Goldstone bosons in crystalline chiral phases

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The QCD Phase Diagram

Temperature $T$ [MeV]

Early universe

Deconfinement and chiral transition

Critical point?

Hadrons

Quarks and Gluons

Net Baryon Density

Color Superconductor?

Neutron stars

RHIC, LHC

FAIR-SIS 300

Nuclei

(GSI)
Focus on chiral symmetry

- Spontaneously broken in vacuum
- Order parameter: chiral condensate $\langle \bar{q}q \rangle$
- Believed to have first-order phase transition for low temperatures
- Critical endpoint

Most calculations: order parameter constant in space
Simpler Phase Diagram

Focus on chiral symmetry

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What happens if we allow space dependence?
Inhomogeneous Phase

- Space dependent order parameter
- Popular for some time
  - Pion Condensation
  - (Color-) superconductivity
- Studied more recently in lower dimensional models (1+1 D Gross-Neveu model)
  [Schön, Thies, PRD (2000)]
Inhomogeneous Phase in Nambu–Jona-Lasinio Model

Nambu–Jona-Lasinio model
[Nickel, PRD (2009)]

► Critical endpoint replaced by Lifschitz point
► first order phase transition replaced by inhomogeneous region
### Inhomogeneous Phase in Nambu–Jona-Lasinio Model

Nambu–Jona-Lasinio model

[Nickel, PRD (2009)]

- Critical endpoint replaced by Lifschitz point
- First order phase transition replaced by inhomogeneous region

Here:

- Consider fluctuations around mean fields
- Goldstone bosons from spontaneous broken symmetries (chiral and spatial)
- Important for transport properties (e.g. cooling in neutron stars)
- Could lead to instabilities (Landau-Peierls instability)
Nambu–Jona-Lasinio Model

- NJL Lagrangian

\[ \mathcal{L} = \bar{\psi} \left( i \partial_\mu - m \right) \psi + G_S \left( \left( \bar{\psi} \psi \right)^2 + \left( \bar{\psi} i \gamma_5 \tau^a \psi \right)^2 \right) \]

- Derive thermodynamic properties from grand potential \( \Omega \)

- Mean-field approximation

\[ S(\vec{x}) = \langle \bar{\psi} \psi \rangle, \quad P(\vec{x}) = \langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle \]

- keep space dependence, but neglect time dependence

\[ \mathcal{L}_{MF} = \bar{\psi} S^{-1} \psi + V(\vec{x}) \]

\[ S^{-1} = i \partial_\mu - \left( m - 2 G_S (S(\vec{x}) + i \gamma_5 \tau^3 P(\vec{x})) \right), \quad V(\vec{x}) = - G_S \left[ S^2(\vec{x}) + P^2(\vec{x}) \right] \]

\[ =: M(\vec{x}) \]
Space dependent Mass

- **Space dependent mass**
  \[ M(\vec{x}) = m - 2G_S \left( S(\vec{x}) + iP(\vec{x}) \right) \]

- **Crystal with unit cell vectors** \( \vec{n}_i, \ i = 1, 2, 3 \)

- **Periodicity in mass**
  \[ M(\vec{x}) = M(\vec{x} + \vec{n}_i) \]

- **Fourier transformation**
  \[ M(\vec{x}) = \sum_{\vec{q}_k} M_{\vec{q}_k} e^{i\vec{q}_k \cdot \vec{x}} \]

- **Wave vector** \( \vec{q}_k \) spans reciprocal lattice: \( \vec{q}_k \cdot \vec{n}_i = 2\pi N_{ki}, \quad N_{ki} \in \mathbb{Z} \)
Grand Potential

Arrive at grand potential

\[ \Omega = -N_C N_F \frac{1}{V} \sum_{E\lambda} T \ln \left[ 2 \cosh \left( \frac{E\lambda - \mu}{2T} \right) \right] + \frac{1}{V} \int d^3x \frac{|M(\vec{x}) - m|^2}{4G_S} \]

with eigenvalues \( E\lambda \) of \( H \) in momentum space

\[ H(\vec{p}_m, \vec{p}_{m'}) = \begin{pmatrix} -\vec{\sigma} \vec{p}_m \delta_{\vec{p}_m,\vec{p}_{m'}} & -\sum \vec{q}_k M\vec{q}_k \delta_{\vec{q}_k, (\vec{p}_m - \vec{p}_{m'})} \\ -\sum \vec{q}_k M\vec{q}_k \delta_{\vec{q}_k, (\vec{p}_{m'} - \vec{p}_m)} & \vec{\sigma} \vec{p}_m \delta_{\vec{p}_m,\vec{p}_{m'}} \end{pmatrix} \]

→ Matrix in momentum space
Grand Potential

Arrive at grand potential

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with eigenvalues \( E_\lambda \) of \( H \) in momentum space

\[
H(\vec{p}_m, \vec{p}_m') = \begin{pmatrix}
-\vec{\sigma} \vec{p}_m \delta \vec{p}_m, \vec{p}_m' & -\sum_{q_k} M_{q_k} \delta q_k, (\vec{p}_m - \vec{p}_m') \\
-\sum_{q_k} M_{q_k} \delta q_k, (\vec{p}_m' - \vec{p}_m) & \vec{\sigma} \vec{p}_m \delta \vec{p}_m, \vec{p}_m'
\end{pmatrix}
\]

→ Matrix in momentum space

Propagator has different incoming and outgoing momenta

\[
S^{-1}(p_{out}, p_{in}) = \gamma^0 (p_0 \delta_{p_{out}, p_{in}} - H(p_{out}, p_{in}))
\]
Gap Eqution

\[ p_{out} p_{in} = p_{out} p_{in} + p_{out} p'_{\text{inter}} p_{\text{inter}} p_{in} \]

Gap equation

\[ S^{-1}(p_{out}, p_{in}) = S_0^{-1}(p_{out}, p_{in}) - \Sigma(p_{out}, p_{in}) \]

Self energy

\[ \Sigma(p_{out}, p_{in}) = 2G_S \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \left[ \text{Tr}(1 iS(p_1, p_2)) + i\gamma^5 \tau_3 \text{Tr}(i\gamma_5 \tau^3 iS(p_1, p_2)) \right] \delta(p_{out} + p_2 - p_{in} - p_1) \]
**Modulated Order Parameter**

Different (periodical) modulations:

- **Chiral Density Wave**
  
  \[ M(z) = M \exp(iqz) \]

- **Solitonic modulation**
  
  \[ M(z) = \Delta \nu \, \text{sn}(\Delta z | \nu) \]

- **2D 'Egg' carton**
  
  \[ M(x, y) = M \cos(qx) \cos(qy) \]

[Carignano, Buballa, PRD (2012)]
Results

[Carignano, Buballa, PRD (2012)]
Phonons and Goldstone modes

Homogeneous phase

Rotational and translational symmetry broken
- fluctuations in amplitude
- fluctuations in space

Inhomogeneous Phase

Chiral symmetry broken
Pions as Goldstone modes

Details in: [Lee, Nakano, Tsue, Tatsumi, Friman, PRD (2015)]
Long range correlations
[Lee, Nakano, Tsue, Tatsumi, Friman, PRD (2015)]

In Ginzburg-Landau Model (valid for $M \approx 0$)
Long range correlations ($\phi = M/2G_S$)

$$f_{ij}(x) = \langle \phi_i(x)\phi_j^*(0) \rangle$$

in $z$-direction

$$\langle \phi(z\vec{e}_z) \cdot \phi^*(0) \rangle \propto \frac{1}{2}\Delta^2 \cos(qz) \left( \frac{z}{z_0} \right)^{-T/T_0}$$

and longitudinal

$$\langle \phi(x_\perp\vec{e}_\perp) \cdot \phi^*(0) \rangle \propto \frac{1}{2}\Delta^2 \left( \frac{x_\perp}{x_0} \right)^{-2T/T_0}$$

Algebraic decay
Full NJL model

Inverse propagator

\[ S^{-1} = i\gamma^\mu \partial_\mu - \frac{1}{2} \left[ (1 + \gamma_5 \tau^3)M(\vec{x}) + (1 - \gamma_5 \tau^3)M^*(\vec{x}) \right] \]

Allow small perturbations in space with bosonic field \( \vec{u}(x) \) up to quadratic order

\[ M(\vec{x}) \rightarrow M_u(\vec{x}) = M(\vec{x} + \vec{u}(x)) \approx \sum_{\{\vec{q}\}} M_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} \left( 1 + i\vec{q} \cdot \vec{u}(x) - \frac{1}{2}(\vec{q} \cdot \vec{u}(x))^2 \right) \]

Partition function

\[ Z = \int i\mathcal{D}\psi^\dagger \int \mathcal{D}\psi \int \mathcal{D}\vec{u} \exp \left( \int_{\mathcal{X}_E} \overline{\psi} S^{-1}[\vec{u}]\psi + V[\vec{u}] \right), \quad V[\vec{u}] = \frac{|M_u(\vec{x})|^2}{4G_S} \]
Integrate out Fermions

\[
Z = \int i D\psi^\dagger \int D\psi \int D\tilde{u} \exp \left( \int_{x_E} \overline{\psi} S^{-1}[\tilde{u}] \psi + V[\tilde{u}] \right)
\]

\[
= \int D\tilde{u} \exp \left( \text{Tr} \ln (S_{MF}^{-1} + \Sigma_1[\tilde{u}] + \Sigma_2[\tilde{u}^2]) + \int_{x_E} V_{MF}(\tilde{x}) + V_1[\tilde{u}] + V_2[\tilde{u}^2] \right)
\]

expand logarithm

\[
\ln (S_{MF}^{-1} + \Sigma) = \ln (S_{MF}^{-1}(1 + S_{MF}\Sigma))
\]

\[
= \ln S_{MF}^{-1} + S_{MF}\Sigma_1[\tilde{u}] + S_{MF}\Sigma_2[\tilde{u}^2] - \frac{1}{2} S_{MF}\Sigma_1[\tilde{u}]S_{MF}\Sigma_1[\tilde{u}] + \mathcal{O}(u^3)
\]

Separate different orders of \( u \) in the partition function

\[
Z = \int D\tilde{u} \exp(\omega_{MF} + \omega_1[\tilde{u}] + \omega_2[\tilde{u}^2])
\]
Individual Contributions

\[ \omega_1[u] = 0 \text{ from gap equation} \]

\[ \omega_2[u^2] = \text{Tr} \left[ S_{MF} \Sigma_2[u^2] \right] + \int_{x_E} \frac{1}{2} \frac{M(\partial_2^2 M^*)u^*^2 + M^*(\partial_2^2 M)u^2}{4G_S} = 0 \text{ gap equation} \]

\[ - \frac{1}{2} \text{Tr} (S_{MF} \Sigma_1[u]S_{MF} \Sigma_1[u]) + \int_{x_E} \frac{|(\partial_2 M)u|^2}{4G_S} \]

second part needs calculation of

\[ \text{Tr} (S_{MF} \Sigma_1[u]S_{MF} \Sigma_1[u]) \]

similar object to polarization loops for meson calculations
Mesons in NJL

Start from Bethe-Salpeter equation

\[ D_{MN}(p, p') = \frac{2G_s}{1 - 2G_s J_{MN}(p, p')} \]

one can get the meson propagator

with polarization loop \( J_{MN}(p, p') \)

\[ \propto \text{Tr} [\Gamma_M S \Gamma_N S] \]
Chiral transformation for CDW

Hamiltonian Chiral Density Wave

\[ H(p, p') = -\gamma_0 \gamma^k p_k \delta(p - p') \]
\[ + \gamma_0 \frac{M}{2} \left[ (1 + \gamma_5 \tau^3) \delta(p - p' + q) + (1 - \gamma_5 \tau^3) \delta(p - p' - q) \right] \]
Chiral transformation for CDW

Hamiltonian Chiral Density Wave

\[ H(p, p') = -\gamma_0 \gamma^k p_k \delta(p - p') \]
\[ + \gamma_0 \frac{M}{2} \left[ (\mathbb{1} + \gamma_5 \tau^3) \delta(p - p' + q) + (\mathbb{1} - \gamma_5 \tau^3) \delta(p - p' - q) \right] \]

Apply a rotation in chiral space [Dautry, Nyman, Nucl. Phys. (1979)]

\[ U(k, k') = \frac{1}{2} \left[ (\mathbb{1} + \gamma_5 \tau^3) \delta(k - k' - q/2) + (\mathbb{1} - \gamma_5 \tau^3) \delta(k - k' + q/2) \right] \]

to the Hamiltonian

\[ H'(p, p') = U^\dagger HU = \gamma_0 \left[ -\gamma^k p_k - \gamma^k \gamma_5 \tau^3 q_k / 2 + M \right] \delta(p - p') \]

From this get eigenvalues

\[ E_{\pm}(\vec{p}) = \sqrt{\vec{p}^2 + M^2 + q^2 / 4} \pm \sqrt{(\vec{p} \cdot \vec{q})^2 + q^2 M^2} \]
CDW: analytic expression for propagator

Inverse propagator

\[ S^{-1}(p, p') = \gamma^0 \left( p_0 \delta(p - p') - H(p, p') \right) \]

apply chiral transformation

\[ S = US'U \]

with

\[ S'(k) = \frac{1}{N(k)} \left[ A(k) + \gamma_5 \tau^3 B(k) + \gamma_\mu C^\mu(k) + \gamma_5 \tau^3 \gamma_\mu D^\mu(k) + \gamma_5 \tau^3 \gamma_\mu \gamma_\nu E^{\mu\nu}(k) \right] \]

diagonal in momentum space
Pions in the inhomogeneous phase

For the CDW the $\pi_3$ (with $\Gamma_{\pi_3} = i\gamma_5\tau^3$) is no NG boson. Instead

$$\Gamma_{\pi_1} = i\gamma_5\tau^1, \quad \Gamma_{\pi_2} = i\gamma_5\tau^2, \quad \Gamma_{\bar{\pi}}(z) = -1 \sin qz + i\gamma_5\tau^3 \cos qz$$

$$J_{\bar{\pi}\pi}(p, p') = i \left( \prod_{j=1}^{4} \int \frac{d^4 k_j}{(2\pi)^4} \right) \text{Tr} \left[ \Gamma_{\bar{\pi}}(p + k_4 - k_1)S(k_1, k_2)\Gamma_{\bar{\pi}}(k_2 - k_3 - p')S(k_3, k_4) \right]$$
Integrating \( \delta \)-functions

Fourier transformation

\[
\Gamma_{\pi}(p) = \frac{i}{2} \left[ (-\mathbb{1} + \gamma_5 \tau^3) \delta(p - q) + (\mathbb{1} + \gamma_5 \tau^3) \delta(p + q) \right]
\]

Inserting \( S = US' U \)

Integrating over internal momenta

\[
J_{\bar{\pi}\pi}(p, p') = i \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ i\gamma_5 \tau^3 S'(k) i\gamma_5 \tau^3 S'(k - p) \right] \delta(p - p')
\]

yields NG Boson via gap equation

\[
\lim_{p = p' \to 0} \left( 1 - 2G_S J_{\pi \pi}(p, p') \right) = 0
\]

\[
\Rightarrow \quad p_0 = m_{\pi} = 0
\]
Mean-Field Calculations:

- Crystalline phase replaces first order phase transition and critical endpoint in phase diagram
- One dimensional modulations favored over two dimensional

Bosonic excitations

- Explicit construction of Goldstone modes in CDW
- Conceptual difficulty due to non-diagonal structure of propagator
- Calculations simplified by chiral transformations
- Goldstone mode identified

Outlook

- Calculate dispersion relations of Goldstone Bosons
- Derive transport properties
- Applications for beyond mean-field calculations
Thank you
Full CDW propagator

Of course more complicated for full CDW propagators

\[ S' = \frac{1}{N} \left[ A + \gamma^5 \tau^3 B + \gamma_\mu C^\mu + \gamma^5 \tau^3 \gamma_\mu D^\mu + \gamma^5 \tau^3 \gamma_\mu \gamma_\nu E^{\mu\nu} \right] \]

\[ A = M \left( k^2 - M^2 - \frac{1}{4} q^2 \right), \quad B = -M q \cdot k \]

\[ C_\mu = k_\mu \left( k^2 - M^2 + \frac{1}{4} q^2 \right) - \frac{1}{2} q_\mu \cdot q \cdot k \]

\[ D_\mu = -k_\mu \cdot q \cdot k + \frac{1}{2} q_\mu \left( k^2 + M^2 + \frac{1}{4} q^2 \right) \]

\[ E_{\mu\nu} = q_\mu k_\nu M \]

\[ N = \left( k^2 - M^2 - \frac{1}{4} q^2 \right)^2 + q^2 k^2 - (q \cdot k)^2 \]
Spinodials

[Nickel, PRD (2009)]