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Phase transitions and the Renormalization Group

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Problem set 2

Discussion of problems: Monday, May 23

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Problem 3: Transfer matrix mathod for the 2d Ising model

In this problem we will generalize the transfer matrix formalism to the two dimensional Ising model for vanishing external magnetic field on a square lattice. Suppose that there are N_r rows and N_c columns. We denote the spin on lattice site (i,j) by σ_{ij}. We will require that $N_c \to \infty$ while here we will calculate the transfer matrix only for $N_r = 1$ and $N_r = 2$. Periodic boundary conditions apply in both directions, so that our system has effectively the topology of a torus. The Hamiltonian H is given by

$$-\beta H = k \sum_{i=1}^{N_r} \sum_{j=1}^{N_c} \left[\sigma_{ij}\sigma_{i+1j} + \sigma_{ij}\sigma_{ij+1}\right] \tag{1}$$

1. For the case $N_c \to \infty, N_r = 1$ show that the transfer matrix is a 2×2 matrix, and show that its eigenvalues are

   $$\lambda_1 = 1 + x^2, \quad \lambda_2 = x^2 - 1 \tag{2}$$

   where $x \equiv e^k$.

2. Now consider the case $N_r = 2$. We need to extend the transfer matrix formalism. Consider the vector

   $$\mathbf{v}_j = (\sigma_{1j}\ \sigma_{2j} \ldots \sigma_{ij}) \tag{3}$$

   This vector gives the configuration of a column j. Show that

   $$H = \sum_{j=1}^{N_c} \left[E_1\left(\mathbf{v}_j, \mathbf{v}_{j+1}\right) + E_2\left(\mathbf{v}_j\right)\right] \tag{4}$$

   where E_1 is the energy of interaction between neighbouring columns and E_2 is the energy of a single column. Hence show that the partition function can be written in the form

   $$Z = \sum_{\mathbf{v}_1 \cdots \mathbf{v}_{N_c}} T_{\mathbf{v}_1\mathbf{v}_2} T_{\mathbf{v}_2\mathbf{v}_3} \cdots T_{\mathbf{v}_{N_c}\mathbf{v}_1} \tag{5}$$

   where T is a transfer matrix of dimensions $2^{N_r} \times 2^{N_r}$.

3. Calculate T for the case $N_r = 2$. Show that the two largest eigenvalues are

$$\lambda_1 = \frac{1}{2}\left(x^4 + 2 + x^{-4} + \sqrt{x^8 + x^{-8} + 14}\right), \quad \lambda_2 = x^4 - 1. \tag{6}$$

Problem 4: Van der Waals equation and the law of corresponding states

In the lecture we derived the Van der Waals equation of state by taking into account an excluded volume of size b per particle due to the hard-core repulsion of particles at short distances and a long-range mean field interaction term charaterized by the parameter a:

$$F(V,T) = -Nk_BT\log(V - Nb) - a\frac{N^2}{V}, \quad \text{i.e.} \quad P = \frac{Nk_BT}{V - Nb} - \frac{N^2 a}{V^2}. \tag{7}$$

Due to its mean-field character, the Van der Waals equation is not able to correctly describe the critical behavior of real fluids. However, it is possible to understand nontrivial fundamental properties of fluids with very modest theoretical effort.

1. Study the equation of state around the critical point. Use the fact that at $T = T_c$ the equation of state has an inflection point (why?), i.e.

$$\left.\frac{\partial P}{\partial V}\right|_{T=T_c} = \left.\frac{\partial^2 P}{\partial V^2}\right|_{T=T_c} = 0. \tag{8}$$

   Determine the critical temperature T_c, the critical pressure P_c and critical volume V_c by expressing them in terms of the microscopic parameters a and b. Why is the result remarkable?

   Hint: Use the fact that the equation of state can be written as a cubic polynomical in V and that this polynomical must take the following form at the critical point (why?): $(V - V_c)^3 = 0$.

2. Show that the equation of state can be expressed in the form

$$\left(P_R + \frac{3}{V_R^2}\right)(3V_R - 1) = 8T_R \tag{9}$$

   with $P_R = P/P_c$, $V_R = V/V_c$ and $T = T/T_c$. This relation is called the *law of corresponding states*. Discuss the significance of this law. What do you obtain for the ratio $P_cV_c/(k_BT_c)$?

3. Calculate the critical exponents of the liquid-gas phase transition and compare your results with the exponents of the nearest neighbor Ising model in one dimension:

$$P - P_c \sim |V - V_c|^\delta, \quad \kappa_T = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_T \sim |T - T_c|^{-\gamma}, \quad V_{\text{gas}} - V_{\text{liquid}} \sim |T - T_c|^\beta. \tag{10}$$

   Hint: To obtain the exponent β expand the equation of state around V_c and T_c and apply the Maxwell construction for the coexistence region to determine the volumes V_{gas} and V_{liquid}. What's the underlying physics idea of the Maxwell construction?