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Phase transitions and the Renormalization Group

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Problem set 3

Discussion of problems: Wednesday, June 14

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Problem 5: Dimensional analysis

Dimensional analysis is a powerful tool and can be applied to various problems in physics. It is based on the fact that in any problem involving a number of dimensionful quantities, the relationship between them can be expressed by forming all possible independent dimensionless quantities $\Pi, \Pi_1, \Pi_2, ...\Pi_n$. The solution for Π can then be expressed in the form

$$\Pi = f(\Pi_1, ..., \Pi_n). \tag{1}$$

1. Assume there is only one dimensionless combination of variables in a given problem. What follows for Π? Can we say anything about the value of Π?

2. Derive the characteristic size of the radius and ground state energy for a hydrogen atom using dimensional analysis. Compare your results with the exact values. Are the dimensionless constants of natural size?

3. Prove Pythagoras' theorem using dimensional analysis. For this task, use only the fact that the area of a right-angle triangle can be expressed as a function of the hypothenuse and one of the acute angles of the triangle (don't use trigonometry!). What happens if you consider non-Euclidian geometries?

   HINT: It is useful to add a well-chosen line to the right-angle triangle.

Problem 6: Ginzburg-Landau-Wilson effective field theory

The construction of effective field theories is key for understanding the basic ideas that lie at the heart of *Wilson's Renormalization Group* formulation. There exist cases for which this task can be performed *exactly* starting from a microscopic theory. Here we will perform this exercise via the *Hubbard-Stratonovich transformation* (see problem 4) for a general Ising model for N spins on a three-dimensional lattice with a lattice spacing a:

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} S_i J_{ij} S_j - \sum_{i=1}^{N} B_i S_i. \tag{2}$$

Here $J_{ij} = J_{ji}$ is a positive symmetric matrix which denotes the couplings between spins i and j and B_i is the external magnetic field at site i.

1. Prove the following relation for the interaction term:

$$\exp\big(\frac{1}{2}\sum_{i,j} S_i J_{ij} S_j\big) \int \mathcal{D}[z] \exp\big(-\frac{1}{2}\sum_{i,j} z_i J^{-1}_{ij} z_j\big) = \int \mathcal{D}[z] \exp\big(-\frac{1}{2}\sum_{i,j} z_i J^{-1}_{ij} z_j + \sum_i z_i S_i\big) \tag{3}$$

with

$$\int \mathcal{D}[z] = \prod_i \int_{-\infty}^{\infty} \frac{dz_i}{\sqrt{2\pi}}.$$

Hint: Introduce new integration variables $z_i' = z_i - \sum_j J_{ij} S_j$.

2. Show that the partition function can be expressed in the following form:

$$Z = \left[\int \mathcal{D}[z] \exp\left(-S[z]\right)\right] \left[\int \mathcal{D}[z] \exp\big(-\frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j\big)\right]^{-1} = \frac{1}{\sqrt{\det \beta \mathbf{J}}} \int \mathcal{D}[z] \exp\left(-S[z]\right), \tag{4}$$

with

$$S[z] = \frac{1}{2\beta}\sum_{i,j} z_i J^{-1}_{ij} z_j - \sum_i \ln[2\cosh(\beta B_i + z_i)].$$

3. Show that the expectation values of the variables z_i are given by the following relation:

$$\big\langle \beta \sum_j J_{ij} S_j \big\rangle = Z^{-1} \mathrm{Tr} \sum_j (\beta J_{ij} S_j) e^{-\beta H} = \langle z_i \rangle \equiv \frac{\int \mathcal{D}[z] z_i \exp\big(-S[z]\big)}{\int \mathcal{D}[z] \exp\big(-S[z]\big)}. \tag{5}$$

Based on this result, show that the expectation values of the new variables ϕ_i defined by

$$\phi_i \equiv \beta^{-1} \sum_j J^{-1}_{ij} z_j, \quad \text{i.e.} \quad z_i = \beta \sum_j J_{ij} \phi_j \tag{6}$$

corresponds to the magnetization per lattice site. Show that the partition function can be written in terms of the *effective action* $S[\phi]$ in the following form:

$$Z = \sqrt{\det \beta \mathbf{J}} \int \mathcal{D}[\phi] e^{-S[\phi]} \;\; \text{with} \;\; S[\phi] = \frac{\beta}{2}\sum_{i,j} \phi_i J_{ij} \phi_j - \sum_i \ln\left[2\cosh\big(\beta(B_i + \sum_j J_{ij}\phi_j)\big)\right]. \tag{7}$$

Hint: For the derivation of relation (5) you can use the technique of *external sources*:

$$z_i = \lim_{\mathbf{a}\to 0} \frac{\partial}{\partial a_i} \exp\big(\sum_j a_j z_j\big). \tag{8}$$

4. The relation (7) is an *exact* representation of the partition function of the Ising model and hence is in general very complicated to solve. In order to simplify the expression we consider a system close to the critical point and assume that the partition function is dominated by small values of ϕ_i. Show that the effective action takes the following form up to order $\mathcal{O}(\phi_i^6)$:

$$S[\phi] = -N\log 2 + \frac{\beta}{2}\sum_{i,j}\phi_i J_{ij}\phi_j - \frac{\beta^2}{2}\sum_i \big(B_i + \sum_j J_{ij}\phi_j\big)^2 + \frac{\beta^4}{12}\sum_i \big(B_i + \sum_j J_{ij}\phi_j\big)^4 \tag{9}$$

Perform the continuum limit $N \to \infty : \phi_i \to \phi(\mathbf{r}), J_{ij} \to J(\mathbf{r} - \mathbf{r}')$, and represent the variables in momentum space, i.e.:

$$\phi(\mathbf{r}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \phi(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}}, \quad \delta(\mathbf{r}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} \tag{10}$$

Show that the effective action takes the following form up to order $\mathcal{O}(\phi^6, B^2, B\phi^3)$ for a homogeneous external field $B_i \to B(\mathbf{r}) = B$:

$$\begin{aligned} S[\phi(\mathbf{p})] &= -N \log 2 - \beta^2 \frac{B}{(2\pi)^3} J(0)\phi(0) + \frac{\beta}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} J(\mathbf{p})(1 - \beta J(\mathbf{p}))\phi(\mathbf{p})\phi(-\mathbf{p}) \\ &+ \frac{\beta^4}{12} \frac{1}{(2\pi)^9} (\prod_{i=1}^{4} \int d^3\mathbf{p}_i) J(\mathbf{p}_1)J(\mathbf{p}_2)J(\mathbf{p}_3)J(\mathbf{p}_4)\phi(\mathbf{p}_1)\phi(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4)\delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \end{aligned} \tag{11}$$

5. We are particularly interested in the *long-wavelength* contributions to the partition function. For this we limit the momentum integrals to wave numbers below a scale Λ and expand the function $J(\mathbf{p})$ for small momenta in powers of $\mathbf{p}$. Show that the final form of the partition function can be written in the form $Z = \int \mathcal{D}[\phi] e^{-S_\Lambda[\phi]}$ with:

$$\begin{aligned} S_\Lambda[\phi(\mathbf{p})] &= aV + bB\phi(0) + \frac{1}{2} \int_{\mathbf{p}} (c_0 + c_1 \mathbf{p}^2)\phi(\mathbf{p})\phi(-\mathbf{p}) \\ &+ \frac{d}{4!} \int_{\mathbf{p}_1} \int_{\mathbf{p}_2} \int_{\mathbf{p}_3} \int_{\mathbf{p}_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4)\phi(\mathbf{p}_1)\phi(\mathbf{p}_2)\phi(\mathbf{p}_3)\phi(\mathbf{p}_4) \end{aligned} \tag{12}$$

Here we used the notation $\int_{\mathbf{p}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Theta(\Lambda - |\mathbf{p}|)$. Discuss the physical meaning of the scale Λ. How are the couplings constants a, b, c_0, c_1 and d related to the constants in (11). Show that in coordinate space the effective action takes the following form:

$$S_\Lambda[\phi(\mathbf{r})] = \int d^3\mathbf{r} \left[a + bB\phi(r) + \frac{c_0}{2}\phi^2(\mathbf{r}) + \frac{c_1}{2}(\nabla\phi(\mathbf{r}))^2 + \frac{d}{4!}\phi^4(\mathbf{r}) \right]. \tag{13}$$