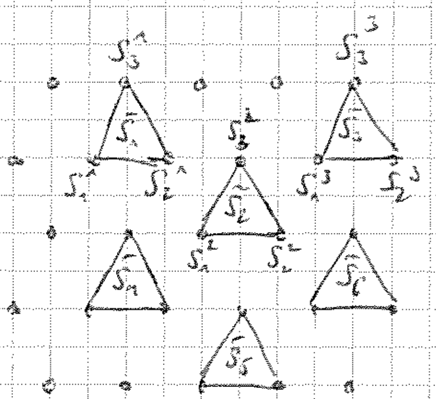
V Applications of the Renormalization Group

1. RG for 2d Ising model on triangular lattice

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - B \sum_i S_i$$



construct explicitly block spins by grouping together 3 spins and assign spin according to majority rule

$$\tilde{S}_I = \text{sign}\left(S_1^I + S_2^I + S_3^I \right)$$

the coarse grained Hamiltonian H' is then given by

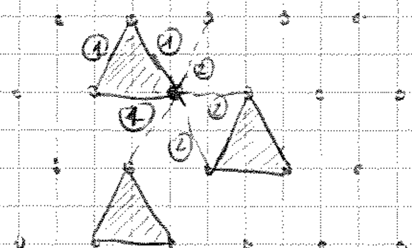
$$e^{-\beta H'(\{\tilde{S}_I\})} = \sum_{\{S_i^I\}} e^{-\beta H(\{S_i\})}$$

sum over all configurations in block spin $\tilde{S}_I$

$\tilde{S}_I$	S_1^I	S_2^I	S_3^I	$e^{-\beta H_0}$
+	+	+	+	e^{3K}
+	−	+	+	e^{-K}
+	+	−	+	e^{-K}
+	+	+	−	e^{-K}
−	−	−	−	e^{3K}
−	+	−	−	e^{-K}
−	−	+	−	e^{-K}
−	−	−	+	e^{-K}

- initially N spins
  ↳ after RG transformation $\frac{N}{3}$ block spins

there are two different types of interaction contributions in a RG transformation:



① interactions within one block spin (intra-cell interactions)

② interactions among different blocks (inter-cell interactions)

- split Hamiltonian into two parts (set $B=0$ for the moment)

$$H_0 = -J \sum_I \sum_{i,j \in I} S_i^I S_j^I = -J \sum_I \left(S_1^I S_2^I + S_2^I S_3^I + S_3^I S_1^I \right)$$

$$V = -J \sum_{I \neq J} \sum_{\substack{i \in I \\ j \in J}} S_i^I S_j^J$$

- in order to truncate number of induced interactions by the RG transformation we will treat V as a perturbation using the cumulant expansion (Niemeyer/van Leeuwen) $\rightarrow$ strictly not justified!

$$\Rightarrow \quad e^{-\beta H'(\{\tilde{S}_I\})} = \sum_{\{S_i^I\}} e^{-\beta H_0(\{S_i\}) - \beta V(\{S_i\})}$$

$$= \left\langle e^{-\beta V} \right\rangle_0 \sum_{\{S_i^I\}} e^{-\beta H_0(\{S_i\})} = \left\langle e^{-\beta V} \right\rangle_0 Z_0^{\frac{N}{3}}$$

$$\text{with} \quad \langle A \rangle_0 = \frac{\sum_{\{S_i^I\}} e^{-\beta H_0(\{S_i\})} A(\{S_i\})}{\sum_{\{S_i^I\}} e^{-\beta H_0(\{S_i\})}}$$

$$Z_0(\beta J) = Z_0(k) = \sum_{S_1, S_2, S_3}' e^{k(S_1 S_2 + S_2 S_3 + S_3 S_1)} = 3e^{-k} + e^{3k}$$

for a given value of $\tilde{S}_I$

Z_0 is independent of $\tilde{S}_I$!

for evaluation of $\langle e^{-\beta V} \rangle_0$ we use the cumulant expansion:

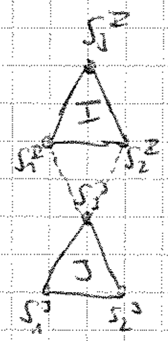
$$\left\langle e^{-\beta V} \right\rangle_0 = \exp\left[-\langle \beta V \rangle_0 + \frac{1}{2}\left(\langle (\beta V)^2 \rangle_0 - \langle \beta V \rangle_0^2 \right) + o(V^3) \right]$$

$$\Rightarrow \quad \beta H'(\{\tilde{S}_I\}) = -\frac{N}{3} \log Z_0(k) + \langle \beta V \rangle_0 - \frac{1}{2}\left(\langle (\beta V)^2 \rangle_0 - \langle \beta V \rangle^2 \right) + o(V^3)$$

show here the calculation of this term

consider two adjacent block spins I and J :

interaction V_{IJ} between two blocks is mediated by spin S_3^J.



$$V = \sum_{I \neq J} V_{IJ} \,, \quad V_{IJ} = -J\, S_3^J (S_1^I + S_2^I)$$

$$\Rightarrow \langle V_{IJ} \rangle_0 = -J \left(\langle S_3^J S_1^I \rangle_0 + \langle S_3^J S_2^I \rangle_0 \right)$$

$$= -2J \langle S_3^J S_1^I \rangle_0 \quad = -2J \langle S_3^J \rangle_0 \langle S_1^I \rangle_0$$

$\uparrow$ H_0 does not couple different blocks!

$$\Rightarrow \text{need to evaluate } \langle S_3^J \rangle_0 = Z_0(K)^{-1} \sum_{\{S_i^J\}} S_3^J \, e^{K(S_1^J S_2^J + S_2^J S_3^J + S_3^J S_1^J)}$$

$$= \tilde{S}_J \underbrace{\frac{e^{-K} + e^{3K}}{e^{3K} + 3e^{-K}}}_{\Phi(K)} \longrightarrow \underline{\text{exercise}}\text{: verify!}$$

$$\Rightarrow \langle V \rangle_0 = -2J \left(\Phi(K) \right)^2 \sum_{\langle IJ \rangle} \tilde{S}_I \tilde{S}_J$$

$$\Rightarrow \beta H'(\{\tilde{S}_I\}) = -\frac{N}{3} \log Z_0(K) - 2K \left(\Phi(K) \right)^2 \sum_{\langle IJ \rangle} \tilde{S}_I \tilde{S}_J$$

↳ exactly the same form like $\beta H(\{S_i\})$ (apart from a new constant term $\sim \log Z_0(K)$)

$$\Rightarrow \boxed{K' = 2K \left(\Phi(K) \right)^2} , \quad \text{RG transformation for coupling constant}$$

↳ note: $(B = 0)$

fixed point:

$$K^* = 2K^* \left(\Phi(K^*)\right)^2$$

$\Rightarrow$ a) $K^* = 0$

b) $K^* = \infty$

c) $K^* = \frac{1}{4} \log\left(1 + 2\sqrt{2}\right) \sim 0{,}34$ exercise: verify!

- compare to exact solution for non-trivial fixed point $K^*_{exact} = \frac{1}{4} \log 3 \sim 0{,}27$

- relevant RG eigenvalue $\lambda_1 = \left.\frac{\partial K'}{\partial K}\right|_{K^*} = 1{,}62$ (exact: $\lambda = \sqrt{3} \sim 1{,}73$)

- calculations can be improved by taking into account second-order term in cumulant expansion
  ↳ leads to induction of more couplings (second- and third nearest neighbor couplings)
  ↳ $\lambda_1 \sim 1{,}77$ (getting closer to exact solution...)

- however: note that cumulant expansion here is not rigorous:
  not obvious that V can be treated as perturbation
  $\longrightarrow$ convergence pattern is thus also not uniform

- inclusion of external magnetic field B leads to a second relevant eigenvalue $\lambda_2^{(0)} = \frac{3}{\sqrt{2}} \sim 2{,}12$

$\lambda_2^{(i)} \leftarrow$ order in V $\lambda_2^{(1)} \sim 3{,}06$

$\lambda_2^{(2)} \sim 2{,}76$ exact: $\lambda_2^{exact} \sim 2{,}8$

- discuss phase diagram graphically again (slides)

2. Kosterlitz-Thouless transition

- we have shown that for systems with a discrete symmetry and short range interactions, there is no long-range order at finite T for $D<2$ (Section II/3 and II/4)

- for system with continuous symmetry and short-range interactions, there is no long-range order at finite T for $D \leq 2$ (Mermin-Wagner theorem)

- discuss experimental results by Chester/Yang/Stephens (1972)
  * use quartz crystal with a thin film of ^4He adsorbed to crystal
  * at low T film is superfluid, crystal oscillates whereas film is at rest
  * for larger T superfluidity gets reduced and eventually disappears, film is locked to crystal and now also oscillates (oscillation frequency changes due to additional mass $\left[\frac{\rho}{\rho_0}\right]$)
  * shift of resonance frequency is measure of superfluid density

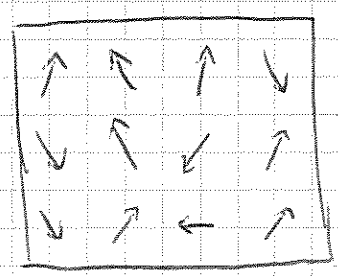
$\Rightarrow$ experimental evidence of phase transition in effectively 2d systems! how to reconcile with Mermin-Wagner theorem?

- superfluidity is characterized by a complex $U(1)$ order parameter (phase), equivalent to a spin theory with two real components
  $\rightarrow$ X-Y - model

X-Y model consists of two-component spins: $\vec{S}_i = (S_i^x, S_i^y)$
with $\vec{S}_i^2 = (S_i^x)^2 + (S_i^y)^2 = 1$
the Hamiltonian for a nearest neighbor interaction takes the form

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \qquad \text{(continuous symmetry!)}$$

in 2 spatial dimensions system represents a plane consisting of 2d rotors



$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$

$$= -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

$$= -J \sum_{\langle i,j \rangle} \left(1 - \frac{(\theta_i - \theta_j)^2}{2} + \dots\right)$$

$$\rightarrow E_0 + \frac{J}{2} \int d^2\vec{r} \, \left(\nabla \theta(\vec{r})\right)^2 + \dots$$

continuum limit
+ expansion for small $(\theta_i - \theta_j)$

E_0 : energy of completely aligned ground state of N rotors (uninteresting in the following, set $E_0 = 0$ without loss of gen.)

$$Z = e^{-\beta E_0} \int \mathcal{D}\theta \, \exp\left[-\frac{\beta J}{2} \int d^2\vec{r} \, (\nabla \theta)^2\right]$$

find local minima of measure: $\frac{\delta H}{\delta \theta(\vec{r})} = 0 \Rightarrow \underline{\nabla^2 \theta(r) = 0}$

two solutions: 1) $\theta_1(\vec{r}) = \text{const}$ $\longrightarrow$ unstable w.r.t. thermal fluctuations!

2.) $\theta_2(\vec{r}) = n \tan^{-1}\left(\frac{r_y - r_y^0}{r_x - r_x^0}\right)$ $\longrightarrow$ stable due to topological structure!

note that $\oint d\vec{\ell} \cdot \nabla \theta = 2\pi n$ "vortex", topological excitation

$\oint$: closed curve that constains $\vec{r}_0$

n: topological charge, $n > 0$: vortex
$n < 0$: anti-vortex

$\boxed{\Longrightarrow \text{show slides}}$

else $\oint d\vec{\ell} \cdot \nabla \theta = 0$ $\Rightarrow$ general solution:

$\oint$: closed curve that does NOT contain $\vec{r}_0$

$$\theta_2(\vec{r}) = \sum_{i=1}^{N} n_i \tan^{-1}\left(\frac{r_y - r_y^i}{r_x - r_x^i}\right)$$

what is the energy of a single vertex!

$$\oint d\vec{l} \cdot \nabla\theta(\vec{r}) \underset{\uparrow}{=} 2\pi r |\nabla\theta| = 2\pi n \qquad \Rightarrow |\nabla\theta| = \frac{n}{r}$$

for a circular path (symmetry)

$$\Rightarrow \quad E = \frac{J}{2}\int d^2\vec{r}\, |\nabla\theta(\vec{r})|^2 = \frac{J}{2} n^2 \int_0^{2\pi} \int_a^L r\,dr \frac{1}{r^2} = \frac{2\pi n^2}{2} \log\frac{L}{a}$$

L → linear size of system

a → short distance cutoff (e.g. lattice spacing)

energy of single vortex diverges,
i.e. is macroscopically large for $L \to \infty$!
→ excitation of single vortex energetically not favoured

how about a vortex ($n=1$) – antivortex ($n=-1$) pair?

$$E_{pair} \sim 2\pi J \log\frac{R}{a} \qquad R: \text{distance of pair}$$

can be shown by noting that there is a one-to-one correspondence to 2d coulomb system (note $\nabla^2\Theta = 0$, Poisson law)

$\Rightarrow$ can be small for small R, at low T vortex-antivortex pairs can be excited!

idea of Kosterlitz and Thouless: at low T vortices are bound in pairs, at higher $T > T_c$ become unbound (plasma of vortices)
$\Rightarrow$ KT transition

consider stability of system w.r.t. creation of single vortices ($n=1$)

$$F = E - TS \quad , \quad S = k_B \log\left(\underbrace{\left(\frac{L}{a}\right)^2}\right)$$

number of possible lattice sites vortex can occupy

$$\Rightarrow F = (\pi J - 2k_B T)\log\frac{L}{a}$$

for $T > T_c = \frac{\pi J}{2k_B}$ it is favorable to create isolated vortices $\Rightarrow$ unbinding

crucial: this can only happen in 2d since here E and S have the same $\log\frac{L}{a}$ dependence!

for general number of spatial dimensions D the correlation function

$\langle S(\vec{r})\, S(0)\rangle$ has the following form for $r\to\infty$ and low T

$$\langle S(\vec{r})\, S(0)\rangle = \begin{cases} e^{-cT} = \text{finite} & D>2 \quad \text{long-range order} \\ \left(\frac{r}{L}\right)^{-\eta} & D=2 \quad \text{algebraic behavior} \\ e^{-\frac{Tr}{2Ja}} & D=1 \quad \text{no long-range order} \end{cases}$$

at large T system should behave like paramagnet:

$$\langle S(\vec{r})\, S(0)\rangle = e^{-r/\xi}$$

$\Rightarrow$ transition from algebraic behavior to exponential indicates phase transition

$\Rightarrow$ note that $\langle S(\vec{r}) S(0)\rangle$ behaves algebraic for all $T < T_{KT}$ (not just at $T = T_c$)

– it can be shown that $\quad \xi(T) \sim \exp\left(\frac{c}{(T-T_{KT})^{\frac{1}{2}}}\right) \quad$ for $T > T_{KT}$

KT transition is transition of infinite order

$\boxed{V/9}$

all our considerations so far neglected interaction effects between vortices

→ RG calculations by Kosterlitz and Thouless showed that the conclusions above are not changed by interaction effects!

→ an interesting prediction of KT theory is the existence of a universal number:

$$\underbrace{\frac{m^2}{\hbar^2}}_{^4\text{He mass}} \frac{k_B T_{KT}^-}{\rho_s(T_{KT})} = \frac{\pi}{2}$$

jump in superfluid density $\rho_s(T)$

independent of used substrate, layer thickness etc. !

⇒ show slides