Structure of Trapped Degenerate Fermi Gases

Robert Roth
CATS Workshop
April 2002
Overview

- The World of Trapped Atomic Fermi Gases

- Description of Trapped Degenerate Fermi Gases
  - The Many-Body Problem
  - Correlations & Effective Interaction
  - Mean-Field & Thomas-Fermi Approximation
  - Energy Functional

- Structure of Single- and Two-Component Fermi Gases
  - Energy Landscapes & Density Profiles
  - Mean-Field Induced Collapse
  - Component Separation
  - Phase Diagram
Onset of Fermi Degeneracy in a Trapped Atomic Gas

B. DeMarco and D. S. Jin

An evaporative cooling strategy that uses a two-component Fermi gas was employed to cool a magnetically trapped gas of $7 \times 10^5$ $^{40}$K atoms to 0.5 of the Fermi temperature $T_F$. In this temperature regime, where the state occupying the lowest energies has increased from essentially zero to nearly 60 percent, quantum degeneracy was achieved through evaporative cooling and as a modification of the two-component Fermi gas mixture. The $^{40}$K has fractional total spin: fermion $F^I = 4 \pm 1/2 = \frac{9}{2}, \frac{7}{2}$.

- $N \approx 10^5 ... 10^6$
- $\ell \approx 1 \mu m$
- $T \approx 300 \text{nK} \approx 0.5 \varepsilon_F$
- $\tau \approx 300 \text{s}$
- $\rho \approx 10 \mu \text{m}^{-3}$

two-component mixture

$|F = \frac{9}{2}, m_F = \frac{9}{2}\rangle$
$|F = \frac{9}{2}, m_F = \frac{7}{2}\rangle$
**Observation of Fermi Pressure in a Gas of Trapped Atoms**

Andrew G. Truscott, Kevin E. Strecker, William I. McAlexander,*
Guthrie B. Partridge, Randall G. Hulet†

Simultaneous trapping of

\[
\begin{align*}
\text{\textsuperscript{7}}\text{Li} & \rightarrow F = 2 \rightarrow \text{boson} \\
\text{\textsuperscript{6}}\text{Li} & \rightarrow F = \frac{3}{2} \rightarrow \text{fermion}
\end{align*}
\]

Evaporative cooling of the bosons \( \rightarrow \) sympathetic cooling of the fermions

\[
N_B \approx N_F \approx 10^5 \ldots 10^6
\]

\[
T \approx 240 \text{ nK} \\
\approx 0.25 \varepsilon_F
\]
### Fermion Experiments — Today

#### Two-Component Fermi Gases

<table>
<thead>
<tr>
<th>Date</th>
<th>Isotope</th>
<th>Temperature</th>
<th>Fermions</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>09/1999</td>
<td>$^{40}\text{K}$</td>
<td>$T = 0.5\varepsilon_F$</td>
<td>$N_F \sim 10^6$</td>
<td>JILA, Boulder/Colorado, B. DeMarco, D.S. Jin</td>
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<tr>
<td>11/2001</td>
<td>$^6\text{Li}$</td>
<td>$T = 0.5\varepsilon_F$</td>
<td>$N_F \sim 10^5$</td>
<td>Duke Univ., Durham/North Carolina, S.R. Granade,..., J.E. Thomas</td>
</tr>
</tbody>
</table>

#### Binary Boson-Fermion Mixtures

<table>
<thead>
<tr>
<th>Date</th>
<th>Isotopes</th>
<th>Temperature</th>
<th>Fermions</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>03/2001</td>
<td>$^7\text{Li}/^6\text{Li}$</td>
<td>$T = 0.25\varepsilon_F$</td>
<td>$N_F \sim 10^5$</td>
<td>Rice Univ., Houston/Texas, A.G. Truscott,..., R.G. Hulet</td>
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<td>07/2001</td>
<td>$^7\text{Li}/^6\text{Li}$</td>
<td>$T = 0.2\varepsilon_F$</td>
<td>$N_F \sim 10^4$</td>
<td>ENS, Paris F. Schreck,..., C. Salomon</td>
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<tr>
<td>08/2001</td>
<td>$^{87}\text{Rb}/^{40}\text{K}$</td>
<td>—</td>
<td>$N_F \sim 10^7$</td>
<td>JILA, Boulder/Colorado J. Goldwin,..., D.S. Jin</td>
</tr>
<tr>
<td>12/2001</td>
<td>$^{23}\text{Na}/^6\text{Li}$</td>
<td>$T = 0.5\varepsilon_F$</td>
<td>$N_F \sim 10^6$</td>
<td>MIT, Cambridge/Massachusetts Z. Hadzibabic,..., W. Ketterle</td>
</tr>
</tbody>
</table>
Theoretical Description of Trapped Degenerate (Fermi) Gases

- The Many-Body Problem
- Correlations & Effective Interaction
- Mean-Field & Thomas-Fermi Approximation
- Energy Functional
Route Through the Many-Body Problem

**Hamiltonian**

\[ H = \sum_i U(\vec{x}_i) + \frac{1}{2m} \sum_i \vec{p}_i^2 + \sum_{i<j} V_{ij} \]

**Model Space**

mean-field states: antisym. product of single-particle states

**Energy Functional**

energy expectation value as functional of the density

**Functional Variation**

ground state density is obtained by energy minimization

**Thomas-Fermi Approx.**

neglect all gradients of the density in the energy functional
Short-Range Correlations

**Interaction**
many realistic two-body interactions show a strong short-range repulsion
(e.g. atom-atom or nucleon-nucleon interactions)

**Correlations**
core induces strong short-range correlations in many-body state
(e.g. correlation hole in two-body density)

**Product States**
short-range correlations cannot be described by product-type states
(e.g. mean-field, superposition of few product states, ...)

---

[v(r)]

\[
v(r) = \begin{cases} 
1 & \text{for } r < r_0 \\
0 & \text{for } r \geq r_0 
\end{cases}
\]

\[
\rho^{(2)}(r) = \begin{cases} 
\rho_0 & \text{for } r < r_0 \\
0 & \text{for } r \geq r_0 
\end{cases}
\]

\[
\rho^{(2)}_{\text{prod}}(r) = \begin{cases} 
\rho_0 & \text{for } r < r_0 \\
0 & \text{for } r \geq r_0 
\end{cases}
\]

---

nuclear matter \( \rho_0 = 0.17 \text{ fm}^{-3} \)
liquid \(^4\text{He} \) (bosonic) \( \rho_0 = 0.022 \text{ Å}^{-3} \)
The Problem

Short-Range Correlations

Interaction
many realistic two-body interactions show a strong short-range repulsion
(e.g. atom-atom or nucleon-nucleon interactions)

Correlations
core induces strong short-range correlations in many-body state
(e.g. correlation hole in two-body density)

Product States
short-range correlations cannot be described by product-type states
(e.g. mean-field, superposition of few product states, ...)

Effective Interaction
replace the full potential by a tamed effective interaction

Effective Contact Interaction

Correlated States
include correlations in many-body model-space
A Suitable Effective Interaction...

- System is very **dilute** and **cold**
  - $\rho^{-1/3} \gg$ range of interaction
  - $q^{-1} \gg$ range of interaction

- Treat the many-body problem in a restricted **model-space** that does not contain correlations

- Looking for the structure of **non-selfbound states** in an external potential

- **Hermitean** interaction operator that obeys standard symmetries (translation, rotation,...)

**Effective Contact Interaction (ECI)**

- Zero-range potential (for each partial wave)
- Expectation value in two-body model-states equals the energy shift induced by the full interaction

$$\langle \phi_n^{\text{mod}} | v^{\text{ECI}} | \phi_n^{\text{mod}} \rangle \overset{!}{=} \Delta E_n$$

<table>
<thead>
<tr>
<th>$E_n$</th>
<th>without interaction</th>
<th>with interaction</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\Delta E_n$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
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<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(project out bound states)
Energy Shift

- relative two-body wave function w/o and with interaction (outside the range of \( v(r) \))
  \[
  \phi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\vartheta, \varphi)
  \]
  \[
  R_{nl}(r) \propto j_l(q_{nl}r)
  \]
  \[
  \bar{R}_{nl}(r) \propto j_l(\bar{q}_{nl}r) - \tan \eta_l(\bar{q}_{nl}) n_l(\bar{q}_{nl}r)
  \]
- auxiliary boundary condition \( R_{nl}(\Lambda) = 0 \) to obtain discrete momentum spectrum
  \[
  q_{nl} \Lambda = \pi (n + \frac{l}{2})
  \]
  \[
  \bar{q}_{nl} \Lambda = \pi (n + \frac{l}{2}) - \eta_l(\bar{q}_{nl}) - \pi n_l^{\text{bound}}
  \]
- momentum shift
  \[
  \Delta q_{nl} \Lambda = (\bar{q}_{nl} - q_{nl}) \Lambda
  \]
  \[
  = -[\eta_l(q_{nl}) - \pi n_l^{\text{bound}}] =: -\hat{\eta}_l(q_{nl})
  \]
- relative energy shift
  \[
  \frac{\Delta E_{nl}}{E_{nl}} = -\frac{2}{q_{nl} \Lambda} \hat{\eta}_l(q_{nl})
  \]

Interaction Operator

- ansatz for a nonlocal contact interaction for the \( l \)th partial wave
  \[
  \mathbf{v}_l^{\text{ECI}} = (\vec{q} \cdot \vec{r})^l g_l \delta^{(3)}(\vec{r}) (\vec{\bar{r}} \cdot \vec{q})^l
  \]
  \[
  = \int d^3 r |\vec{r}\rangle \frac{\partial^l}{\partial r^l} g_l \delta^{(3)}(\vec{r}) \frac{\partial^l}{\partial \bar{r}^l} |\vec{\bar{r}}\rangle
  \]
- expectation value in non-interacting two-body states
  \[
  \left\langle \phi_{nlm} | \mathbf{v}_l^{\text{ECI}} | \phi_{nlm} \right\rangle = \frac{1}{\Delta E_{nl}}
  \]
- interaction strengths \( g_l \) determined by \( \hat{\eta}_l(q) \)
  \[
  g_l = -\frac{4\pi}{2m_{\text{red}}} \left[ \frac{(2l + 1)!!}{l!} \right]^2 \hat{\eta}_l(q) q^{2l+1}
  \]
- parametrization of \( \hat{\eta}_l(q) \) in terms of the scattering lengths \( a_l \) for \( |qa_l| \ll 1 \)
  \[
  g_l = \frac{4\pi}{2m_{\text{red}}} \frac{(2l + 1)}{(l!)^2} a_l^{2l+1} + O(q^2)
  \]
A Model for a Trapped Degenerate Fermi Gas

- trapped gas of $\Xi$ distinguishable fermionic species ($\xi = 1, \ldots, \Xi$) interacting via the s- and p-wave contact interaction
- for simplicity: equal trapping potentials and s- and p-wave scattering lengths, $a_0$ and $a_1$, for all components

**Hamiltonian**

$$H = \sum_i U(\vec{x}_i) + \frac{1}{2m} \sum_i \vec{p}_i^2 + \frac{4\pi a_0}{m} \sum_{i<j} \delta^{(3)}(\vec{r}_{ij}) + \frac{12\pi a_1^3}{m} \sum_{i<j} \left( \vec{q}_{ij} \cdot \frac{\vec{r}_{ij}}{r_{ij}} \right) \delta^{(3)}(\vec{r}_{ij}) \left( \frac{\vec{r}_{ij}}{r_{ij}} \cdot \vec{q}_{ij} \right)$$

- trap kinetic s-wave p-wave

**Mean-Field States (homogeneous)**

- $N$-body state $|\Psi\rangle$ is an antisymmetrized product of single-particle momentum eigenstates $|\vec{k}_i, \xi_i\rangle$
  $$|\Psi\rangle = \mathcal{A} \left( |\vec{k}_1, \xi_1\rangle \otimes \cdots \otimes |\vec{k}_N, \xi_N\rangle \right)$$
- for each component $\xi$ all momenta $|\vec{k}|$ up to the Fermi momentum $\kappa_{i\xi}$ appear

**Thomas-Fermi Approximation**

- energy density of the trapped gas is locally given by the energy density of the homogeneous system
  $$\mathcal{E}_{\text{hom}}(\kappa_1, \ldots, \kappa_\Xi) = \frac{1}{V} \langle \Psi | H_{\text{hom}} | \Psi \rangle$$
- i.e. the Fermi momenta $\kappa_{i\xi}$ are replaced by local Fermi momenta $\kappa_{i\xi}(\vec{x})$
Energy-Density for Trapped Fermions

**Single-Component System**

$$\mathcal{E}_1[\kappa](\vec{x}) =$$

$$= \frac{1}{6\pi^2} U(\vec{x}) \kappa^3(\vec{x})$$

$$+ \frac{1}{20\pi^2 m} \kappa^5(\vec{x})$$

$$+ \frac{a_1^3}{30\pi^3 m} \kappa^8(\vec{x})$$

- trap
- kinetic
- s-wave
- p-wave

**Two-Component System**

$$\mathcal{E}_2[\kappa_1, \kappa_2](\vec{x}) =$$

$$= \frac{1}{6\pi^2} U(\vec{x}) \left[ \kappa_1^3(\vec{x}) + \kappa_2^3(\vec{x}) \right]$$

$$+ \frac{1}{20\pi^2 m} \left[ \kappa_1^5(\vec{x}) + \kappa_2^5(\vec{x}) \right]$$

$$+ \frac{a_0}{9\pi^3 m} \kappa_1^3(\vec{x}) \kappa_2^3(\vec{x})$$

$$+ \frac{a_1^3}{30\pi^3 m} \left[ \kappa_1^8(\vec{x}) + \kappa_2^8(\vec{x}) + \frac{1}{2} \kappa_1^3(\vec{x}) \kappa_2^5(\vec{x}) + \frac{1}{2} \kappa_1^5(\vec{x}) \kappa_2^3(\vec{x}) \right]$$

- energy expectation value

$$E_\Xi[\kappa_1, \ldots, \kappa_\Xi] = \int d^3x \, \mathcal{E}_\Xi[\kappa_1, \ldots, \kappa_\Xi](\vec{x})$$

- density

$$\rho_\xi(\vec{x}) = \frac{1}{6\pi^2} \kappa_\xi^3(\vec{x})$$

- particle number

$$N[\kappa_\xi] = \frac{1}{6\pi^2} \int d^3x \, \kappa_\xi^3(\vec{x})$$
Ground State — Variationally

Functional Variation

minimization of the energy \( E[\kappa_1, ..., \kappa_\Xi] \) for fixed numbers of particles \( N_1, ..., N_\Xi \) gives the ground state density profile

- **chemical potentials**: implement constraints on the particle numbers via a set of Lagrange multipliers \( \mu_1, ..., \mu_\Xi \)
- unconstraint minimization of the transformed energy functional

\[
F[\kappa_1, ..., \kappa_\Xi] = E[\kappa_1, ..., \kappa_\Xi] - \sum_{\xi=1}^{\Xi} \mu_\xi N[\kappa_\xi]
\]

\[
= \int d^3x \ F[\kappa_1, ..., \kappa_\Xi](\vec{x})
\]

- stationary points of the energy density are solutions of the Euler-Lagrange equations

\[
\frac{\partial}{\partial \kappa_\xi(\vec{x})} F[\kappa_1, ..., \kappa_\Xi](\vec{x}) = 0, \quad \forall \xi
\]

- since \( F \) is local (does not depend on gradients) the ground state has to minimize \( F \) for each \( \vec{x} \)

Recipe

ground-state densities at some \( \vec{x} \) are given by the minimum of the transformed energy density \( F[\kappa_1, ..., \kappa_\Xi](\vec{x}) \) for this \( \vec{x} \)
Structure of a Trapped Degenerate Two-Component Fermi Gas

- Energy Landscapes & Density Profiles
- Mean-Field Induced Collapse
- Component Separation
- Phase Diagram
assume a spherical symmetric parabolic trapping potential

\[ U(\vec{x}) = \frac{m\omega^2}{2} x^2 = \frac{1}{2m\ell^4} x^2 \]

determine the densities for \( \mu_1, \mu_2 \) chosen such that the desired particle numbers are reproduced

- \( a_0 > 0 \): repulsive interactions flatten the density profile
- \( a_0 < 0 \): attractive interactions enhance the central density
- outside a certain range of scattering lengths \( a_0 \) no solutions of this type exist anymore

for a typical trap with \( \ell = 1 \mu m \):

\[ a_0 = 200 a_{\text{Bohr}} \rightarrow a_0/\ell = 0.01 \]
\[ a_0 = 2000 a_{\text{Bohr}} \rightarrow a_0/\ell = 0.1 \]
Two-Component Fermi Gas

Energy-Density Landscape: $a_0 < 0$

- minimum of $F_2$ is only local for attractive interactions ($a_0 < 0$ or $a_1 < 0$)
- NB: physically the state is metastable for all signs of the scattering lengths
- local minimum vanishes if the attractive s-wave interaction exceeds a critical strength

attractive interactions can induce a collapse of the Fermi gas towards high densities
Two-Component Fermi Gas

Collapse — Critical Particle Number

- Assume parabolic trapping potential with mean oscillator length $\ell$
- Obtain the density profile for the critical chemical potential $\mu_{cr}$ and calculate $N_{cr}$

Abs. stabilization due to p-wave repulsion $a_1/|a_0| > 2/(3\pi^{2/3})$

P-wave attraction lowers critical particle number substantially

P-wave induced collapse and interference with separation

\[
\begin{align*}
\text{Log}_{10} N_{cr} & \quad a_0/\ell \\
-0.12 & \quad +0.030 \\
-0.1 & \quad +0.025 \\
-0.08 & \quad +0.020 \\
-0.06 & \quad +0.015 \\
-0.04 & \quad +0.010 \\
-0.02 & \quad +0.000 \\
0 & \quad -0.010 \\
0.02 & \quad -0.020 \\
0.04 & \quad -0.030 \\
0.06 & \quad -0.040 \\
0.08 & \quad -0.050
\end{align*}
\]
Two-Component Fermi Gas

Energy-Density Landscape: $a_0 > 0$

- **overlapping configuration:** for moderate repulsive s-wave interactions a unique minimum exists at $\kappa_1 = \kappa_2$

- **separation:** beyond a critical interaction strength two separate minima emerge at $\kappa_1 = 0$, $\kappa_2 > 0$ and $\kappa_1 > 0$, $\kappa_2 = 0$
Two-Component Fermi Gas

Separation — Density Distributions

\[ \rho_1(r, z) = \rho_2(r, -z) \]

\(a_0/\ell = 0\)

\(a_0/\ell = 0.06\)

\(a_0/\ell = 0.066\)

\(N_1 = N_2 = 10^7\)

\(a_1/\ell = 0\)

\(a_0/\ell = 0.07\)

\(a_0/\ell = 0.08\)

\(a_0/\ell = 0.10\)
Two-Component Fermi Gas
Separation — Critical Particle Number

- assume parabolic trapping potential with mean oscillator length $\ell$
- obtain the density profile for the critical chemical potential $\mu_{cr}$ and calculate $N_{cr}$

interference with collapse induced by $p$-wave attraction

$p$-wave attraction lowers critical particle number substantially

abs. stabilization due to $p$-wave repulsion $a_1/a_0 > 2^{4/3}/(3\pi^{2/3})$

![Graph showing $\log_{10} N_{cr}$ versus $a_0/\ell$ for different values of $a_1/\ell$.]
Two-Component Fermi Gas

Stability Map

overlapping conf. is stable for all particle numbers

p-wave stabilized high-density phase above $N_{cr}$

mean-field collapse above critical particle number

components separate above critical particle number

$\log_{10} N_{cr}$
• **Feshbach resonances** allow to tune the strength of the atom-atom interaction (scattering lengths) via an external magnetic field.

simultaneous s- and p-wave Feshbach resonance predicted for a two-component $^{40}\text{K}$ system with $F = \frac{9}{2}$, $m_F = -\frac{9}{2}, -\frac{7}{2}$

Summary

Strategy

- developed a simple framework to describe interacting degenerate quantum gases
- effective contact interaction + mean-field states + Thomas-Fermi approximation → energy functional
- investigated the influence of s- and p-wave interactions on structure and stability of degenerate Fermi gases

Results

- s- and p-wave interactions have strong influence on the density profiles and the stability of the gas
- **collapse**: attractive interactions can induce a collapse of the dilute gas towards high densities
- **separation**: repulsive interactions can cause a spatial separation of the different components
- in all cases a complex interplay between s- and p-wave interactions is observed

...have a look at [http://theory.gsi.de/~trap](http://theory.gsi.de/~trap)