III Mean field theory ; Landau theory

only very few interacting systems can be solved exactly in statistical physics

even "just" extending the simple calculations of the nearest neighbor Ising model from 1d to 2d is a highly nontrivial task

→ Lars Onsager, Phys Rev 65, 117

one of the most famous papers in theoretical physics

⇒ need methods that allow to solve general interacting systems in an approximate way, two options:

1.) mean field theory: replace interaction with other particles by an averaged interaction ("mean field")

simplest approximation, calculations usually straightforward

2.) Renormalization group: much more systematic framework that allows to treat interactions beyond the mean-field level (more later!)

1) Mean field solution of the 1d Ising model (nearest neighbor)

$$H = -J \sum_{\langle ij \rangle} S_i S_j - B \sum_i S_i$$

for $J=0$ we have $H = -B \sum_i S_i$, an effective one-body problem

⇒ partition function factorizes into product of single-particle partition functions:

$$Z_c(J=0, B) = \prod_{i=1}^{N} \left(e^{\beta B} + e^{-\beta B} \right) = \left[2\cosh(\beta B) \right]^N = (Z_c^1)^N$$

$$\Rightarrow \quad \underset{m}{\cancel{M}} = -\frac{1}{N} \frac{\partial F}{\partial B} = \tanh(\beta B)$$

basic idea of mean field theory:

transform H for $J \neq 0$ into a form $H = - B^i_{eff} \sum_i S_i$

↑ site-dependent effective field due to magnetic moments of all other spins

self consistent problem:

- B^i_{eff} generated by magnetic moments of all other spins $B^i_{eff} \sim m$
- however: m is unknown a priori
- each spin i feels external field B plus mean field
- spin i is not special, must itself have magnetic moment

write $B^i_{eff} = \sum_j J_{ij} S_j + B$ → couples only nearest neighbors, restricts sum to two terms

$$= \underbrace{\sum_j J_{ij} \langle S_j \rangle}_{\text{mean field}} + \cancel{\underbrace{\sum_j (S_j - \langle S_j \rangle)}_{\text{fluctuations}}} + \underbrace{B}_{\text{external field}}$$

↓ neglect in mean field theory

$$= J \cdot 2 \cdot m + B$$

⇒ from free Ising model $m = \tanh(\beta(2Jm + B))$

↳ implicit equation for m, self consistency

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exercise: Show that this solution minimizes the free energy [write $S_i = m + \delta S_i$ and neglect quadratic terms in δS

$* \Rightarrow \frac{\partial F}{\partial m} = 0 \Rightarrow -2\beta m + 2\beta J \tanh(\beta(2Jm + B))$] $= 0$

$\boxed{III/3}$

study spontaneous magnetization: $B=0$

$$\overset{m}{\cancel{M}} = \tanh(2\beta J \overset{m}{\cancel{M}})$$

for solutions $m \neq 0$ we need $\tanh(2\beta J m) > m$ for small m

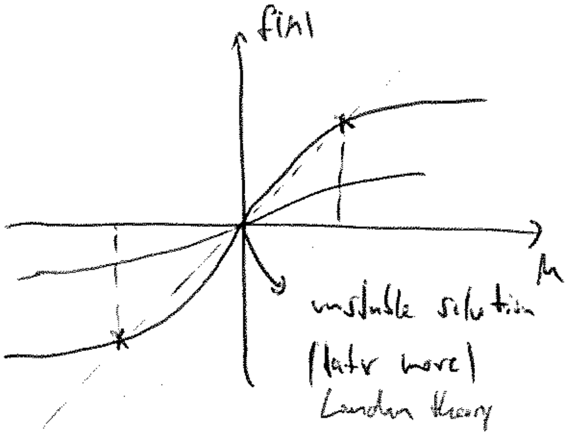
$$\tanh x \sim x - \frac{x^3}{3} + \dots$$

$$\cancel{M}\, m = \frac{2Jm}{k_B T_c} \Rightarrow T_c = \frac{2J}{k_B}$$

$\downarrow$

phase transition

(in contrast to exact solution!)



unstable solution

(later more)

Landau theory

$\boxed{\text{replace } M \text{ by } m \downarrow}$

determine critical exponents

$$m \sim |T-T_c|^\beta$$

$$B \sim m^\delta \quad (\text{at } T=T_c)$$

$$\chi_T = \frac{\partial m}{\partial B}\Big|_T \sim |T-T_c|^{-\gamma}$$

expand equation of state $m = \tanh(\beta(B + 2Jm))$ around $T=T_c$

$$= \tanh\left(\frac{B}{k_B T} + m\tau\right) \qquad \tau = \frac{T_c}{T}$$

for small m and B we obtain $m = \left(\frac{B}{k_B T} + m\tau\right) - \frac{1}{3}\left(\frac{B}{k_B T} + m\tau\right)^3 + \dots \quad (*)$

for $B=0$ we get $m(1-\tau) = -\frac{1}{3} m^3 \tau^3 \Rightarrow m^2 = -\frac{3(1-\tau)}{\tau^3} \sim (T-T_c)$

$$\Rightarrow \boxed{\beta = \frac{1}{2}}$$

for δ: $m(1-\tau) = \frac{B}{k_B T} - \frac{1}{3}\left(\frac{B}{k_B T} + m\tau\right)^3$

$\downarrow$

0

$$= \frac{B}{k_B T} - \frac{1}{3} m^3 \tau^3 - \cancel{\frac{1}{3}\frac{B^3}{(k_B T)^3}} - \cancel{\frac{3}{3}\frac{B}{k_B T} m^2 \tau^2} \dots$$

$\Rightarrow \frac{B}{k_B T} \sim m^3 \quad \boxed{\delta = 3}$

$\hookrightarrow$ all terms involve higher power in B and $B \sim m^3$ from first two terms

$\xrightarrow{\text{justify a posteriori}}$

for γ: start from eq. (*)

$$\left.\frac{\partial M}{\partial B}\right|_T = \chi_T = \frac{1}{k_B T} + \chi_T \cdot \tau + h^2 .$$

$$\Rightarrow \chi_T (1-\tau) = \frac{1}{k_B T} \Rightarrow \chi_T \sim \frac{1}{T-T_c} \quad \boxed{\gamma = 1}$$

2. Liquid-gas transition; van der Waals equation

no phase transitions for a free noninteracting gas.

basic idea: modify ideal gas equation of state $p \cdot V = N k_B T$
so that finite volume of particles and averaged interaction
is taken into account

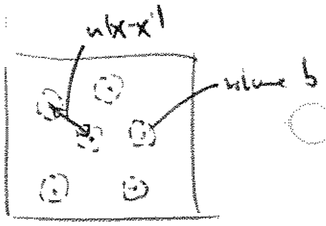
consider semiclassical partition function of a free gas

$$Z_c^{\text{semiclassic, free}} = Z_0 = \frac{V^N}{N! \lambda^{3N}}$$

with thermal wavelength $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ (see exercise)

now consider system with particle interaction $u(\vec{x}_i - \vec{x}_j)$ and particle volume b

$$\Rightarrow Z_c = \underbrace{\frac{1}{N! \lambda^{3N}}}_{\substack{\text{from momentum} \\ \text{integrals (unchanged)} \\ \text{kinetic energy}}} \int d^3x_1 \ldots \int d^3x_N \, e^{-\beta \sum_{i<j} u(\vec{x}_i - \vec{x}_j)}$$



apply mean field approximation for interaction term:

$$-\beta \sum_{i<j} u(\vec{x}_i - \vec{x}_j) = -\frac{\beta}{2} \sum_{i \neq j} u(\vec{x}_i - \vec{x}_j) = -\frac{\beta}{2} \int d^3x \int d^3x' \, \rho(\vec{x}) \, u(\vec{x} - \vec{x}') \, \rho(\vec{x}')$$

$$\rightarrow -\frac{\beta}{2} \int d^3x \int d^3x' \, \langle \rho \rangle^2 \, u(\vec{x} - \vec{x}')$$

with $\rho(\vec{x}) = \sum_i \delta(\vec{x} - \vec{x}_i)$

$$\langle \rho \rangle = \int d^3x \, \rho(\vec{x}) = \frac{N}{V}$$

$$\Rightarrow -\frac{\beta}{2}\int d^3x \int d^3x' \, \langle \rho \rangle^2 u(\vec{x}-\vec{x}')$$

$$= -\frac{\beta}{2}\frac{N^2}{V^2}\int d^3(\vec{x}-\vec{x}') \underbrace{\int d^3(\vec{x}+\vec{x}')}_{\to V} u(\vec{x}-\vec{x}')$$

$$= -\frac{\beta N^2}{2V}\int d^3x \, u(x) = \overbrace{\beta \underbrace{\frac{N^2}{V}\cdot a}_{\text{energy}}}^{\text{dimensionless}} \quad , \quad a = -\frac{1}{2}\int d^3x \, u(x)$$

$$\Rightarrow Z_c = \frac{1}{N!\lambda^{3N}}\int d^3x_1 \ldots \int d^3x_N \, e^{\beta\frac{N^2}{V}a} = \frac{(V-bN)^N}{N!\lambda^{3N}} e^{\beta\frac{N^2}{V}a}$$

i.e for the equation of state we obtain

$$F = -k_B T \log Z_c \overset{\text{Stirling}}{\approx} \underbrace{-N k_B T \log\left(\frac{V-Nb}{N\lambda^3}\right)}_{\text{ideal gas term with } V \to V-bN \to \text{(see exercise)}} - k_B T \beta \frac{N^2}{V}\cdot a$$

$$\Rightarrow p = -\frac{\partial F}{\partial V}\bigg|_{T,N} = \frac{N k_B T}{V-Nb} - \frac{N^2}{V^2}a$$

Critical behaviour of Van-der Waals liquids

liquid-gas transition	Ising model
$v_g - v_e \sim \lvert T-T_c \rvert^{\beta}$	$m = -\frac{1}{N}\frac{\partial F}{\partial B}\Big\vert_T \sim \lvert T-T_c \rvert^{\beta}$
$\lvert p-p_c \rvert \sim \lvert V-V_c \rvert^{\delta}$	$B \sim m^{\delta}$, $m \sim B^{\frac{1}{\delta}}$
$\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial p}\Big\vert_T \sim \lvert T-T_c \rvert^{-\gamma}$	$\chi_T = \frac{\partial m}{\partial B}\Big\vert_T \sim \lvert T-T_c \rvert^{-\gamma}$

In mean field approx we found

$\beta = \frac{1}{2}$

$\delta = 3$

$\gamma = 1$

Van-der Vaals equation leads to the same critical exponents ←

↳ exercise

3. Landau theory of phase transitions

there is a deeper reason why these critical exponents are identical for ferromagnetic transitions and liquid-gas transitions

consider equations of state close to T_c:

Ising model

$$\frac{B}{k_B T} = m(1-\tau) + m^3\left(\tau - \tau^2 + \frac{\tau^3}{3} + \dots\right)$$

$$= \eta t + \eta^3 + O(t\eta^3)$$

$$\left[\tau = \frac{T_c}{T},\ \eta = m,\ t = \frac{T-T_c}{T_c}\right]$$

$$\tau^{-1} = \frac{T}{T_c} = t+1 \qquad = \frac{1}{\tau} - 1$$

$$\boxed{\begin{array}{c}\tau = \dfrac{1}{1+t}\\ \downarrow\\ \tau \sim 1-t\end{array}}$$

Van der Waals

$$p_R = \frac{8T_R}{3V_R - 1} - \frac{3}{V_R^2} \rightarrow \underline{\text{exercise}}$$

$$\left[p_R = \frac{p}{p_c},\ V_R = \frac{V}{V_c},\ T_R = \frac{T}{T_c}\right]$$

$$p_R = \frac{8(1+t)}{3(1+\phi)-1} - \frac{3}{(1+\phi)^2}$$

$$= 1 + 4t - 6t\phi - \frac{3}{2}\phi^3 + O(t\phi^2, \phi^4)$$

$$\left[t = \frac{T-T_c}{T_c},\ \phi = \frac{V-V_c}{V_c}\right]$$

mapping:

$$\left.\begin{array}{lcl}\dfrac{B}{k_B T} & \Leftrightarrow & p_R - 1 - 4t\\[2ex] m & \Leftrightarrow & \underbrace{-\dfrac{V_e - V_c}{V_c}}\end{array}\right\} \Rightarrow \text{identical equation of state!}$$

$$\frac{\rho_e}{\rho_c} - 1 = \frac{V_c - V}{V} \sim \frac{V_c - V}{V_c} = -\frac{V - V_c}{V_c}$$

the equations of state can be derived from an energy function $L(p,t,\phi)$ resp $L(B,t,h)$ by <u>minimization</u> with respect to $\underline{\phi}$ resp $\underline{m}$ (order parameter), compare mean field calculation for Ising model

$$L(p,t,\phi) = L_0(p,t) + c\left((p_R - 1 - 4t)\phi + 3t\phi^2 + \frac{3}{8}\phi^4\right)$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow p_R - 1 - 4t + 6t\phi + \frac{3}{2}\phi^3 = 0$$

[Similarly for $L(B,t,M) = L_0(B,t) + c\left(-\frac{\eta B}{k_B T} + \frac{t\eta^2}{2} + \frac{\eta^4}{4}\right)$]

__Note:__ Energy functions L contain the most general leading terms in the order parameter for $\eta \to 0$ consistent with the symmetries of the system (illustrate with Ising model)

Basic idea of __Landau theory__:

Postulate a free energy ("Landau free energy") L that depends on a set of coupling constants $\{K_i\}$ (e.g. B, T and J for the Ising model) and an order parameter η, whereas ground state of system is given by global minimum of L ($\frac{\partial L}{\partial \eta} = 0$). Assume that __any__ continuous phase transition can be described in this form.

Constraints on L:

a) L has to be consistent with symmetries of system

b) for $T \to T_c$ we have $\eta \to 0$, assume that L is analytic function of both η and $\{K_i\}$, that means for uniform systems:

$$\mathcal{L} = \frac{L}{V} = \sum_{n=0}^{\infty} a_n(\{k\})\eta^n$$

c) for inhomogeneous systems the order parameter becomes a function of position $\eta(\vec{x})$, $\frac{L}{V}$ is a local function, i.e. is a function of $\eta(\vec{x})$ plus gradients of $\eta(\vec{x})$

d) In disordered phase we have $\eta = 0$ and $\eta \neq 0$ in the ordered phase at $T < T_c$

Construction of $\mathcal{L}$: Ising model

- for $B=0$ the Ising model has the symmetry $S_i \to -S_i$, hence $\mathcal{L}(\eta) = \mathcal{L}(-\eta)$

$$\Rightarrow \mathcal{L} = a_0 + \underset{\substack{\updownarrow \\ 0}}{\cancel{a_1\eta}} + a_2\eta^2 + \underset{\substack{\downarrow \\ 0}}{\cancel{a_3\eta^3}} + a_4\eta^4 + \dots$$

↘ also since $\eta=0$ for $T>T_c$ (i.e. even without $\eta \leftrightarrow -\eta$ symmetry it has to vanish, by ordering below independent parameter a_n

- If we truncate at a_4 we need $a_4 > 0$, otherwise we can minimize $\mathcal{L}$ by $\eta \to \pm\infty$
- each parameter a_n generally depends on T: $a_i(T)$
- at large T we have $\eta=0$ and hence $\mathcal{L}(T \gg T_c) = a_0(T)$

↑ smooth background term does not involve order parameter

$\Rightarrow$ can set $a_0 = 0$ without loss of generality

- expand a_2 and a_4 in temperature around T_c:

$$a_2 = a_2^0 + \frac{T-T_c}{T_c} a_2^1 + O((T-T_c)^2)$$

$$a_4 = a_4^0 + \frac{T-T_c}{T_c} a_4^1 + O((T-T_c)^2)$$

$$\Rightarrow \mathcal{L} = \left(a_2^0 + \frac{T-T_c}{T_c} a_2^1\right)\eta^2 + \left(a_4^0 + \frac{T-T_c}{T_c} a_4^1\right)\eta^4$$

$$\frac{\partial\mathcal{L}}{\partial\eta} = 0 \Rightarrow 2\left(a_2^0 + \frac{T-T_c}{T_c} a_2^1\right)\eta + 4\left(a_4^0 + \frac{T-T_c}{T_c} a_4^1\right)\eta^3 = 0$$

for $T<T_c$ we have $\eta \neq 0$ $\Rightarrow$ $\eta = \sqrt{\dfrac{-\left(a_2^0 + \frac{T-T_c}{T_c} a_2^1\right)}{2\left(a_4^0 + \cancel{\frac{T-T_c}{T_c} a_4^1}\right)}}$

for real solution we need $a_2^0 = 0$

and $a_2^1 > 0$. a_4^1 term not leading, can set $a_4^1 = 0$

$$\Rightarrow \eta = \sqrt{\frac{-\left(\frac{T-T_c}{T_c}\right)a_2^1}{2a_4^0}}, \quad \mathcal{L} = a_2^1 t \eta^2 + a_4^0\eta^4$$

- for $B \neq 0$ we obtain an additional term in $\mathcal{L}$:

$$\mathcal{L} = a_2^1 \cdot t \cdot \eta^2 + a_4^0\eta^4 - B\eta$$

microscopic parameters a_i cannot be determined within Landau ↳ since $H \sim -B\sum_i S_i$

Show plots of $\mathcal{L}$ for various values of T and B
(Goldenfeld p 143)

critical exponents

- $\eta \sim |T-T_c|^{\beta}$: we found $\eta(T) = \sqrt{\frac{-a_2 t}{2a_4}} \Rightarrow \underline{\beta = \frac{1}{2}}$
- for $B \neq 0$ we have $2a_2^1 t\eta + 4a_4^0\eta^3 = B$
  at $T = T_c$ we have $t = 0 \Rightarrow B \sim \eta^3 \Rightarrow \underline{\delta = 3}$
- $\chi_T(B) = \frac{\partial\eta}{\partial B}\Big|_T \Rightarrow 2a_2^1 t\frac{\partial\eta}{\partial B} + 12a_4^0\eta^2\frac{\partial\eta}{\partial B} = 1$

$$\chi_T = \frac{1}{2}\left(a_2^1 t + 6a_4^0\eta^2\right)^{-1} \sim |T-T_c|^{-1}, \Rightarrow \underline{\gamma = 1}$$

scales $\sim t$ for $t>0$

First order phase transitions

consider now a more general form for $\mathcal{L}$

$$\mathcal{L} = a_2^1 t\eta^2 + a_4^0\eta^4 + \underset{\substack{\uparrow\\ \text{new}}}{a_3^0\eta^3} \underset{\substack{\uparrow\\ \text{consider } B=0 \text{ case}}}{-B\eta}$$

remember: linear term in η with microscopic parameter a_1 not allowed since $\eta = 0$ for $T > T_c$

$$\frac{\partial\mathcal{L}}{\partial\eta} = 0 \Rightarrow \eta = 0 \quad \text{or} \quad \eta = -c \pm \sqrt{c^2 - \frac{a_2^1 t}{2a_4^0}} \qquad \left[c = \frac{3a_3^0}{8a_4^0}\right]$$

$\downarrow$

real solution for $c^2 > \frac{a_2^1 t}{2a_4^0} \Leftrightarrow t < \underbrace{\frac{2a_4^0 c^2}{a_2^1 t}}_{>0,\text{ i.e. } T>T_c} = t^*$

- lowering t below t^* leads to second minimum in $\mathcal{L}$
- lowering t below t_1 leads to a new global minimum
  $\Rightarrow$ value for η jumps discontinuously from $\eta = 0$ to $\eta(t_1)$
  $\hookrightarrow$ first order phase transition

$\Rightarrow$ in general a cubic term in $\mathcal{L}$ leads to a first order phase transition, absence of cubic term guarantees a continuous phase trans.

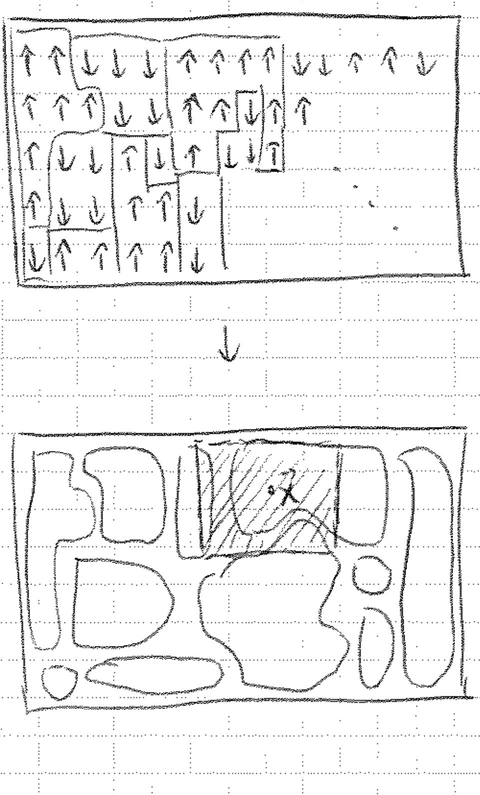
however: note that Landau theory is strictly *not* valid for 1st order phase transitions! (why? $\eta \not\to 0$ for $T \to T_c$, η not necessarily small)

Study of tricritical point in Landau theory $\longrightarrow$ *exercise*

4. Landau theory and coarse graining

- for the physical interpretation of the Landau function it is very instructive to allow for spatially varying order parameters, like for the Ising model $m(\vec{x})$
- furthermore it makes it possible to connect microscopic theories with classical effective theories, illustrates the role of degrees of freedom at different length scales and their systematic treatment $\Rightarrow$ lies at the heart of the RG

Consider a magnetic system in a regime with correlation length ξ



assign a *local magnetization* $m(\vec{x})$ to each point in space (coarse graining):

$$m_\Lambda(\vec{x}) = \frac{1}{N_\Lambda(\vec{x})} \sum_{i \in \vec{x}_\Lambda} \langle S_i \rangle$$

$\uparrow$

$N_\Lambda(\vec{x}) = \frac{\Lambda^{-d}}{a^d}$: number of spins in block of size Λ^{-1} and location $\vec{x}$

$\nearrow$ lattice spacing

linear dimension Λ^{-1} is chosen such that

$$a \ll \Lambda^{-1} \leq \xi(T)$$

$\Rightarrow m_\Lambda(\vec{x})$ is a *smooth* and *slowly varying* function in space (does not contain Fourier components of wave numbers $k > \Lambda$)

- $m(\vec{x})$ <u>not</u> uniquely defined, depends on the choice of Λ!

<u>What is the general form of the Landau function for a system given by $m(\vec{x})$?</u>

- a function of the form $L = \sum_{\vec{x}_i} \mathcal{L}(m_\Lambda(\vec{x}_i))$ <u>cannot</u> be complete since minimization of L with respect to $m(\vec{x})$ would just result in an independent minimization at each point
- need to take into account that domain walls, i.e. differences in $m(\vec{x})$ in adjacent blocks cost energy, simplest way for this is a term of the form

$$\left[\sum_{\vec{x}_i}\underset{\downarrow}{\sum_{\vec{\delta}}}\left(\frac{m_\Lambda(\vec{x}_i)-m_\Lambda(\vec{x}_i+\vec{\delta})}{\Lambda^{-1}}\right)^2\right] \longrightarrow \int d^d\vec{x}\,\left(\nabla m_\Lambda(\vec{x})\right)^2$$

↳ slowly varying and $a \to 0$

vector of length Λ^{-1}, pointing to nearest neighbor block

$$\Rightarrow \quad L = \int d^d\vec{x}\left[\mathcal{L}(m_\Lambda(\vec{x})) + \frac{\gamma}{2}\left(\nabla m_\Lambda(\vec{x})\right)^2\right]$$

L is a functional of $m_\Lambda(\vec{x})$, i.e. depends on entire function at all $\vec{x}$, also called <u>effective Hamiltonian</u>: short distance physics integrated out, effective degree of freedom $m_\Lambda(\vec{x})$ low-energy / long distance degrees of freedom

How is L related to the Hamiltonian and the thermodynamic free energy F?

$$F = -k_B T \log Z_c \Rightarrow \underline{e^{-\beta F} = \mathrm{Tr}\, e^{-\beta H}}$$

$$e^{-\beta L[m_\Lambda(\vec{x})]} = \mathrm{Tr}\left[e^{-\beta H[S_i]}\,\delta\left(\sum_{i\in\vec{x}_\Lambda} S_i - m_\Lambda(\vec{x})N_\Lambda(\vec{x})\right)\right]$$

$$= \mathrm{Tr}'\, e^{-\beta H[S_i]}$$

↳ <u>partial trace</u>: only configurations with local magnetization $m_\Lambda(\vec{x})$ are included

That means by introducing $L[m_\Lambda(\vec{x})]$ we have divided the sum over all states ("Tr") into two steps:

$$Z = e^{-\beta F} = \sum_{\{S_i\}} e^{-\beta H} = \sum_{\{S_i^*\}} \sum_{\{\vec{S}_i\}} e^{-\beta H} = \sum_{\{S_i^*\}} e^{-\beta L[m_\Lambda(\vec{x})]}$$

$\{S_i^*\}$: remaining configurations not consistent with $m_\Lambda(\vec{x})$

$\{\vec{S}_i\}$: all microscopic configurations consistent with magnetization profile $m_\Lambda(\vec{x})$

the sum $\sum_{\{S_i^*\}}$ in the continuum limit amounts to the sum/integral over all coarse grained functions $m_\Lambda(\vec{x})$ and is written as

$$Z = \int Dm_\Lambda \, e^{-\beta L[m_\Lambda(\vec{x})]}$$

functional integral

Dm_Λ involves an infinite number of integrals in continuum limit; a discretized version takes the form

$$\int Dm_\Lambda \rightarrow \prod_{i=1}^{N} \int dm(\vec{x}_i)$$

only smoothly varying (at scale a) functions, all Fourier components $k > \Lambda$ are already included $m_\Lambda(\vec{x})$

in practice the integration over coarse grained degrees of freedom is usually done in momentum space:

$$m(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3\vec{k} \; m(\vec{k}) \, e^{i\vec{k}\cdot\vec{x}} \;, \quad m(\vec{k}) = \int d^3\vec{x} \; m(x) \, e^{-i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \int Dm_\Lambda \rightarrow \prod_{|\vec{k}_i| < \Lambda} \int dm_\Lambda(\vec{k}_i)$$

the functional $L_\Lambda[m(\vec{x})]$ can be computed (in principle) starting from a microscopic Hamiltonian

↳ exercise: for general Ising model $H = -\sum_i B_i S_i - \sum_{ij} J_{ij} S_i S_j$ via a Hubbard-Stratonovich transformation

close to the critical point $L[m(\vec{x})]$ takes the form

$$\underline{L_\Lambda[m(\vec{x})] = \int d^3\vec{x} \left(a_0 - B\, m_\Lambda(\vec{x}) + a_2 m_\Lambda^2(\vec{x}) + \frac{\gamma}{2} (\nabla m_\Lambda(\vec{x}))^2 + a_4 m_\Lambda^4(\vec{x}) \right)}$$

or in momentum space

$$L_\Lambda[m(\vec{p})] = V \cdot a_0 - \underset{\substack{\uparrow \\ \text{for constant } B}}{B\, m_\Lambda(0)} + \int^{\Lambda} \frac{d^3\vec{p}}{(2\pi)^3} \underbrace{\left(a_2 + \frac{\gamma}{2} p^2\right)}_{\substack{\text{from expansion of} \\ J_{ij} \to J(\vec{p}) \sim c_1 + c_2 \vec{p}^2 \text{ for} \\ \text{small wave numbers}}} m_\Lambda(\vec{p})\, m_\Lambda(-\vec{p})$$

$$+ a_4 \int^{\Lambda} dp_1 \int^{\Lambda} dp_2 \int^{\Lambda} dp_3 \int^{\Lambda} dp_4\, m_\Lambda(\vec{p}_1)\, m_\Lambda(\vec{p}_2)\, m_\Lambda(\vec{p}_3)\, m_\Lambda(\vec{p}_4)\, \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)$$

Ginzburg-Landau-Wilson action

$$\int^{\Lambda} dp \equiv \int \frac{d^3 p}{(2\pi)^3}\, \Theta(\Lambda - |p|)$$

- describes long-wavelength order-parameter fluctuations, integration over $m_\Lambda(\vec{p})$ yields contributions from long-wavelength fluctuations to partition function
- effect of short-wavelength contributions encoded in the couplings a, b, c, d, either compute them explicitly as function of Λ via coarse graining or treat them as free parameters
- quantity $e^{-\beta L_\Lambda[m]}$ proportional to the probability density for observing an order-parameter distribution specified by the field configuration $m_\Lambda(\vec{x})$
- partition function given by averaging $e^{-\beta L_\Lambda}$ over all possible configurations $m(\vec{x})$
- term $m(\vec{p})\, m(-\vec{p})$ describes kinetic term, $m(\vec{p}_1)\, m(\vec{p}_2)\, m(\vec{p}_3)\, m(\vec{p}_4)$ interaction of modes at different

if we only consider <u>constant</u> configurations $m(\vec{x}) = \bar{m}$, $\nabla m(x) = 0$ we obtain

$$Z \sim \int_{-\infty}^{\infty} d\bar{m} \; e^{-\beta L(\bar{m})}$$

approximating the integral further by the most likely configuration (saddle point approximation) leads to

$$\left.\frac{dL(\bar{m})}{d\bar{m}}\right|_{\bar{m}=\bar{m}^*} = 0 \quad \Rightarrow \text{mean field result, Landau theory}$$

<u>Interlude</u>: classical physics $\Leftrightarrow$ quantum physics (path integral formulation)

$$U(x_b, t_b; x_a, t_a) = \langle x_b | e^{-i(t_b - t_a)\frac{\hat{H}}{\hbar}} | x_a \rangle$$

matrix element of time evolution operator $\langle x_b | e^{-i(t_b-t_a)\hat{H}/\hbar} | x_a \rangle$

$$= \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t) \; e^{iS[x(t)]/\hbar}$$

with $\int_{t_a}^{t_b} dt \; L(x(t), \dot{x}(t)) = S[x(t)]$

<u>action</u>: T-V

classical physics results from saddle point approximation (most likely path): $\underline{\delta S[x(t)]} = 0$

"mean field result"

variational principle

$\Rightarrow$ Euler-Lagrange equations

5. Validity and breakdown of mean-field theory

- mean field theory amounts to evaluating the partition function in saddle point approximation for a constant magnetization
- approximation can be improved by including fluctuations to quadratic order (Gaussian approximation)

consider Ginzburg-Landau-Wilson functional for $B=0$ and after dropping the quartic terms:

$$L_\Lambda = \int d^3\vec{x} \left(\frac{\gamma}{2} (\nabla m(\vec{x}))^2 + a_2 m^2(\vec{x}) \right) + a_0 V$$

we insert the Fourier expansion for $m(\vec{x})$: $m(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} m(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}$

$$\Rightarrow L_\Lambda = \int d^3\vec{x} \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} \left[\frac{\gamma}{2} (-\vec{k}\cdot\vec{k}') + a_2 \right] m(\vec{k})\, m(\vec{k}')\, \underbrace{e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{\to \delta(\vec{k}+\vec{k}')} + a_0 V$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \left[\frac{\gamma}{2} \vec{k}^2 + a_2 \right] m(\vec{k})\, m(-\vec{k}) + a_0 V \qquad (*)$$

- mean field theory only takes into account contributions at $\vec{k}=0$ (uniform state)
- relation $(*)$ is called Gaussian approximation, takes fluctuations at finite $\vec{k}$ into account, contributions at different $\vec{k}$ independent (non-interacting fluctuations)
- Gaussian approximation can be solved exactly

- $m(\vec{x})$ is not defined on length scales shorter than Λ^{-1}
  $\Rightarrow$ only integrate over modes with $|\vec{k}| < \Lambda$

$$e^{-\beta F} = \int_{-\infty}^{\infty} \prod_{|\vec{k}|<\Lambda} dm_\Lambda(\vec{k})\; e^{-\beta L_\Lambda} \underset{\substack{\uparrow \\ \text{Gauss-Integral} \\ \text{(see exercise)}}}{=} \exp\left[\frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3} \log\left(\frac{2\pi V k_B T}{2a_2 + \gamma k^2} \right) \right] \cdot e^{-\beta a_0 V}$$

- mean-field + Gaussian approximation leads to the <u>same</u> critical exponents like mean-field theory (which are known to be inconsistent with exp)

* study if ~~no~~ Gaussian approximation can be systematically improved by perturbatively include effects from interactions, ie $m^4(\vec{x})$ resp $\iiiint_{k_1 - k_4} m(k_1) m(k_2) m(k_3) m(k_4) \delta(\ )$

the functional for $B = 0$ ~~ratt~~ in D dimensions reads (setting $a_0 = 0$)

$$L_\Lambda = \int d^D \vec{x} \left(a_2 m^2(\vec{x}) + \frac{\gamma}{2} \left(\nabla m(\vec{x}) \right)^2 + a_4 m^4(\vec{x}) \right)$$

~~@~~ close to T_c we know from Landau theory that

$$a_2 \sim a_2^1 \cdot t$$
$$\hookrightarrow \frac{T - T_c}{T_c}$$

We perform a dimensional analysis of different terms in L_Λ:

- introduce $\bar{m} = \sqrt{\beta \gamma}\, m$, $\frac{a_2^1 \cdot t}{\gamma} = \frac{r_0}{2}$, $\frac{a_4}{\beta \gamma^2} = \frac{u_0}{4}$

$$\Rightarrow \beta L_\Lambda = \int d^D \vec{x} \left(\frac{(\nabla \bar{m})^2}{2} + \frac{r_0}{2} \bar{m}^2 + \frac{u_0}{4} \bar{m}^4 \right)$$

* $[\beta L_\Lambda] = 1 \Rightarrow \left[\int d^D \vec{x} \, (\nabla \bar{m})^2 \right] = 1 = L^D L^{-2} [\bar{m}]^2 \Rightarrow [\bar{m}] = L^{1 - \frac{D}{2}}$

* $[r_0] = \underset{([\int d^d x \, r_0 \bar{m}^2] = 1)}{L^{-2+D} \cdot L^{-D}} = L^{-2}$; $[u_0] = \underset{([\int d_1^d x \, u_0 \bar{m}^4] = 1)}{L^{-4+2D} \cdot L^{-D}} = L^{D-4}$

* ~~we~~ in mean field theory we find $\xi \sim |T - T_c|^{-\frac{1}{2}}$ ($\nu = \frac{1}{2}$)

$\Rightarrow r_0 \sim T - T_c \sim \frac{1}{\xi(T)^2}$ (see next section)

* define dimensionless fields and couplings:

$$\phi = \frac{\bar{m}}{L^{1 - \frac{D}{2}}} , \quad \vec{x} = \frac{\vec{x}}{L} , \quad \bar{u}_0 = \frac{u_0}{L^{D-4}} , \quad \underset{\downarrow}{L = r_0^{-\frac{1}{2}}}$$

measure all lengths in terms of ξ

$$\Rightarrow Z = \int D\phi \, e^{-H_0 - H_{int}} \underset{\uparrow}{=} \int D\phi \, e^{-H_0} \left(1 - H_{int} + \frac{1}{2!} H_{int}^2 - \dots \right)$$

perturbative expansion in H_{int}

$$H_0 = \int d^D \vec{x} \left(\frac{(\nabla \phi)^2}{2} + \frac{\phi^2}{2} \right) , \quad H_{int} = \int d^D \vec{x} \, \frac{1}{4} \bar{u}_0 \phi^4$$

here $\bar{u}_0 = u_0 r_0^{(D-4)/2} \sim t^{\frac{D-4}{2}}$

$\Rightarrow$ interaction term becomes arbitrarily large as we approach the critical point ($\xi \to \infty$, $t \to 0$) <u>for $D < 4$</u>, no matter how small the original coupling u_0

$\Rightarrow$ it is <u>NOT</u> possible to systematically improve Gaussian approximation by adding interaction effects <u>perturbatively</u>, non perturbative methods necessary

Conclusions: (for the Ising universality class, i.e. the form of the Landau functional close to T_c)

(a) for $D > 4$ Landau theory is reliable and gives correct critical exponents

(b) for $D < 4$ Landau theory is not valid

(c) for $D = 4$ fluctuations give logarithmic corrections to mean field results

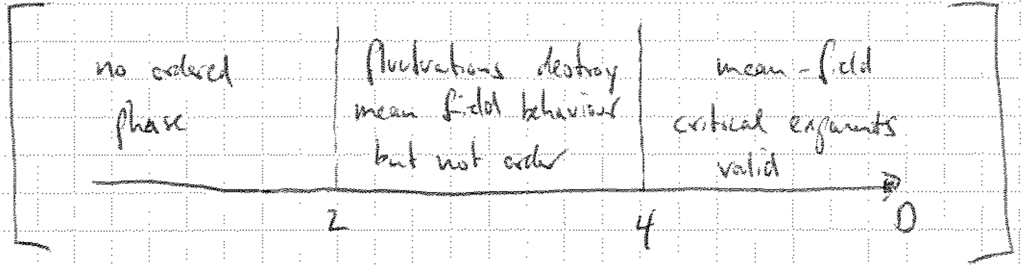
$D_{crit} = 4$ is called the <u>upper critical dimension</u> and is for general systems given by the relation

$$D_{crit} = \frac{2\beta + \gamma}{\nu} \leftarrow \text{critical exponents of Landau theory}$$

not shown explicitly in this lecture

for Ising model we found $\beta = \frac{1}{2}$, $\gamma = 1$, $\nu = \frac{1}{2}$

$$\left[M \sim |T - T_c|^{\beta},\ \chi_T \sim |T - T_c|^{\gamma},\ \xi \sim |T - T_c|^{\nu} \right]$$



show slide with experimental values vs RG vs Landau (keynote)

6. Anomalous dimension

study in some more detail the physical implications of the value of the critical exponent ν (~~[illegible]~~ or alternatively η)

define anomalous dimension θ: $\nu = \frac{1}{2} - \theta$

use dimensional analysis: we have $[\xi] = L$ and $[r_0] = L^{-2}$ (see previous ~~ch~~ section) and $r_0 \sim t$

hence: $\xi \sim r_0^{-\frac{1}{2}} \sim t^{-\frac{1}{2}} \Rightarrow \nu = \frac{1}{2}, \theta = 0$

based on these very general arguments ν should always take the value $\nu = \frac{1}{2}$ and anomalous dimension $\theta = 0$

based on keynote table however that is not true, ~~how~~

how is this possible! any other value seems to violate dimensional analysis

answer: there must be another length scale in the system!

despite the fact that physics at long wavelength dominates physics close to critical points (ie scales $\sim \xi$) also microscopic length scales show up in critical exponents

consider lattice spacing a as an additional scale:

$$[\xi] = L \ , \quad [a] = L \qquad [r_0] = L^{-2}$$

$$\hookrightarrow \sim t$$

generalizing arguments above leads to (see also exercise set 3)

$$\xi = r_0^{-\frac{1}{2}} f(r_0 \cdot a^2)$$

$\uparrow$ function to be determined

what happens close to critical point $(t \to 0)$?

assume $f(x) \sim x^{\theta}$ for $x \to 0$ (non analytic behaviour!)

$$\Rightarrow \quad \xi \sim t^{-\frac{1}{2}+\theta} a^{2\theta}$$

this result is remarkable, even though $\frac{a}{\xi} \lll 1$ we <u>cannot</u> in general replace a function $\Phi\left(\frac{a}{\xi}\right)$ by $\Phi(0)$ (then a would not appear in any final result

quantities at very different length scales contribute to the physics of critical phenomena!