Towards an effective relativistic density functional for dense matter in supernovae and compact stars

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Outline

• Introduction
  Astrophysics and Equation of State, Nuclear and Stellar Matter, Constraints, Correlations, Relativistic Density Functional

• Nuclear Correlations in Matter
  Generalized Relativistic Density Functional, Light and Heavy Clusters, Low-Density Limit, Scattering Correlations, Neutron Matter

• Coulomb Correlations in Matter
  Coulomb Interaction in Matter, One-Component Plasma, Gas/Liquid Phase, Solid Phase

• Summary
Introduction
Astrophysics and Equation of State

- essential ingredient in astrophysical model calculations:

  **Equation(s) of State (EoS) of dense matter**

  ⇒ dynamical evolution of **supernovae**
  ⇒ static properties of **neutron stars**
  ⇒ conditions for **nucleosynthesis**
  ⇒ energetics, **chemical composition**, transport properties, . . .
Astrophysics and Equation of State

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  **Equation(s) of State (EoS) of dense matter**
  
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- **timescale of reactions ≪ timescale of system evolution**

  ⇒ **equilibrium** (thermal, chemical, . . .)
  
  ⇒ application of **EoS** reasonable
EoS Parameters

standard choice:

- **density**:  
  \[10^{-9} \lesssim \frac{\rho}{\rho_{\text{sat}}} \lesssim 10\]  
  with nuclear saturation density  
  \[\rho_{\text{sat}} \approx 2.5 \cdot 10^{14} \text{ g/cm}^3\]  
  \[(n_{\text{sat}} = \frac{\rho_{\text{sat}}}{m_n} \approx 0.15 \text{ fm}^{-3})\]

- **temperature**:  
  \[0 \text{ MeV} \leq k_B T \lesssim 50 \text{ MeV}\]  
  \[(\approx 5.8 \cdot 10^{11} \text{ K})\]

- **electron fraction**:  
  \[0 \leq Y_e \lesssim 0.6\]

sometimes other choices
more appropriate:
e.g. crust of neutron stars  
(density → pressure)
EoS Constituents

most relevant particles: (at low temperatures and not too high densities)

- neutrons, protons
- nuclei
- electrons, (muons) (charge neutrality!)
- neutrinos (often not in equilibrium, treated independently of EoS)

more particles under extreme conditions: e.g. high densities, high temperatures (hyperons, mesons, . . . )
EoS for Astrophysical Applications

- many EoS developed in the past:
  from simple parametizations to sophisticated models
- many investigations of detailed aspects:
  often restricted to particular conditions
⇒ only few realistic global EoS used in astrophysical simulations
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  covering of full parameter space in a single model
⇒ combination of different features/approaches required
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  - effect of correlations
    ⇒ formation and dissolution of clusters
    ⇒ phase transition: gas/liquid ↔ solid
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- important distinction:
  nuclear matter ↔ stellar matter
  ⇒ very different systems
Nuclear Matter

- only strongly interacting particles
- no electromagnetic interaction, no charge neutrality
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- many-body correlations due to short-range nuclear interaction
  ⇒ clustering ⇒ liquid-gas phase transition in thermodynamic limit
  ⇒ balance attraction ↔ repulsion ⇒ feature of saturation
- characteristic nuclear matter parameters $\rho_{\text{sat}}$, $E_{\text{sat}}/A$, $K$, $J$, $L$, ...
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\begin{itemize}
  \item “non-congruent” phase transition
\end{itemize}
Stellar Matter

- both hadrons and leptons
- strong and electromagnetic interaction
- condition: charge neutrality
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  - formation of inhomogeneous matter and finite-size structures
  - clustering ⇒ new particle species (nuclei) ⇒ change of chemical composition
  - lattice formation ⇒ phase transition: liquid/gas ↔ solid
  - “pasta phases”
  - modification of thermodynamic properties
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aim:
- consider these (and more) features by extending
  relativistic mean-field (RMF) model for nuclei
- theoretical formulation as “density functional”
  with well-constrained parameters
nuclear physics

- nuclei (binding energy, radii, charge formfactor, spin-orbit splittings, ...)

Constraints

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  - heavy-ion collisions (flow, particle production, fragment yields, . . .)
- **astrophysics**
  - compact stars (static properties, cooling, . . .)

Correlations

• interacting many-body system
  ⇒ information on correlations in spectral functions
Correlations

- interacting many-body system
  \[\Rightarrow\] information on correlations in spectral functions

- approximation: quasiparticles with self-energies
  - change of particle properties
  - reduction of residual correlations
  - definition of chemical composition?
  - extreme case: uncorrelated quasiparticles
Correlations

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- quasiparticle concept very successful in **nuclear physics**
  ⇒ **phenomenological mean-field models** (e.g. Skyrme, Gogny, relativistic)
  with only nucleons as degrees of freedom
Correlations

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- Low densities: clusters as new degrees of freedom
  ⇒ Benchmark: virial equation of state
  (see e.g. C. J. Horowitz, A. Schwenk, Nucl. Phys. A 776 (2006) 55)
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⇒ transition in unified model?
• **constituents**: nucleons $\Rightarrow \psi_i \ (i = n, p)$ Dirac spinors
Relativistic Density Functional

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- **interaction**:  
  - strong $\Rightarrow$ meson fields $A_m \ (m = \sigma, \omega, \rho$, convenient auxiliary fields$)$
  - electromagnetic $\Rightarrow A_\gamma$
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  - electromagnetic $\Rightarrow A_\gamma$
- **energy of nucleus**
  
  $$E = \int d^3r \varepsilon(\vec{r}) + E_{cm} + E_{pair} + \ldots$$

with energy density functional

$$
\varepsilon = \sum_i w_i \left[ t_i + (m_i - \Gamma_i \sigma A_\sigma) n_i^{(s)} + (\Gamma_{i\omega} A_\omega + \Gamma_{i\rho} A_\rho + \Gamma_{i\gamma} A_\gamma) n_i \right] + \frac{1}{2} \left( m_\sigma^2 A_\sigma^2 + \nabla A_\sigma \cdot \nabla A_\sigma - m_\omega^2 A_\omega^2 - \nabla A_\omega \cdot \nabla A_\omega - m_\rho^2 A_\rho^2 - \nabla A_\rho \cdot \nabla A_\rho - \nabla A_\gamma \cdot \nabla A_\gamma \right)
$$

- **single-particle densities** $t_i = \bar{\psi}_i \vec{\gamma} \cdot \vec{p} \psi_i$  
  $n_i^{(s)} = \bar{\psi}_i \psi_i$  
  $n_i = \bar{\psi}_i \gamma_0 \psi_i$
- **occupation numbers** $w_i$
Relativistic Density Functional

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\[ + \frac{1}{2} \left( m_\sigma^2 A_\sigma^2 + \vec{\nabla} A_\sigma \cdot \vec{\nabla} A_\sigma - m_\omega^2 A_\omega^2 - \vec{\nabla} A_\omega \cdot \vec{\nabla} A_\omega - m_\rho^2 A_\rho^2 - \vec{\nabla} A_\rho \cdot \vec{\nabla} A_\rho - \vec{\nabla} A_\gamma \cdot \vec{\nabla} A_\gamma \right) \]

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- **occupation numbers**
  \[ w_i \]

- **density dependent meson-nucleon couplings**
  \[ \Gamma_{im} = g_{im} \Gamma_m(q) \quad q = n_n + n_p \]
  ⇒ medium dependent interaction
  ⇒ rearrangement contributions to self-energies
  \[ \Gamma_{i\gamma} = Q_i \Gamma_\gamma \] with charge number $Q_i$
Nuclear Correlations in Matter
Theoretical Approaches

- ideal mixture of independent particles, no interaction
  ⇒ **Nuclear Statistical Equilibrium/Law of Mass Action**
  most simple approach, suppression of nuclei ⇒ **excluded volume mechanism**
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- mixture of interacting particles/correlations
  ⇒ **Virial Equation of State**
  model-independent low-density benchmark
Theoretical Approaches

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  ⇒ Virial Equation of State
  model-independent low-density benchmark

• considering medium effects with increasing density
  ⇒ Quantum Statistical/Generalized Beth-Uhlenbeck Approach
  correlations of quasiparticles with medium-dependent properties,
  microscopic origin of cluster dissolution/Mott effect (action of Pauli principle)
Theoretical Approaches

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• interpolation from low to high densities around nuclear saturation
  \[\Rightarrow \text{Generalized Relativistic Density Functional}\]
  correct limits, formation and dissolution of nuclei
• include **new degrees of freedom** with medium-dependent properties:
  ○ light nuclei (**deuteron**, **triton**, **helion**, **α-particle**)
  ○ nucleon-nucleon scattering correlations (**nn**, **pp**, **np channels**)
  ○ heavy nuclei (**$A > 4$**)
  ⇒ interaction via minimal coupling to mesons/photon with scaled strengths
Generalized Relativistic Density Functional

- include new degrees of freedom with medium-dependent properties:
  - light nuclei (deuteron, triton, helion, \( \alpha \)-particle)
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- model parameters
  - vacuum masses of nucleons, electrons, nuclei
  - effective resonance energies and degeneracy factors
  - density-dependent meson-nucleon/nucleus couplings, fitted to properties of atomic nuclei
  - medium-dependent mass shifts of clusters (bound and continuum states)

Details:
Light Nuclei

shift of binding energies/masses

- solve in-medium Schrödinger equation with realistic nucleon-nucleon potentials
- parametrization of shifts $\Delta m_i$
- main effect: Pauli principle
  $\Rightarrow$ blocking of states in the medium!
Light Nuclei

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- example: symmetric nuclear matter, nuclei at rest in medium
- in vacuum: experimental binding energies
- nuclei become unbound ($B_i < 0$) with increasing density of medium
- dissolution of clusters at high densities $\Rightarrow$ Mott effect
inhomogeneous matter at low densities

- comparison with uniform matter
  \[ \Rightarrow \text{increase in binding energy} \]
inhomogeneous matter at low densities

- comparison with uniform matter
  ⇒ increase in binding energy
- spherical Wigner-Seitz cell calculation
  - generalized rel. density functional
  - extended Thomas-Fermi approximation
  - electrons for charge compensation
  - heavy nucleus surrounded by gas of nucleons
- self-consistent calculation with interacting nucleons, electrons

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graph showing particle number density over radius.}
\end{figure}

\begin{align*}
A_{\text{heavy}} &= 147.1 \\
Z_{\text{heavy}} &= 62.3
\end{align*}

T = 5 \text{ MeV} \\
n = 0.01 \text{ fm}^{-3} \\
Y_p = 0.4
inhomogeneous matter at low densities

- comparison with uniform matter
  ⇒ increase in binding energy
- spherical Wigner-Seitz cell calculation
  - generalized rel. density functional
  - extended Thomas-Fermi approximation
  - electrons for charge compensation
  - heavy nucleus surrounded by gas of nucleons and light clusters
- self-consistent calculation with interacting nucleons, electrons and light nuclei
- increased probability of finding light clusters at surface of heavy nucleus
Heavy Nuclei II

- traditional approach in EoS tables:
  - single-nucleus approximation (SNA)
    (one representative heavy nucleus)
  - no distribution of nuclei
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  - full table of nuclei included (c.f. NSE calculations)
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  - medium-dependent shift of binding energies from SNA
- medium effects:
  - relative stabilization of heavier and exotic nuclei
  - dissolution of nuclei depending on density, temperature, np-asymmetry
- parametrization of mass shifts $\Delta m_i$,
  only preliminary results

AME2011: G. Audi, W. Meng (private communication)
Low-Density Limit I

- only two-body correlations relevant

- comparison of generalized relativistic density functional with virial Equation of State (model-independent benchmark, depends only on experimental binding energies and phase shifts $\delta^{(ij)}_l$)
Low-Density Limit I

- only **two-body correlations** relevant

- **comparison** of generalized relativistic density functional with virial Equation of State (model-independent benchmark, depends only on experimental binding energies and phase shifts $\delta_{(ij)}$)

- **fugacity expansion** of thermodynamic potential $\Omega$
  \[ C_m = \frac{\Gamma_m^2}{m_m^2} \quad (m = \omega, \sigma, \rho, \delta) \]
  ⇒ consistency relations with *virial coefficients* and zero-density meson-nucleon couplings
  \[ E_{ij}(T) \quad (i, j = n, p) \]
  representing NN scattering correlations
  ⇒ **effective degeneracy factors** $g_{ij}^{(\text{eff})}(T)$
  (cf. treatment of excited states of nuclei)
  ⇒ relativistic corrections
• zero temperature limit of consistency relations without scattering correlations

\[ C_\omega - C_\sigma = \frac{\pi}{2m} \left[ a_{nn}(^1S_0) + a_{pp}(^1S_0) + a_{np}(^1S_0) + 3a_{np}(^3S_1) \right] \]

\[ C_\rho - C_\delta = \frac{\pi}{2m} \left[ a_{nn}(^1S_0) + a_{pp}(^1S_0) - a_{np}(^1S_0) - 3a_{np}(^3S_1) \right] \]

with scattering lengths \( a_{ij} \) and assuming \( m = m_n = m_p \)
Low-Density Limit II

- zero temperature limit of consistency relations without scattering correlations
  
  \[ C_\omega - C_\sigma = \frac{\pi}{2m} \left[ a_{nn}(^1S_0) + a_{pp}(^1S_0) + a_{np}(^1S_0) + 3a_{np}(^3S_1) \right] \]
  
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- comparison of experiment with RMF parametrizations

<table>
<thead>
<tr>
<th></th>
<th>exp.</th>
<th>DD2 [1]</th>
<th>DD-ME(\delta) [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_\omega - C_\sigma ) [fm(^2)]</td>
<td>-14.15</td>
<td>-5.39</td>
<td>-4.90</td>
</tr>
<tr>
<td>( C_\rho - C_\delta ) [fm(^2)]</td>
<td>-9.61</td>
<td>2.48</td>
<td>2.55</td>
</tr>
</tbody>
</table>


⇒ conventional mean-field models don’t reproduce effect of correlations at very-low densities
⇒ explicit scattering correlations needed
**NN Scattering Correlations**

- **Effective resonance energies**

\[
\sum_l g_l^{(ij)} \int \frac{dE}{\pi} \frac{d\delta_l^{(ij)}}{dE} \exp \left( -\frac{E}{T} \right) = \pm g_0^{(ij)} \exp \left( -\frac{E_{ij}}{T} \right)
\]

**Effective-range expansion** for s-wave phase shifts:

\[
k \cot \delta_0^{(ij)} = -\frac{1}{a_{ij}} + \frac{r_{ij} k^2}{2}
\]

⇒ analytical results

low \( T \):

\[
I_0^{(ij)}(T) \rightarrow -a_{ij} \sqrt{\mu_{ij} T/(2\pi)}
\]

unitary limit: \( E_{ij}(T) = T \ln 2 \)
NN Scattering Correlations

- effective resonance energies

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- effective degeneracy factors

\[ \sum_l g_l^{(nn)} \int \frac{dE}{\pi} \frac{d\delta_l^{(nn)}}{dE} \exp \left( -\frac{E}{T} \right) = g_{nn}^{(eff)} \exp \left( -\frac{E_{nn}}{T} \right) - g_n \frac{\lambda_{nn}^3}{\lambda_n^6} \frac{C_+}{2T} \]

\[ C_+ = C_\omega - C_\sigma + C_\rho - C_\delta \]

\[ \lambda_i = \sqrt{2\pi/(m_i T)} \]
Neutron Matter at Low Densities I

**comparison:** different effects

- nonrelativistic ideal gas

![Graph showing internal energy per nucleon vs. density](image)

**internal energy per nucleon** \( E/A \)

(ideal gas: \( E/A = 3T/2 \))

- T = 10 MeV
**comparison:** different effects

- nonrelativistic ideal gas
  \[ \downarrow \text{rel. kinematics + quantum statistics} \]
- relativistic Fermi gas

internal energy per nucleon \( E/A \)

(ideal gas: \( E/A = 3T/2 \))

![Graph showing internal energy per nucleon as a function of density for different models.](image)

- ideal gas
- relativistic Fermi gas
**comparison:** different effects
- nonrelativistic ideal gas
  ↓ rel. kinematics + quantum statistics
- relativistic Fermi gas
  ↓ two-body correlations
- virial EoS with relativistic correction

internal energy per nucleon \( E/A \)
(ideal gas: \( E/A = 3T/2 \))

![Diagram showing internal energy per nucleon vs density](image-url)
Neutron Matter at Low Densities I

**comparison:** different effects

- **nonrelativistic ideal gas**
  ▼ rel. kinematics + quantum statistics
- **relativistic Fermi gas**
  ▼ two-body correlations
- **virial EoS with relativistic correction**
  (not included in standard virial EoS)
  ▼ mean-field effects
- **standard RMF model with density dependent couplings**

internal energy per nucleon $E/A$
(ideal gas: $E/A = 3T/2$)

![Graph showing internal energy per nucleon $E/A$ vs. density $n$ at $T = 10$ MeV, with lines for different models: ideal gas, relativistic Fermi gas, relativistic virial EoS, and standard RMF.](graph.png)
Neutron Matter at Low Densities I

comparison: different effects

- nonrelativistic ideal gas
  ↓ rel. kinematics + quantum statistics
- relativistic Fermi gas
  ↓ two-body correlations
- virial EoS with relativistic correction
  (not included in standard virial EoS)
  ↓ mean-field effects
- standard RMF model with
density dependent couplings
  ↓ two-body correlations
- generalized relativistic density
  functional (gRDF) with contributions
  from nn scattering

internal energy per nucleon $E/A$
(ideal gas: $E/A = 3T/2$)

![Graph showing internal energy per nucleon vs density for different models: ideal gas, relativistic Fermi gas, relativistic virial EoS, standard RMF, and gRDF. The graph compares the energy at $T = 10$ MeV across various densities.](image)
comparison: $p/n$ in different models (ideal gas: $p/n = T$)

Light Clusters and Continuum Correlations

- particle fractions
  \[ X_i = A_i \frac{n_i}{n_b} \quad n_b = \sum_i A_i n_i \]

- low densities:
  two-body correlations most important

- high densities:
  dissolution of clusters
  \( \Rightarrow \) Mott effect

generalized relativistic density functional

\( T = 10 \text{ MeV} \)
\( Y_p = 0.4 \)

(without heavy clusters)
Light Clusters and Continuum Correlations

- particle fractions
  \[ X_i = A_i \frac{n_i}{n_b} \quad n_b = \sum_i A_i n_i \]

- low densities:
  two-body correlations most important

- high densities:
  dissolution of clusters
  \( \Rightarrow \) Mott effect

- effect of NN continuum correlations
  - dashed lines: without continuum
  - solid lines: with continuum
  \( \Rightarrow \) reduction of deuteron fraction,
    redistribution of other particles

- correct limits with generalized relativistic density functional

(Without heavy clusters)
Coulomb Correlations in Matter
Coulomb Interaction in Matter

- explicit potential $A_\gamma$ only in systems with spatially inhomogeneous charge distribution, homogeneous approaches for EoS $\Rightarrow$ effective treatment of Coulomb effects
Coulomb Interaction in Matter

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- crystal:
  lattice-periodic Coulomb potential $\rightarrow$ potential in Wigner-Seitz approximation:
  single nucleus and electron background in spherical cell with size such that total charge vanishes $\Rightarrow$ screening of Coulomb potential
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- analytical solution for homogeneously charged sphere (ion, radius $R$, charge $Q_e$) and constant electron density $n_e = 3/(4\pi R^3) = Q n_{ion}$

$\Rightarrow$ Coulomb energy $E_C^{(WS)} = E_C^{(sph)} + \Delta E_C^{(WS)}$
Coulomb Interaction in Matter

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  $\Rightarrow$ Coulomb energy $E_C^{(WS)} = E_C^{(sph)} + \Delta E_C^{(WS)}$

  with

  $E_C^{(sph)} = \frac{3}{5} \frac{Q^2 e^2}{R}$ part of energy of nucleus

  $\Delta E_C^{(WS)} = -\frac{9}{10} \frac{Q^2 e^2}{R_e} \left(1 - \frac{R^2}{3R_e^2}\right)$ energy shift with finite-size correction

  $\Rightarrow$ approximation for lattice Coulomb energy, often applied in EoS models in liquid phase (?)
One-Component Plasma (OCP) I

- $N$ ions (point particles, charge $Q_e > 0$) in homogeneous background of electrons (density $n_e = 3/(4\pi a_e^3)$) at temperature $T$
One-Component Plasma (OCP) I

- *N ions* (point particles, charge $Qe > 0$) in homogeneous background of *electrons* (density $n_e = 3/(4\pi a_e^3)$) at temperature $T$
- *classical model with screened Coulomb interaction* between ions (calculation: Ewald method)
- *internal energy* of ions: $U_{\text{ion}} = U_{\text{kin}} + U_{\text{pot}}$ with $U_{\text{kin}} = \frac{3}{2}NT$
One-Component Plasma (OCP) I

- \( N \) ions (point particles, charge \( Qe > 0 \)) in homogeneous background of electrons (density \( n_e = 3/(4\pi a_e^3) \)) at temperature \( T \)
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- Monte Carlo simulation, only one relevant parameter \( \Gamma = \frac{Q^2e^2}{a_eT} \) for \( U_{\text{pot}}/(NT) \)
One-Component Plasma (OCP) I

- $N$ ions (point particles, charge $Qe > 0$) in homogeneous background of electrons (density $n_e = 3/(4\pi a_e^3)$) at temperature $T$
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- Monte Carlo simulation, only one relevant parameter $\Gamma = \frac{Q^2e^2}{a_eT}$ for $U_{\text{pot}}/(NT)$
- example: 1024 ions in $8 \times 8 \times 8$ bcc lattice
One-Component Plasma (OCP) II

- limits:
  \[ \Gamma \to 0 : \text{liquid phase} \]
  \[ U^{(L)}_{\text{pot}}/(NT) \to -\frac{\sqrt{3}}{2} \Gamma^{3/2} \]
  (Debye-Hückel)

  \[ \Gamma \to \infty : \text{solid phase} \]
  \[ U^{(S)}_{\text{pot}}/(NT) \to \frac{3}{2} + C_M \Gamma \]
  \( C_M^{(bcc)} = -0.895929255682 \) Madelung constant)
• limits:
  \[ \Gamma \to 0 : \text{liquid phase} \quad U_{\text{pot}}^{(L)}/(NT) \to -\frac{\sqrt{3}}{2} \Gamma^{3/2} \] (Debye-Hückel)
  \[ \Gamma \to \infty : \text{solid phase} \quad U_{\text{pot}}^{(S)}/(NT) \to \frac{3}{2} + C_M \Gamma \]  
  \( (C_M^{(bcc)} = -0.895929255682 \text{ Madelung constant}) \)
• parametrization of Monte Carlo results

One-Component Plasma (OCP) II

- **limits:**
  - $\Gamma \to 0$: liquid phase
    \[ U^{(L)}_{\text{pot}}/(NT) \to -\frac{\sqrt{3}}{2} \Gamma^{3/2} \]
  (Debye-Hückel)
  - $\Gamma \to \infty$: solid phase
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  ($C_M^{(\text{bcc})} = -0.895929255682$ Madelung constant)

- **parametrization** of Monte Carlo results

- **free energies:** $F^{(L)}_{\text{pot}}, F^{(S)}_{\text{pot}}$ from integration

  \[ \frac{F^{(L)}_{\text{pot}}(\Gamma)}{NT} = \int_0^\Gamma d\Gamma' \frac{U_{\text{pot}}(\Gamma')}{NT} \quad \frac{F^{(S)}_{\text{pot}}(\Gamma)}{NT} = \ldots \]

  $\Rightarrow F^{(L)}, F^{(S)}$ (integration constants!)

One-Component Plasma (OCP) II

- **limits:**
  - $\Gamma \to 0$ : liquid phase \quad $U_{pot}^{(L)}/(NT) \to -\sqrt{3/2} \Gamma^{3/2}$ (Debye-Hückel)
  - $\Gamma \to \infty$ : solid phase \quad $U_{pot}^{(S)}/(NT) \to \frac{3}{2} + C_M \Gamma$

  $(C_M^{(bcc)} = -0.895929255682$ Madelung constant)

- **parametrization** of Monte Carlo results

- **free energies**: $F_{pot}^{(L)}$, $F_{pot}^{(S)}$ from integration

  \[
  \frac{F_{pot}^{(L)}(\Gamma)}{NT} = \int_0^\Gamma d\Gamma' \frac{U_{pot}(\Gamma')}{NT} \quad \frac{F_{pot}^{(S)}(\Gamma)}{NT} = \ldots
  \]

  $\Rightarrow F^{(L)}$, $F^{(S)}$ (integration constants !)

- **melting point**: $F^{(L)}(\Gamma_m) = F^{(S)}(\Gamma_m)$

  $\Rightarrow \Gamma_m \approx 175$

  - very sensitive to Coulomb correlations
  - Wigner-Seitz approximation fails

Gas/Liquid Phase I

constituents \((i)\):
- baryons \((n, p, \Lambda, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^-, \ldots) \Rightarrow\) fermions \((\sigma_i = +1)\)
- mesons \((\pi^+/\pi^-, \pi^0, K^+/K^-, K^0/\bar{K}^0, \omega, \rho, \ldots) \Rightarrow\) bosons \((\sigma_i = -1)\)
- light nuclei \((^2\text{H}, ^3\text{H}, ^3\text{He}, ^4\text{He}) \Rightarrow\) fermions/bosons
- heavy nuclei \((^A_iZ_i), \text{NN scattering correlations} \Rightarrow\) classical particles \((\sigma_i = 0)\)
- leptons \((e^-/e^+, \mu^-/\mu^+, \nu_e/\bar{\nu}_e, \nu_\mu/\bar{\nu}_\mu, \ldots) \Rightarrow\) fermions
- photons \((\gamma) \Rightarrow\) bosons
Gas/Liquid Phase I

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- baryons \((n, p, \Lambda, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^-, \ldots)\) \(\Rightarrow\) fermions \((\sigma_i = +1)\)
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- photons \((\gamma)\) \(\Rightarrow\) bosons

- consider particles \((\eta_i = +1)\) and antiparticles \((\eta_i = −1)\)
- degeneracy factors \(g_i\)
- distinguish individual constituents \((g_i = \text{const.}, i \in \mathcal{I})\)
  and effective constituents \((g_i(T, n_j), i \in \mathcal{E})\)
Gas/Liquid Phase I

constituents $(i)$:
- baryons $(n, p, \Lambda, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^-, \ldots) \Rightarrow$ fermions ($\sigma_i = +1$)
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- consider particles ($\eta_i = +1$) and antiparticles ($\eta_i = -1$)
- degeneracy factors $g_i$
- distinguish individual constituents ($g_i = \text{const.}, i \in I$) and effective constituents ($g_i(T, n_j), i \in E$)
- quasi-particles with relativistic energy

$$e_i^{(\eta_i)}(k) = \sqrt{k^2 + (m_i - S_i)^2} + \eta_i V_i$$

$S_i$ scalar potential, $V_i$ vector potential, $m_i$ rest mass in vacuum, $k$ momentum
interaction

- Lorentz scalar mesons \( m \in S = \{\sigma, \delta, \sigma_\star, \ldots\} \)
- Lorentz vector mesons \( m \in V = \{\omega, \rho, \phi, \ldots\} \)
Gas/Liquid Phase II

interaction

• Lorentz scalar mesons $m \in S = \{\sigma, \delta, \sigma_*, \ldots\}$
• Lorentz vector mesons $m \in V = \{\omega, \rho, \phi, \ldots\}$

○ represented by (classical) fields $A_m$ with mass $m_m$
○ coupling to constituents: $\Gamma_{im} = g_{im}\Gamma_m$
  with scaling factors $g_{im}$ and density dependent $\Gamma_m = \Gamma_m(\varrho)$, $\varrho = \sum_i B_i n_i$
Interaction

- Lorentz scalar mesons $m \in S = \{\sigma, \delta, \sigma_*, \ldots\}$
- Lorentz vector mesons $m \in V = \{\omega, \rho, \phi, \ldots\}$

- Represented by (classical) fields $A_m$ with mass $m_m$
- Coupling to constituents: $\Gamma_{im} = g_{im} \Gamma_m$
  with scaling factors $g_{im}$ and density dependent $\Gamma_m = \Gamma_m(\varrho)$, $\varrho = \sum_i B_i n_i$

- Scalar potential
  $$S_i = \sum_{m \in S} \Gamma_{im} n_m^{(\text{source})} - \Delta m_i$$
  with medium-dependent mass shift $\Delta m_i(T, n_j)$

- Vector potential
  $$V_i = \sum_{m \in V} \Gamma_{im} n_m^{(\text{source})} + V_i^{(\text{em})} + V_i^{(r)}$$
interaction

- Lorentz scalar mesons \( m \in S = \{\sigma, \delta, \sigma_{*}, \ldots\} \)
- Lorentz vector mesons \( m \in V = \{\omega, \rho, \phi, \ldots\} \)

- represented by (classical) fields \( A_m \) with mass \( m_m \)
- coupling to constituents: \( \Gamma_{im} = g_{im} \Gamma_m \)
  with scaling factors \( g_{im} \) and density dependent \( \Gamma_m = \Gamma_m(\varrho), \varrho = \sum_i B_i n_i \)

- scalar potential \( S_i = \sum_{m \in S} \Gamma_{im} n^{(source)}_m - \Delta m_i \)
  with medium-dependent **mass shift** \( \Delta m_i(T, n_j) \)

- vector potential \( V_i = \sum_{m \in V} \Gamma_{im} n^{(source)}_m + V^{(em)}_i + V^{(r)}_i \)
  with electromagnetic contribution \( V^{(em)}_i = T f_L(\Gamma_i) \) from fit of OCP data
  assuming linear mixing rule \( (\Gamma_i = Q_i^{5/3} \Gamma_Q, \Gamma_Q = e^2/(a_Q T), a_Q = [3/(4\pi n_Q)]^{1/3}) \)
  and **rearrangement contribution** \( V^{(r)}_i = B_i V^{(r)} + U^{(mass)}_i + U^{(em)}_i + U^{(deg)}_i \)
  \( V^{(r)} = \sum_{m \in V} \Gamma'_m A_m n^{(source)}_m - \sum_{m \in S} \Gamma'_m A_m n^{(source)}_m, \Gamma'_m = d \Gamma_m / d \varrho \)
effective density functional

- grand canonical potential density

\[ \omega(L) = \omega_{qp}(L) + \omega_{\text{strong}}(L) + \omega_{\text{em}}(L) \]
**Effective Density Functional**

- Grand canonical potential density

\[ \omega^{(L)} = \omega_{qp}^{(L)} + \omega_{\text{strong}}^{(L)} + \omega_{\text{em}}^{(L)} \]

- Contribution of quasi-particles

\[ \omega_{qp}^{(L)} = \sum_{i \in I} g_i \left( \omega_i^{(r)} + \omega_i^{(p)} \delta_{\sigma_i,+1} + \omega_i^{(c)} \delta_{\sigma_i,-1} \right) + \sum_{i \in \mathcal{E}} \left( g_i \omega_i^{(r)} - U_{i}^{(\text{deg})} n_i \right) \]
**Gas/Liquid Phase III**

**effective density functional**

- **grand canonical potential density**
  \[ \omega^{(L)} = \omega_{qp}^{(L)} + \omega_{\text{strong}}^{(L)} + \omega_{\text{em}}^{(L)} \]

- **contribution of quasi-particles**
  \[ \omega_{qp}^{(L)} = \sum_{i \in I} g_i \left( \omega_i^{(r)} + \omega_i^{(p)} \delta_{\sigma_i,+1} + \omega_i^{(c)} \delta_{\sigma_i,-1} \right) + \sum_{i \in E} \left( g_i \omega_i^{(r)} - U_i^{\text{(deg)}} n_i \right) \]

  - **regular contribution**
    \[ \omega_i^{(r)} = -\frac{T}{\sigma_i} \int \frac{d^3k}{(2\pi)^3} \sum_{\eta_i} \ln[1 + \sigma_i \exp(-E_i^{(\eta_i)}/T)] \]

    with \( E_i^{(\eta_i)} = e_i^{(\eta_i)} - \mu_i \)

  - **pairing contribution** \( \omega_i^{(p)} = \ldots \)

  - **condensate contribution** \( \omega_i^{(c)} = \ldots \)
Gas/Liquid Phase III

**effective density functional**

- **grand canonical potential density**
  \[ \omega^{(L)} = \omega_{qp}^{(L)} + \omega_{\text{strong}}^{(L)} + \omega_{\text{em}}^{(L)} \]

- **contribution of quasi-particles**
  \[ \omega_{qp}^{(L)} = \sum_{i \in \mathcal{I}} g_i \left( \omega_i^{(r)} + \omega_i^{(p)} \delta_{\sigma_i,+1} + \omega_i^{(c)} \delta_{\sigma_i,-1} \right) + \sum_{i \in \mathcal{E}} \left( g_i \omega_i^{(r)} - U_i^{(\text{deg})} n_i \right) \]
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    with \( E_i^{(\eta_i)} = e_i^{(\eta_i)} - \mu_i \)
  - **pairing contribution** \( \omega_i^{(p)} = \ldots \)
  - **condensate contribution** \( \omega_i^{(c)} = \ldots \)

- **contribution from strong interaction**
  \[ \omega_{\text{strong}}^{(L)} = \sum_{m \in \mathcal{S}} m_m^2 A_m^2 - \sum_{m \in \mathcal{V}} m_m^2 A_m^2 - V^{(r)} q - \sum_{i \in \mathcal{I} \cup \mathcal{E}} U_i^{(\text{mass})} n_i \]
Gas/Liquid Phase III

effective density functional

- grand canonical potential density

\[ \omega(L) = \omega_{qp}^{(L)} + \omega_{strong}^{(L)} + \omega_{em}^{(L)} \]

- contribution of quasi-particles

\[ \omega_{qp}^{(L)} = \sum_{i \in \mathcal{I}} g_i \left( \omega_i^{(r)} + \omega_i^{(p)} \delta_{\sigma_i,+1} + \omega_i^{(c)} \delta_{\sigma_i,-1} \right) + \sum_{i \in \mathcal{E}} \left( g_i \omega_i^{(r)} - U_i^{(deg)} n_i \right) \]

- regular contribution

\[ \omega_i^{(r)} = -\frac{T}{\sigma_i} \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 + \sigma_i \exp \left( -E_i^{(\eta)} / T \right) \right] \]

with \( E_i^{(\eta)} = e_i^{(\eta)} - \mu_i \)

- pairing contribution \( \omega_i^{(p)} = \ldots \)

- condensate contribution \( \omega_i^{(c)} = \ldots \)

- contribution from strong interaction

\[ \omega_{strong}^{(L)} = \sum_{m \in \mathcal{S}} m^2 m^2 A_m^2 - \sum_{m \in \mathcal{V}} m^2 A_m^2 - V^{(r)} q - \sum_{i \in \mathcal{I} \cup \mathcal{E}} U_i^{(mass)} n_i \]

- contribution from electromagnetic interaction

\[ \omega_{em}^{(L)} = - \sum_{i \in \mathcal{I} \cup \mathcal{E}} U_i^{(em)} n_i \]
fermions $\Rightarrow$ pairing correlations

- pairing potential $v_i(k, k')$
fermions $\Rightarrow$ pairing correlations

- pairing potential $v_i(k, k')$

- pairing contribution to $\omega_{qp}^{(L)}$

$$\omega_i^{(p)} = \int \frac{d^3k}{(2\pi)^3} \sum \eta_i \left\{ \frac{1}{2} [e_i^{(\eta_i)}(k) - \mu_i - E_i^{(\eta_i)}(k)] + \Delta_i^{(\eta_i)}(k) \nu_i^{(\eta_i)}(k) \right\}$$

$$+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \sum \eta_i \nu_i^{(\eta_i)}(k) \nu_i^{(\eta_i)}(k, k') \nu_i^{(\eta_i)}(k')$$

$$E_i^{(\eta_i)} = \pm \sqrt{[e_i^{(\eta_i)}(k) - \mu_i]^2 + [\Delta_i^{(\eta_i)}(k)]^2}, \quad \Delta_i^{(\eta_i)}(k) \text{ pairing gap}$$

$$\nu_i^{(\eta_i)}(k) = \frac{\Delta_i^{(\eta_i)}(k)}{2E_i^{(\eta_i)}(k)} [1 - 2 f_{+1}(E_i^{(\eta_i)}(k))] \text{ anomalous distribution function,}$$

$$f_{+1}(E) = [\exp(E) + 1]^{-1} \text{ Fermi-Dirac distribution function}$$
fermions $\Rightarrow$ pairing correlations

- pairing potential $v_i(k, k')$

- pairing contribution to $\omega^{(L)}_{qp}$

$$\omega_i^{(p)} = \int \frac{d^3k}{(2\pi)^3} \sum_i \left\{ \frac{1}{2} [e_i^{(\eta_i)}(k) - \mu_i - E_i^{(\eta_i)}(k)] + \Delta_i^{(\eta_i)}(k) \nu_i^{(\eta_i)}(k) \right\}$$

$$+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \sum_i \nu_i^{(\eta_i)}(k) \nu_i^{(\eta_i)}(k, k') \nu_i^{(\eta_i)}(k')$$

$$E_i^{(\eta_i)} = \pm \sqrt{[e_i^{(\eta_i)} - \mu_i]^2 + [\Delta_i^{(\eta_i)}]^2}, \Delta_i^{(\eta_i)}(k) \text{ pairing gap}$$

$$\nu_i^{(\eta_i)}(k) = \frac{\Delta_i^{(\eta_i)}(k)}{2E_i^{(\eta_i)}(k)} [1 - 2f_+(E_i^{(\eta_i)})] \text{ anomalous distribution function,}$$

$$f_+(E) = [\exp(E) + 1]^{-1} \text{ Fermi-Dirac distribution function}$$

- $\partial \omega^{(L)} / \partial \Delta_i^{(\eta_i)}(k) = 0 \Rightarrow$ gap equation

$$\Delta_i^{(\eta_i)}(k) + \int \frac{d^3k'}{(2\pi)^3} \nu_i^{(\eta_i)}(k, k') \nu_i^{(\eta_i)}(k') = 0$$
Gas/Liquid Phase V

bosons $\Rightarrow$ condensation

- condensate contribution to $\omega_{qp}^{(L)}$

$$
\omega_i^{(c)} = \frac{1}{2}[\zeta_i^{(\eta_i)}]^{2}[\left(m_i - S_i\right)^2 - \left(\mu_i - V_i\right)^2]
$$

with parameter $\zeta_i^{(\eta_i)}$
Gas/Liquid Phase V

**bosons** ⇒ **condensation**

- condensate contribution to $\omega^{(L)}_{q\mathbf{p}}$

$$\omega_{i}^{(c)} = \frac{1}{2}[\zeta_{i}^{(\eta_{i})}]^{2}[(m_{i} - S_{i})^{2} - (\mu_{i} - V_{i})^{2}]$$

with parameter $\zeta_{i}^{(\eta_{i})}$

- general condition on chemical potential $\mu_{i}$

$$|\mu_{i} - V_{i}| \leq m_{i} - S_{i}$$
**Gas/Liquid Phase V**

**bosons ⇒ condensation**

- condensate contribution to $\omega_{qp}^{(L)}$

$$\omega_i^{(c)} = \frac{1}{2}[\zeta_i^{(\eta_i)}]^2[(m_i - S_i)^2 - (\mu_i - V_i)^2]$$

with parameter $\zeta_i^{(\eta_i)}$

- general condition on chemical potential $\mu_i$

$$|\mu_i - V_i| \leq m_i - S_i$$

- $\partial \omega^{(L)}/\partial \zeta_i^{(\eta_i)} = 0 \Rightarrow$ condition for condensation solutions:
  - $\zeta_i^{(\eta_i)} = 0$: no condensation
  - $\zeta_i^{(\eta_i)} \neq 0, \mu_i = V_i + m_i - S_i$: condensation of particles
  - $\zeta_i^{(\eta_i)} \neq 0, \mu_i = V_i - m_i + S_i$: condensation of antiparticles

value of $\zeta_i^{(\eta_i)}$ determined by density of condensate state
densities ⇒ usual form for quasiparticles

- net particle density

\[ n_i = g_i \sum \eta_i \left\{ \int \frac{d^3k}{(2\pi)^3} \eta_i f_{\sigma i}(\eta_i)(k) + \left[ \xi_{\sigma i}(\eta_i) \right]^2 (\mu_i - \nu_i) \delta_{\sigma i, -1} \right\} \]

\[ f_{\sigma i}(\eta_i) = \frac{1}{2} \left\{ 1 - \frac{e_i(\eta_i)}{E_i(\eta_i)} \left[ 1 - 2 f_{\sigma i}(E_i(\eta_i)) \right] \right\}, \quad f_{\sigma}(E) = \left[ \exp(E) + \sigma \right]^{-1} \]
**Gas/Liquid Phase VI**

**densities** ⇒ usual form for quasiparticles

- **net particle density**

  \[
  n_i = g_i \sum \eta_i \left\{ \int \frac{d^3k}{(2\pi)^3} \eta_i f_i^{(\eta_i)}(k) + \left[ \zeta_i^{(\eta_i)} \right]^2 (\mu_i - V_i) \delta_{\sigma_i, -1} \right\}
  \]

  \[
  f_i^{(\eta_i)} = \frac{1}{2} \left\{ 1 - \frac{e_i^{(\eta_i)} - \mu_i}{E_i^{(\eta_i)}} \right\} [1 - 2 f_{\sigma_i}(E_i^{(\eta_i)})], \quad f_{\sigma}(E) = [\exp(E) + \sigma]^{-1}
  \]

- **net scalar density**

  \[
  n_i^{(s)} = g_i \sum \eta_i \left\{ \int \frac{d^3k}{(2\pi)^3} \sqrt{m_i - S_i} f_i^{(\eta_i)}(k) + \left[ \zeta_i^{(\eta_i)} \right]^2 (m_i - S_i) \delta_{\sigma_i, -1} \right\}
  \]
**Gas/Liquid Phase VI**

**densities** ⇒ usual form for quasiparticles

- **net particle density**

\[
n_i = g_i \sum \eta_i \left\{ \int \frac{d^3k}{(2\pi)^3} \eta_i f_i^{(\eta_i)}(k) + [\zeta_i^{(\eta_i)}]^2(\mu_i - V_i)\delta_{\sigma_i,-1} \right\}
\]

\[
f_i^{(\eta_i)} = \frac{1}{2} \left\{ 1 - \frac{e_i^{(\eta_i)} - \mu_i}{E_i^{(\eta_i)}} \left[ 1 - 2f_\sigma(E_i^{(\eta_i)}) \right] \right\}, \quad f_\sigma(E) = [\exp(E) + \sigma]^{-1}
\]

- **net scalar density**

\[
n_i^{(s)} = g_i \sum \eta_i \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{m_i - S_i}{\sqrt{k^2 + (m_i - S_i)^2}} f_i^{(\eta_i)}(k) + [\zeta_i^{(\eta_i)}]^2(m_i - S_i)\delta_{\sigma_i,-1} \right\}
\]

- **source densities**
  - Lorentz scalar mesons, \( m \in S \)

\[
n^{(\text{source})}_m = \sum_{i \in I \cup E \cup S} g_{im} n_i^{(s)}
\]

  - Lorentz vector mesons, \( m \in V \)

\[
n^{(\text{source})}_m = \sum_{i \in I \cup E \cup S} g_{im} n_i
\]
thermodynamic consistency

- natural variables of $\omega^{(L)}$: $T$, $\mu_i$, $A_m$, $\Delta_i^{(\eta)}(k)$, $\zeta_i^{(\eta_i)}$

but $\omega^{(L)}$ depends explicitly on densities $n_i$, $n_i^{(s)}$ (already defined!)
Gas/Liquid Phase VII

thermodynamic consistency

- natural variables of $\omega^{(L)}$: $T, \mu_i, A_m, \Delta_i^{(\eta_i)}(k), \zeta_i^{(\eta_i)}$

but $\omega^{(L)}$ depends explicitly on densities $n_i, n_i^{(s)}$ (already defined!)

- consistency criterion

$$n_j = - \frac{\partial}{\partial \mu_j} \omega^{(L)}(T, \mu_i, A_m, \Delta_i^{(\eta_i)}(k), \zeta_i^{(\eta_i)}) \bigg|_{T, \mu_i \neq j, A_m, \Delta_i^{(\eta_i)}(k), \zeta_i^{(\eta_i)}}$$
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$\Rightarrow$ definition of rearrangement potentials

- $U_i^{(\text{mass})} = \sum_{j \in I \cup E} \frac{\partial \Delta m_j}{\partial n_i} n_j^{(s)}$
- $U_i^{(\text{em})} = \sum_{j \in I \cup E} \frac{\partial V_j^{(\text{em})}}{\partial n_i} n_j$
- $U_i^{(\text{deg})} = \sum_{j \in E} \frac{\partial g_j}{\partial n_i} \omega_j^{(r)}$
Gas/Liquid Phase VII

thermodynamic consistency

- natural variables of $\omega^{(L)}$: $T, \mu_i, A_m, \Delta_i^{(\eta)}(k), \zeta_i^{(\eta)}$

but $\omega^{(L)}$ depends explicitly on densities $n_i, n_i^{(s)}$ (already defined!)

- consistency criterion

$$n_j \equiv - \frac{\partial}{\partial \mu_j} \omega^{(L)}(T, \mu_i, A_m, \Delta_i^{(\eta)}(k), \zeta_i^{(\eta)}) \bigg|_{T, \mu_i \neq j, A_m, \Delta_i^{(\eta)}(k), \zeta_i^{(\eta)}}$$

$\Rightarrow$ definition of rearrangement potentials

- $U_i^{(\text{mass})} = \sum_{j \in I \cup E} \frac{\partial \Delta m_j}{\partial n_i} n_j^{(s)}$

- $U_i^{(\text{em})} = \sum_{j \in I \cup E} \frac{\partial V^{(em)}}{\partial n_i} n_j$

- $U_i^{(\text{deg})} = \sum_{j \in E} \frac{\partial g_j}{\partial n_i} \omega_j^{(r)}$

- non-standard contributions to entropy density

$$s = - \frac{\partial \omega^{(L)}}{\partial T} \bigg|_{\mu_i, A_m, \Delta_i^{(\eta)}(k), \zeta_i^{(\eta)}}$$
Solid Phase I

combination of models

- homogeneously distributed constituent particles
  - leptons, photons, neutrons, certain nuclei(?), . . .
  - contribution to grand canonical potential as in gas/liquid phase
Solid Phase I

combination of models

- homogeneously distributed constituent particles
  - leptons, photons, neutrons, certain nuclei(?), . . .
  - contribution to grand canonical potential as in gas/liquid phase

- nuclei on lattice sites, excitation of lattice vibrations/phonons
  - Einstein/Debye-like model, three branches ($\lambda = 0, 1, 2$)
Solid Phase I

combination of models

- homogeneously distributed constituent particles
  - leptons, photons, neutrons, certain nuclei(?), . . .
  - contribution to grand canonical potential as in gas/liquid phase

- nuclei on lattice sites, excitation of lattice vibrations/phonons
  - Einstein/Debye-like model, three branches \((\lambda = 0, 1, 2)\)
    - one longitudinal mode:
      \[
      \omega_i(0, \vec{q}) = \alpha_0 \omega_i^{(p)}
      \]
    - two transversal modes:
      \[
      \omega_i(1, \vec{q}) = \alpha_1 \omega_i^{(p)} \frac{q}{k_i^{(D)}} \\
      \omega_i(2, \vec{q}) = \alpha_2 \omega_i^{(p)} \frac{q}{k_i^{(D)}}
      \]

plasma frequency \(\omega_i^{(p)} = \sqrt{4\pi Q_i e^2 n_Q / m_i}\)
Debye wave number \(k_i^{(D)} = (6\pi^2 n_i)^{1/3}\)
parameters \(\alpha_0, \alpha_1, \alpha_2\)
Solid Phase II

- **parameters** $\alpha_0$, $\alpha_1$, $\alpha_2$

  fitted to reproduce known frequency moments

$$\mu_n = \frac{1}{3} \sum_{\lambda, \vec{q}} \omega_i(\lambda, \vec{q}) / \omega_i^{(p)} [n]$$

for $n = 1, 2$

and consistency relation in classical limit ($3\bar{\mu} = \ln(\alpha_0\alpha_1\alpha_2) - 2/3$)
Solid Phase II

- **parameters** $\alpha_0, \alpha_1, \alpha_2$
  fitted to reproduce known frequency moments

$$\mu_n = \frac{1}{3} \sum_{\lambda, \vec{q}} \omega_i(\lambda, \vec{q}) / \omega_i^{(p)} \right] n$$

for $n = 1, 2$

and consistency relation in classical limit $(3\bar{\mu} = \ln(\alpha_0\alpha_1\alpha_2) - 2/3)$

**bcc lattice**

<table>
<thead>
<tr>
<th></th>
<th>exact calculation*</th>
<th>model</th>
<th>significance</th>
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<tbody>
<tr>
<td>$\mu_{-2}$</td>
<td>12.972</td>
<td>12.850</td>
<td>mean square displacement (classical)</td>
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<tr>
<td>$\mu_{-1}$</td>
<td>2.79855</td>
<td>2.79031</td>
<td>mean square displacement (quantal)</td>
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<tr>
<td>$\mu_1$</td>
<td>0.5113875</td>
<td>exact</td>
<td>zero-point oscillation energy</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>1/3</td>
<td>exact</td>
<td>Kohn rule</td>
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<tr>
<td>$\mu_3$</td>
<td>0.25031</td>
<td>0.24905</td>
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<tr>
<td>$\bar{\mu}$</td>
<td>-0.831298</td>
<td>exact</td>
<td>classical limit of free energy</td>
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Solid Phase III

effective density functional
- canonical description $\Rightarrow$ free energy density

$$f^{(S)} = \sum_{i \in S} n_i [m_i + F_i^{(ph)} + F_i^{(em)} + F_i^{(mix)}]$$
Solid Phase III

effective density functional

- canonical description $\Rightarrow$ free energy density

$$f^{(S)} = \sum_{i \in S} n_i \left[ m_i + F_i^{(ph)} + F_i^{(em)} + F_i^{(mix)} \right]$$

- contribution of phonons

$$F_i^{(ph)} = T \left\{ \frac{3}{2} \mu_1 \eta_i + \sum_{\lambda=0}^{2} \ln \left[ 1 - \exp(-\alpha_\lambda \eta_i) \right] - \frac{1}{3} \sum_{\lambda=1}^{2} D_3(\alpha_\lambda \eta_i) \right\}$$

with Debye function $D_3(x)$

essential parameters $\eta_i = \omega_i^{(p)}/T$

$\eta_i \to 0$: classical limit
$\eta_i \to \infty$: quantal effects
effective density functional

- canonical description $\Rightarrow$ free energy density

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$\eta_i \rightarrow \infty$: quantal effects

- contribution of electromagnetic interaction

$$F^{(em)}_i = T [C_{M}^{(bcc)} \Gamma_i + f_S(\Gamma_i)] \quad \text{(from fit to OCP)}$$
effective density functional

- **canonical description** ⇒ **free energy density**
  \[
  f^{(S)} = \sum_{i \in S} n_i [m_i + F^{(ph)}_i + F^{(em)}_i + F^{(mix)}_i]
  \]

  - contribution of **phonons**
    \[
    F^{(ph)}_i = T \left\{ \frac{3}{2} \mu_1 \eta_i + \sum_{\lambda=0}^{2} \ln[1 - \exp(-\alpha_\lambda \eta_i)] - \frac{1}{3} \sum_{\lambda=1}^{2} D_3(\alpha_\lambda \eta_i) \right\}
    \]

    with Debye function \(D_3(x)\)

    essential parameters \(\eta_i = \omega_i^{(p)}/T\)

    \(\eta_i \to 0\): classical limit
    \(\eta_i \to \infty\): quantal effects

  - contribution of **electromagnetic interaction**
    \[
    F^{(em)}_i = T \left[ C^{(bcc)}_M \Gamma_i + f_S(\Gamma_i) \right] \quad \text{(from fit to OCP)}
    \]

  - **mixing** contribution
    \[
    F^{(mix)}_i = T \ln\left( \frac{Q_{ini}}{g_i n_Q} \right) \quad n_Q = \sum_i Q_i n_i
    \]
• EoS of cold outer crust very well known
  \( (\beta \text{ equilibrium}, \ T = 0 \text{ MeV}) \)

\[ \beta \text{ equilibrium, } T = 0 \text{ MeV} \]

- EoS of cold outer crust very well known ($\beta$ equilibrium, $T = 0$ MeV)

- calculation in Wigner-Seitz and Thomas-Fermi approximation (WS-TF) not sufficient

$\beta$ equilibrium, $T = 0$ MeV

- EoS of cold outer crust very well known
  ($\beta$ equilibrium, $T = 0$ MeV)

- calculation in Wigner-Seitz and Thomas-Fermi approximation (WS-TF) not sufficient

- effects of temperature
  - change of chemical composition
  - melting of crystal, solidification of gas/liquid

---

### $\beta$ equilibrium, $T = 0$ MeV

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ERDF - 35
Stefan Tytel
• EoS of cold outer crust very well known ($\beta$ equilibrium, $T = 0$ MeV)

• calculation in Wigner-Seitz and Thomas-Fermi approximation (WS-TF) not sufficient

• effects of temperature
  ○ change of chemical composition
  ○ melting of crystal, solidification of gas/liquid

• general electron fraction
  ○ out of $\beta$ equilibrium
  $\Rightarrow$ global EoS table
Solid Phase IV

- EoS of cold outer crust very well known ($\beta$ equilibrium, $T = 0$ MeV)

- calculation in Wigner-Seitz and Thomas-Fermi approximation (WS-TF) not sufficient

- effects of temperature
  - change of chemical composition
  - melting of crystal, solidification of gas/liquid

- general electron fraction
  - out of $\beta$ equilibrium
  - $\Rightarrow$ global EoS table

- details of phase transitions

- work in progress

$\beta$ equilibrium, $T = 0$ MeV

Summary
construction of **effective relativistic density functional** for dense matter

- extended set of *constituents* ⇒ nucleons, hyperons, mesons, nuclei, leptons, . . .
  ⇒ *quasiparticles* with medium dependent properties
- nuclear interaction ⇒ meson exchange with density dependent couplings
- electromagnetic interaction ⇒ effective potential from Monte Carlo simulations
- formation and dissolution of *clusters*
- rearrangement contributions for thermodynamic consistency
- phase transition liquid/gas ↔ solid
- well constrained *parameters*, correct limits
- work in progress

⇒ preparation of *EoS tables* for astrophysical applications
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  for your attention and patience