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Phase transitions and the Renormalization Group

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Problem set 3

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Problem 5: Dimensional analysis

Dimensional analysis is a powerful tool and can be applied to various problems in physics. It is based on the fact that in any problem involving a number of dimensionful quantities, the relationship between them can be expressed by forming all possible independent dimensionless quantities $\Pi, \Pi_1, \Pi_2, ...\Pi_n$. The solution for Π can then be expressed in the form

$$\Pi = f(\Pi_1, ..., \Pi_n). \tag{1}$$

1. Assume there is only one dimensionless combination of variables in a given problem. What follows for Π? Can we say anything about the value of Π?

SOLUTION: In this case we have

$$\Pi = \text{const} \tag{2}$$

Here the question is: what sets the scale for the constant? Naturalness conjectures this constant to be of order one. If this ration would not be of order one there should be a reason for the smallness of one of the parameters, the parameter would be fine tuned.

2. Derive the characteristic size of the radius and ground state energy for a hydrogen atom using dimensional analysis. Compare your results with the exact values. Are the dimensionless constants of natural size?

SOLUTION: For the hydrogen atom the relevant physical quantities are:

- the reduced mass of the proton-electron system: $\mu = m_e m_p/(m_e + m_p) \sim m_e$
- the charge of the system, the Coulomb potential takes the form: $V(r) = -\frac{C_c e^2}{r}$ (e.g. $C_c = \frac{1}{4\pi\varepsilon_0}$)
- quantum mechanical system: constant $\hbar$ (from Schroedinger equation).

That means the list of dimensionful quantities is here: $m_e, (C_c e^2) = a$ and $\hbar$. The units of these quantities are (E=energy, L = length, T = time):

- $[m_e] = M = EL^{-2}T^2$

- $[a] = E * L = ML^2T^{-2}L = ML^3T^{-2}$
- $[\hbar] = L * (MLT^{-1}) = ML^2T^{-1} = E * T$ (action)

Based on these three buildiung blocks $[ET^2L^{-2}, EL, ET]$ we can build just one quantity with units of energy and one quantity with units of radius:

$$r = C_r \frac{\hbar^2}{m_e a} = C_r \frac{\hbar^2}{m_e(C_c e^2)} = C_r a_0 \tag{3}$$

$$E = C_E m_e (a/\hbar)^2 = C_E \frac{m_e(C_c e^2)^2}{\hbar^2} = C_E \frac{e^2 C_c}{a_0} \tag{4}$$

$$\tag{5}$$

with the Bohr radius $a_0 = \frac{\hbar^2}{m_e e^2 C_c}$. Comparing with the exact results

$$r_{exact} = a_0, \quad E_{exact} = \frac{1}{2}\frac{e^2 C_c}{a_0} \tag{6}$$

we see that the dimensonless constants take the value $C_r = 1, C_E = 1/2$. Pretty close!

3. Prove Pythagoras' theorem using dimensional analysis. For this task, use only the fact that the area of a right-angle triangle can be expressed as a function of the hypothenuse and one of the acute angles of the triangle (don't use trigonometry!). What happens if you consider non-Euclidian geometries?

   HINT: It is useful to add a well-chosen line to the right-angle triangle.

   SOLUTION: We consider a triangle with the hypotenuse a and the sides b and c. The angle between a and b is α. Then the area of this triangle can be written in the form $A = f(a, \alpha)$. Now we draw a line from the top of the triangle perpendicular to the hyponetuse, this creates two new right-angled triangles with the areas $B = f(b, \alpha)$ and $C = f(c, \alpha)$ with

$$A = B + C, \quad f(a, \alpha) = f(b, \alpha) + f(c, \alpha) \tag{7}$$

   Dimensional analysis tells us that $f(x, \alpha) = x^2 \tilde{f}(\alpha)$. Hence we immediately obtain $a^2 = b^2 + c^2$. In non-eucledian geometries the three triangles are not congruent anymore, hence the angles α will be different and the argument does not work anymore.

Problem 6: Ginzburg-Landau-Wilson effective field theory

The construction of effective field theories is key for understanding the basic ideas that lie at the heart of *Wilson's Renormalization Group* formulation. There exist cases for which this task can be performed *exactly* starting from a microscopic theory. Here we will perform this exercise via the *Hubbard-Stratonovich transformation* for a general Ising model for N spins on a three-dimensional lattice with a lattice spacing a:

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} \sigma_i J_{ij} \sigma_j - \sum_{i=1}^{N} B_i \sigma_i. \tag{8}$$

Here $J_{ij} = J_{ji}$ is a positive symmetric matrix which denotes the couplings between spins i and j and B_i is the external magnetic field at site i.

1. Prove the following relation for the interaction term:

$$\exp\big(\frac{1}{2}\sum_{i,j}\sigma_i J_{ij}\sigma_j\big)\int \mathcal{D}[z]\exp\big(-\frac{1}{2}\sum_{i,j} z_i J^{-1}_{ij} z_j\big) = \int \mathcal{D}[z]\exp\big(-\frac{1}{2}\sum_{i,j} z_i J^{-1}_{ij} z_j + \sum_i z_i\sigma_i\big) \quad (9)$$

with

$$\int \mathcal{D}[z] = \prod_i \int_{-\infty}^{\infty} \frac{dz_i}{\sqrt{2\pi}}.$$

Hint: Introduce new integration variables $z'_i = z_i - \sum_j J_{ij}\sigma_j$.

Solution: Inserting the new variables on the right hands side

$$\begin{aligned} & -\frac{1}{2} z_i J^{-1}_{ij} z_j + z_i\sigma_i && (10)\\ &= -\frac{1}{2}(z'_i + \sigma_k J_{ki}) J^{-1}_{ij}(z'_j + J_{jl}\sigma_l) + (z'_i + \sigma_k J_{ki})\sigma_i && (11)\\ &= -\frac{1}{2} z'_i J^{-1}_{ij} z'_j + \frac{1}{2}\sigma_i J_{ij}\sigma_j && (12)\end{aligned}$$

and the relation follows immediately.

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2. Show that the partition function can be expressed in the following form:

$$Z = \left[\int \mathcal{D}[z]\exp\left(-S[z]\right)\right]\left[\int \mathcal{D}[z]\exp\big(-\frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j\big)\right]^{-1} = \frac{1}{\sqrt{\det \beta\mathbf{J}}}\int \mathcal{D}[z]\exp\left(-S[z]\right), \quad (13)$$

with

$$S[z] = \frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j - \sum_i \ln[2\cosh(\beta B_i + z_i)].$$

Solution: We use relation (9) to rewrite the partition function:

$$\begin{aligned} Z &= \operatorname{Tr}\exp\left[\frac{\beta}{2}\sum_{i,j}\sigma_i J_{ij}\sigma_j + \beta\sum_i B_i\sigma_i\right] && (14)\\ &= \left[\int \mathcal{D}[z]\exp\big(-\frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j\big)\operatorname{Tr}\exp\big(\sum_i (z_i + \beta B_i)\sigma_i\big)\right]\left[\int \mathcal{D}[z]\exp\big(-\frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j\big)\right]^{-1} && (15)\\ &= \frac{1}{\sqrt{\det \beta\mathbf{J}}}\int \mathcal{D}[z]\exp\big(-\frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j + \ln 2\cosh\big(\beta\sum_i (z_i + B_i)\big)\big) && (16)\end{aligned}$$

with $\mathrm{Tr} = \sum_{\{\sigma_i\}}$. The determinant relation can be shown by diagonalizing the matrix $\beta^{-1}J^{-1} = O^{-1}\bar{J}O, \bar{z}_i = O^{-1}_{ij}z_j, \bar{J}\bar{z} = \lambda\bar{z}$:

$$\int \mathcal{D}[z]\exp\big(-\frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j\big) = \int \mathcal{D}[\bar{z}]\exp\big(-\frac{1}{2}\sum_i \lambda_i \bar{z}_i^2\big) \tag{17}$$

$$= \prod_i \big(\frac{1}{\lambda_i}\big)^{1/2} = \sqrt{\frac{1}{\det \bar{\mathbf{J}}}} = \sqrt{\frac{1}{\det(\beta^{-1}\mathbf{O}\mathbf{J}^{-1}\mathbf{O}^{-1})}} = \sqrt{\det \beta\mathbf{J}} \tag{18}$$

Here we used that the matrix O is just a rotation and hence do not affect the integration measure: $dz_i = d\bar{z}_i$.

3. Show that the expectation values of the variables z_i are given by the following relation:

$$\big\langle \beta\sum_j J_{ij}\sigma_j\big\rangle = Z^{-1}\mathrm{Tr}\sum_j(\beta J_{ij}\sigma_j)e^{-\beta H} = \langle z_i\rangle \equiv \frac{\int \mathcal{D}[z] z_i \exp\big(-S[z]\big)}{\int \mathcal{D}[z]\exp\big(-S[z]\big)}. \tag{19}$$

   Based on this result, show that the expectation values of the new variables ϕ_i defined by

$$\phi_i \equiv \beta^{-1}\sum_j J^{-1}_{ij} z_j, \quad \text{i.e.} \quad z_i = \beta\sum_j J_{ij}\phi_j \tag{20}$$

   corresponds to the magnetization per lattice site. Show that the partition function can be written in terms of the *effective action* $S[\phi]$ in the following form:

$$Z = \sqrt{\det \beta\mathbf{J}}\int \mathcal{D}[\phi]e^{-S[\phi]} \text{ with } S[\phi] = \frac{\beta}{2}\sum_{i,j}\phi_i J_{ij}\phi_j - \sum_i \ln\left[2\cosh\big(\beta(B_i + \sum_j J_{ij}\phi_j)\big)\right]. \tag{21}$$

   Hint: For the derivation of relation (19) you can use the technique of *external sources*:

$$z_i = \lim_{\mathbf{a}\to 0}\frac{\partial}{\partial a_i}\exp\big(\sum_j a_j z_j\big). \tag{22}$$

Solution:

$$\begin{aligned}
\big\langle \beta\sum_j J_{kj}\sigma_j\big\rangle &= Z^{-1}\mathrm{Tr}\sum_j(\beta J_{kj}\sigma_j)e^{-\beta H} && (23)\\
&= \lim_{\mathbf{a}\to 0}\frac{\partial}{\partial a_k}\frac{\mathrm{Tr}\exp\left[\beta/2(\sigma_i + a_i)J_{ij}(\sigma_j + a_j) + \beta B_i\sigma_i\right]}{\mathrm{Tr}\exp\left[\beta/2\sigma_i J_{ij}\sigma_j + \beta B_i\sigma_i\right]} && (24)\\
&= \lim_{\mathbf{a}\to 0}\frac{\partial}{\partial a_k}\frac{\int \mathcal{D}[z]\exp\big(-\frac{1}{2\beta}z_i J^{-1}_{ij}z_j\big)\mathrm{Tr}\exp\big(\beta B_i\sigma_i + z_i(\sigma_i + a_i)\big)}{\int \mathcal{D}[z]\exp\big(-\frac{1}{2\beta}z_i J^{-1}_{ij}z_j\big)\mathrm{Tr}\exp\big(\beta B_i\sigma_i + z_i\sigma_i\big)} && (25)\\
&= \frac{\int \mathcal{D}[z] z_i\exp\big(-S[z]\big)}{\int \mathcal{D}[z]\exp\big(-S[z]\big)} && (26)\\
&= \langle z_i\rangle && (27)
\end{aligned}$$

where we used relation (9) and replaced $\sigma_i \to \sigma_i + a_i$. Hence the expectation value of ϕ_i is given by:

$$\langle\phi_i\rangle = \beta^{-1}\left\langle J^{-1}_{ij}z_j\right\rangle = \langle\sigma_i\rangle \tag{28}$$

Substituting this variable in (13) we obtain:

$$Z = \left[\int \mathcal{D}[\phi] \exp\left(-S[\phi]\right) \right] \left[\int \mathcal{D}[\phi] \exp\Big(-\frac{\beta}{2} \sum_{i,j} \phi_i J_{ij} \phi_j\Big) \right]^{-1} = \sqrt{\det \beta \mathbf{J}} \int \mathcal{D}[\phi] \exp\left(-S[\phi]\right), \quad (29)$$

with

$$S[\phi] = \frac{\beta}{2} \sum_{i,j} \phi_i J_{ij} \phi_j - \sum_i \ln[2 \cosh \beta (B_i + \sum_j J_{ij} \phi_j)].$$

The factors from the change in the integration measure cancel in numerator and denominator. This equation represents the partition function of the Ising model in terms of an N-dimensional integral over variables ϕ whose expectation values are the magnetization per lattice site. The variables ϕ can therefore be interpreted as the fluctuating magnetization. In the limit $N \to \infty$ the discrete product of integrals $\mathcal{D}[\phi]$ becomes an infinite product of integrals, i.e. a *functional integral*. The resulting effective action $S[\phi]$ defines a classical effective field theory for the order-parameter of the Ising model.

4. The relation (21) is an *exact* representation of the partition function of the Ising model and hence is in general very complicated to solve. In order to simplify the expression we consider a system close to the critical point and assume that the partition function is dominated by small values of ϕ_i. Show that the effective action takes the following form up to order $\mathcal{O}(\phi_i^6)$:

$$S[\phi] = -N \log 2 + \frac{\beta}{2} \sum_{i,j} \phi_i J_{ij} \phi_j - \frac{\beta^2}{2} \sum_i \big(B_i + \sum_j J_{ij} \phi_j\big)^2 + \frac{\beta^4}{12} \sum_i \big(B_i + \sum_j J_{ij} \phi_j\big)^4 \quad (30)$$

Perform the continuum limit $N \to \infty : \phi_i \to \phi(\mathbf{r}), J_{ij} \to J(\mathbf{r} - \mathbf{r}')$, and represent the variables in momentum space, i.e.:

$$\phi(\mathbf{r}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \phi(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}}, \quad \delta(\mathbf{r}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} \quad (31)$$

Show that the effective action takes the following form up to order $\mathcal{O}(\phi^6, B^2, B\phi^3)$ for a homogeneous external field $B_i \to B(\mathbf{r}) = B$:

$$\begin{aligned} S[\phi(\mathbf{p})] &= -N \log 2 - \beta^2 \frac{B}{(2\pi)^3} J(0)\phi(0) + \frac{\beta}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} J(\mathbf{p})(1 - \beta J(\mathbf{p}))\phi(\mathbf{p})\phi(-\mathbf{p}) \quad (32) \\ &+ \frac{\beta^4}{12} \frac{1}{(2\pi)^9} \Big(\prod_{i=1}^4 \int d^3\mathbf{p}_i\Big) J(\mathbf{p}_1)J(\mathbf{p}_2)J(\mathbf{p}_3)J(\mathbf{p}_4)\phi(\mathbf{p}_1)\phi(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4)\delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \end{aligned}$$

SOLUTION: Equation (30) follows directly from expanding the function $\ln 2 \cosh(x)$ for small x. Inserting the Fourier transforms we obtain :

$$\begin{aligned} \phi_i J_{ij} \phi_j &\to \int d^3\mathbf{r} d^3\mathbf{r}' \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3}{(2\pi)^9} \phi(\mathbf{p}_1) J(\mathbf{p}_2) \phi(\mathbf{p}_3) e^{-i\mathbf{p}_1\cdot\mathbf{r}} e^{-i\mathbf{p}_2\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\mathbf{p}_3\cdot\mathbf{r}'} \quad (33) \\ &= \int \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3}{(2\pi)^3} \phi(\mathbf{p}_1) J(\mathbf{p}_2) \phi(\mathbf{p}_3) \delta(\mathbf{p}_1 + \mathbf{p}_2) \delta(\mathbf{p}_2 - \mathbf{p}_3) \quad (34) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} J(\mathbf{p}) \phi(\mathbf{p}) \phi(-\mathbf{p}) \quad (35) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} J(-\mathbf{p}) \phi(\mathbf{p}) \phi(-\mathbf{p}) \quad (36) \end{aligned}$$

since $J_{ij} = J_{ji}$ and hence $J(\mathbf{p}) = J(-\mathbf{p})$

$$\begin{aligned}
B_i J_{ik}\phi_k \;\; &\rightarrow \;\; \int d^3\mathbf{r} d^3\mathbf{r}' \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3}{(2\pi)^9} B\delta(\mathbf{p}_1) J(\mathbf{p}_2)\phi(\mathbf{p}_3) e^{-i\mathbf{p}_1\cdot\mathbf{r}} e^{-i\mathbf{p}_2\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\mathbf{p}_3\cdot\mathbf{r}'} \\
&= B\int d^3\mathbf{r} d^3\mathbf{r}' \frac{d^3\mathbf{p}_2 d^3\mathbf{p}_3}{(2\pi)^9} J(\mathbf{p}_2)\phi(\mathbf{p}_3) e^{-i\mathbf{p}_2\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\mathbf{p}_3\cdot\mathbf{r}'} \qquad (37) \\
&= B\int \frac{d^3\mathbf{p}_2 d^3\mathbf{p}_3}{(2\pi)^3} J(\mathbf{p}_2)\phi(\mathbf{p}_3)\delta(\mathbf{p}_2 - \mathbf{p}_3)\delta(\mathbf{p}_2) \qquad (38) \\
&= \frac{B}{(2\pi)^3} J(0)\phi(0) \qquad (39)
\end{aligned}$$

$$\begin{aligned}
\sum_i (J_{ij}\phi_j)^2 = J_{ij}J_{ik}\phi_j\phi_k \;\; &\rightarrow \;\; \int d^3\mathbf{r} d^3\mathbf{r}' d^3\mathbf{r}'' \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 d^3\mathbf{p}_4}{(2\pi)^{12}} \\
&\quad \times J(\mathbf{p}_1)J(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4) e^{-i\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\mathbf{p}_2\cdot(\mathbf{r}-\mathbf{r}'')} e^{-i\mathbf{p}_3\cdot\mathbf{r}'} e^{-i\mathbf{p}_4\cdot\mathbf{r}''} \\
&= \int \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 d^3\mathbf{p}_4}{(2\pi)^3} J(\mathbf{p}_1)J(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4)\delta(\mathbf{p}_1+\mathbf{p}_2)\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2+\mathbf{p}_4) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} J(\mathbf{p})J(-\mathbf{p})\phi(\mathbf{p})\phi(-\mathbf{p}) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} J(\mathbf{p})J(\mathbf{p})\phi(\mathbf{p})\phi(-\mathbf{p}). \qquad (40)
\end{aligned}$$

$$\begin{aligned}
\sum_i (J_{ij}\phi_j)^4 = \;\; &\rightarrow \;\; \int d^3\mathbf{r} d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{r}_3 d^3\mathbf{r}_4 \prod_{i=1}^{8} \frac{d^3\mathbf{p}_i}{(2\pi)^3} J(\mathbf{p}_1)J(\mathbf{p}_2)J(\mathbf{p}_3)J(\mathbf{p}_4)\phi(\mathbf{p}_5)\phi(\mathbf{p}_6)\phi(\mathbf{p}_7)\phi(\mathbf{p}_8) \\
&\quad e^{-i\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{r}_1)} e^{-i\mathbf{p}_2\cdot(\mathbf{r}-\mathbf{r}_2)} e^{-i\mathbf{p}_3\cdot(\mathbf{r}-\mathbf{r}_3)} e^{-i\mathbf{p}_4\cdot(\mathbf{r}-\mathbf{r}_4)} e^{-i\mathbf{p}_5\cdot\mathbf{r}_1} e^{-i\mathbf{p}_6\cdot\mathbf{r}_2} e^{-i\mathbf{p}_7\cdot\mathbf{r}_3} e^{-i\mathbf{p}_8\cdot\mathbf{r}_4} \\
&= \int d^3\mathbf{r} \int \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 d^3\mathbf{p}_4}{(2\pi)^{12}} J(\mathbf{p}_1)J(\mathbf{p}_2)J(\mathbf{p}_3)J(\mathbf{p}_4)\phi(\mathbf{p}_1)\phi(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4) \\
&\quad e^{-i(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\cdot\mathbf{r}} \\
&= \int \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 d^3\mathbf{p}_4}{(2\pi)^9} J(\mathbf{p}_1)J(\mathbf{p}_2)J(\mathbf{p}_3)J(\mathbf{p}_4)\phi(\mathbf{p}_1)\phi(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4)\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)
\end{aligned}$$

The units of the quantities are:

$$[\beta B] = [\beta J] = [\beta J\phi] \qquad (41)$$

Hence

$$\left[\beta^{-1}\right] = [B] = [J] = E, \quad [\phi] = 1 \qquad (42)$$

5. We are particularly interested in the *long-wavelength* contributions to the partition function. For this we limit the momentum integrals to wave numbers below a scale Λ and expand the function $J(\mathbf{p})$ for small momenta in powers of $\mathbf{p}$. Show that the final form of the partition function can be written in the form $Z = \int \mathcal{D}[\phi] e^{-\sigma_\Lambda[\phi]}$ with:

$$\begin{aligned}
S_\Lambda[\phi(\mathbf{p})] &= aN + bB\phi(0) + \frac{1}{2}\int_{\mathbf{p}} (c_0 + c_1\mathbf{p}^2)\phi(\mathbf{p})\phi(-\mathbf{p}) \\
&\quad + \frac{d}{4!}\int_{\mathbf{p}_1}\int_{\mathbf{p}_2}\int_{\mathbf{p}_3}\int_{\mathbf{p}_4} \delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\phi(\mathbf{p}_1)\phi(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4) \qquad (43)
\end{aligned}$$

Here we used the notation $\int_{\mathbf{p}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\Theta(\Lambda - |\mathbf{p}|)$. Discuss the physical meaning of the scale Λ. How are the couplings constants a, b, c_0, c_1 and d related to the constants in Eq. (32). Show that in coordinate space the effective action takes the following form:

$$S_\Lambda[\phi(\mathbf{r})] = \int d^3\mathbf{r}\left[a + bB\phi(r) + \frac{c_0}{2}\phi^2(\mathbf{r}) + \frac{c_1}{2}(\nabla\phi(\mathbf{r}))^2 + \frac{d}{4!}\phi^4(\mathbf{r})\right]. \tag{44}$$

SOLUTION: We can immediately identify the constants:

$$a = -\log 2 \tag{45}$$

$$b = -\beta^2\frac{BJ(0)}{(2\pi)^3} \tag{46}$$

$$c_0 = \beta J(0)(1 - \beta J(0)) \tag{47}$$

$$c_1 = \beta J'(0)(1 - 2\beta J(0)) \tag{48}$$

$$d = 2\beta^4 \tag{49}$$

The coordinate space representation is obtained by an inverse fourier transform. Note in particular that the p^2 factor ranslates into the gradient term.