow \left.\begin{aligned} k_1' &= \frac{1}{2}\log\cosh(2k_1) \\ k_0' &= \log 2 + k_1' \end{aligned}\right\} \text{"renormalized couplings"}$$

and the RG-transformed Hamiltonian takes the form

$$-\beta H' = k_0' \mathbb{1} + k_1' \sum_{\langle ij\rangle} \sigma_i \sigma_j$$

repeat RG transformation

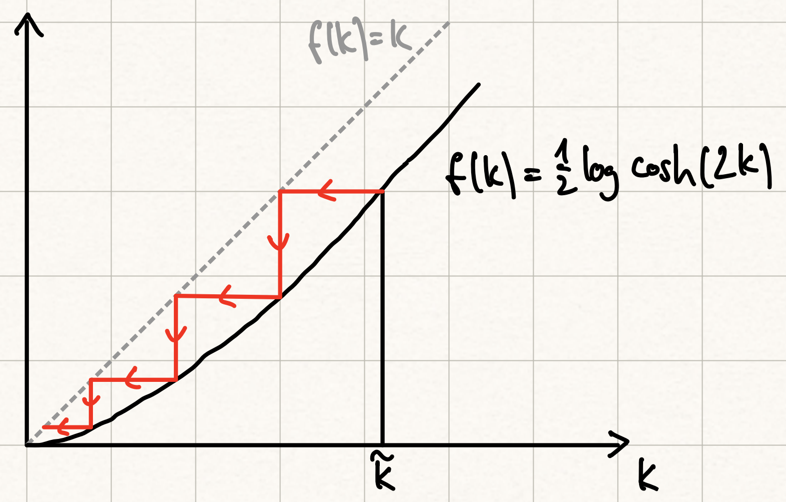
note that k_0' term does not depend on spins and can be treated easily in partition function, see below

after n steps we obtain:

$$k_1^{(n)} = \frac{1}{2}\log\cosh\left(2k_1^{(n-1)}\right)$$
$$k_0^{(n)} = \log 2 + k_1^{(n)}$$

what are the fixed points of this RG transformation?

$$K_n^* = \frac{1}{2} \log \cosh(2 K_n^*)$$



for any finite initial value $\tilde{k}$ $K^{(n)}$ decreasing with every RG step

two fixed points: 1.) $K_n^* = 0$ $(T \to \infty)$: complete disorder

2.) $K_n^* = \infty$ $(T = 0)$: perfect order

partition function:

$$Z_N(K_n) = e^{\frac{N}{2} K_0'} Z_{\frac{N}{2}}(K_n')$$

$$= e^{\frac{N}{2} K_0' + \frac{N}{4} K_0''} Z_{\frac{N}{4}}(K_n'')$$

$$= \exp\left[\frac{N}{2} \sum_{i=1}^{n} \frac{K_0^{(i)}}{2^{i-1}} + \log Z_{N/2^n}(K_n^{(n)})\right]$$

here we can use the fact that $K_n^{(n)}$ is rapidly decreasing with n and we can hence set $K_n^{(n)} \sim 0$ for some large enough n and we obtain for the free energy density:

$$f^{(n)} = -k_B T \left(\frac{1}{2} \sum_{i=1}^{n} \frac{K_0^{(i)}}{2^{i-1}} + \frac{1}{2^n} \log 2 \right)$$

↳ free energy density of the non-interacting Ising model

⇒ rapidly approaches the exact solution

$$f_{exact} = -k_B T \log(2 \cosh K_1)$$

→ Mathematica plots

note that the form of the Hamiltonian does NOT change while applying the RG transformations (apart from the fact that the initial Hamiltonian does not contain a K_0 coupling ($K_0^{(0)} = 0$))

⇒ RG(nearest-neighbor model) = nearest-neighbor model

⇒ RG transformation can be evaluated exactly

these properties are a particular feature of the 1d Ising model and are not true in general!