Equilibration in finite Bose systems

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Topics

1. Introduction

VOLUME 48, NUMBER 15 PHYSICAL REVIEW LETTERS 12 APRIL 1982

Equilibration in Finite Fermion Systems

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(Received 14 October 1981)

A novel nonlinear transport equation for the time-dependent single-particle occupation numbers in an equilibrating fermion system is derived. In the case of constant transport coefficients its analytical solution together with an expression for the equilibration time is obtained. Applications in mean-field theories extended to include particle collisions for the description of low-energy heavy-ion reactions are envisaged.

PACS numbers: 24.60.+m, 24.90.+d, 25.70.Bc

(here at MeV energies, typical for nuclear levels)

Nonlinear partial diff. equation for equilibration in a Fermi system

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \epsilon}\left[v\, n\,(1-n) + n^2 \frac{\partial D}{\partial \epsilon}\right] + D\,\frac{\partial^2 n}{\partial \epsilon^2}$$

$n \equiv n\,(\epsilon, t)$ occupation number probability distribution
$v(\epsilon, t)$ drift coefficient
$D(\epsilon, t)$ diffusion coefficient

The transport coefficients are defined as moments of the transition probability. In the simplified case of constant v and D, the equation becomes

$$\frac{\partial n}{\partial t} = -v\,\frac{\partial}{\partial \epsilon}\left[n\,(1-n)\right] + D\,\frac{\partial^2 n}{\partial \epsilon^2}\ .$$

Although it looks simple, it is difficult to solve analytically due to the nonlinearity. It has the correct Fermi-type equilibrium solution with the temperature T= - D/v :

$$n_{\text{eq}}(\epsilon) = \{1 + \exp[-(v/D)(\epsilon - \epsilon_F)]\}^{-1}$$

The analytical solution of the nonlinear equation...

...is obtained either through a nonlinear transformation and subsequent solution of the resulting linear diffusion equation, or via a linear transformation and solution of the ensuing Burgers' equation.

For a simple theta-function initial distribution $n_{\mathrm{i}}(\epsilon) = \theta(1 - \epsilon/\epsilon_0)$ the analytical result is

$$n(\epsilon,t) = \frac{D}{v}\Bigg\{ \Bigg[e^{\frac{v}{D}(\epsilon_0-\epsilon)}\Bigg(\frac{1}{\sqrt{\pi D t}}\exp\Big(-\frac{(\epsilon_0+a_-)^2}{4Dt}\Big) - \frac{v}{D}\Big[1 - \mathrm{erf}\Big(\frac{\epsilon_0+a_-}{\sqrt{4Dt}}\Big)\Big]\Bigg) - \frac{1}{\sqrt{\pi D t}}\exp\Big(-\frac{(\epsilon_0-a_+)^2}{4Dt}\Big)\Bigg] \Bigg/ \Bigg[\mathrm{erf}\Big(\frac{\epsilon_0-a_+}{\sqrt{4Dt}}\Big) + 1 + e^{\frac{v}{D}(\epsilon_0-\epsilon)}\Big[1 - \mathrm{erf}\Big(\frac{\epsilon_0+a_-}{\sqrt{4Dt}}\Big)\Big]\Bigg]\Bigg\} + 1$$

with $a_+ = vt + \varepsilon$, $a_- = vt - \varepsilon$. The Fermi distribution is the limit for $t \to \infty$.

2. Equilibration in a Fermi system (here at MeV energies)

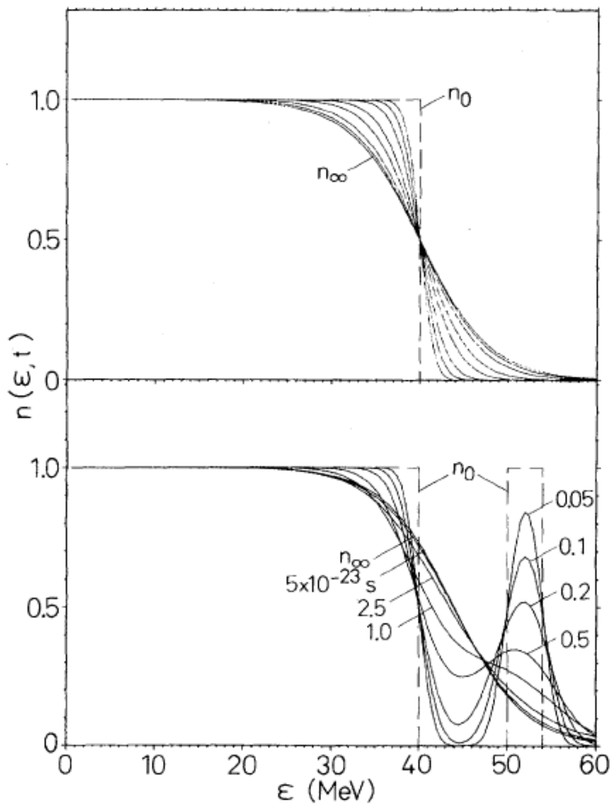


FIG. 1. Analytical solutions of the nonlinear differential equation (7) for the occupation-number distribution in a finite fermion system. The initial distributions (dashed curves) are n_0, the equilibrium distribution is n_∞. The transport coefficients are $D = 20 \times 10^{23}$ MeV^2 s^{-1}, $v = -5 \times 10^{23}$ MeV s^{-1}. Times are in units of 10^{-23} s.

Equilibration in a Fermi system

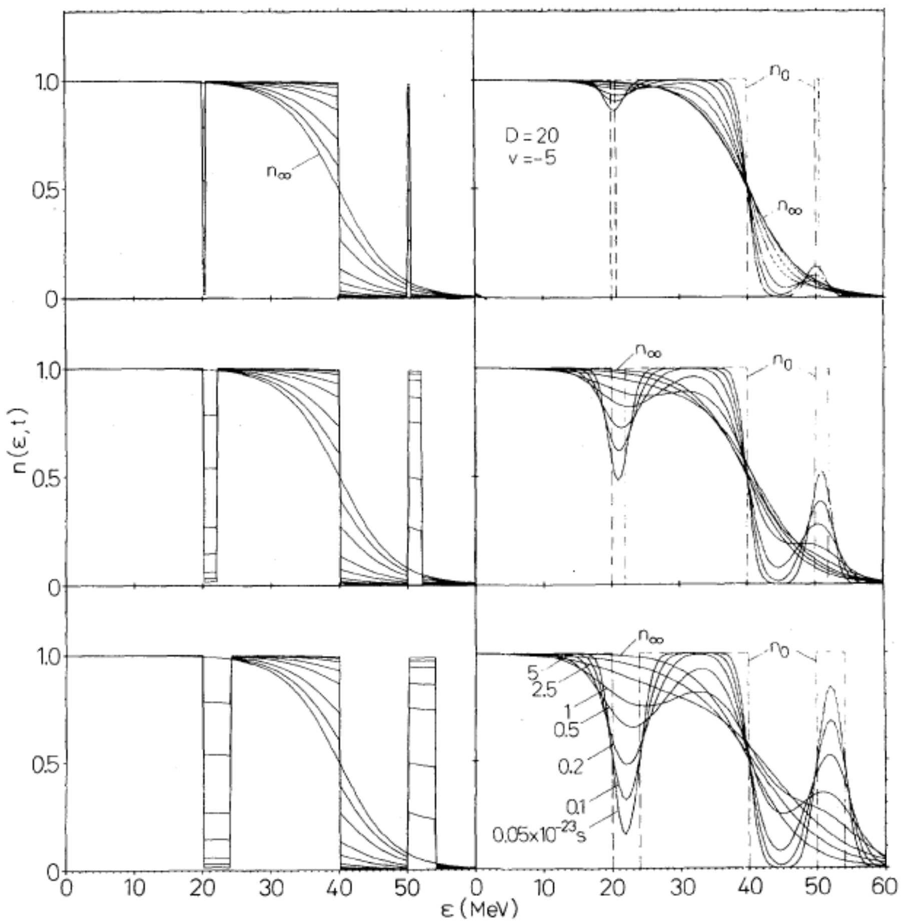
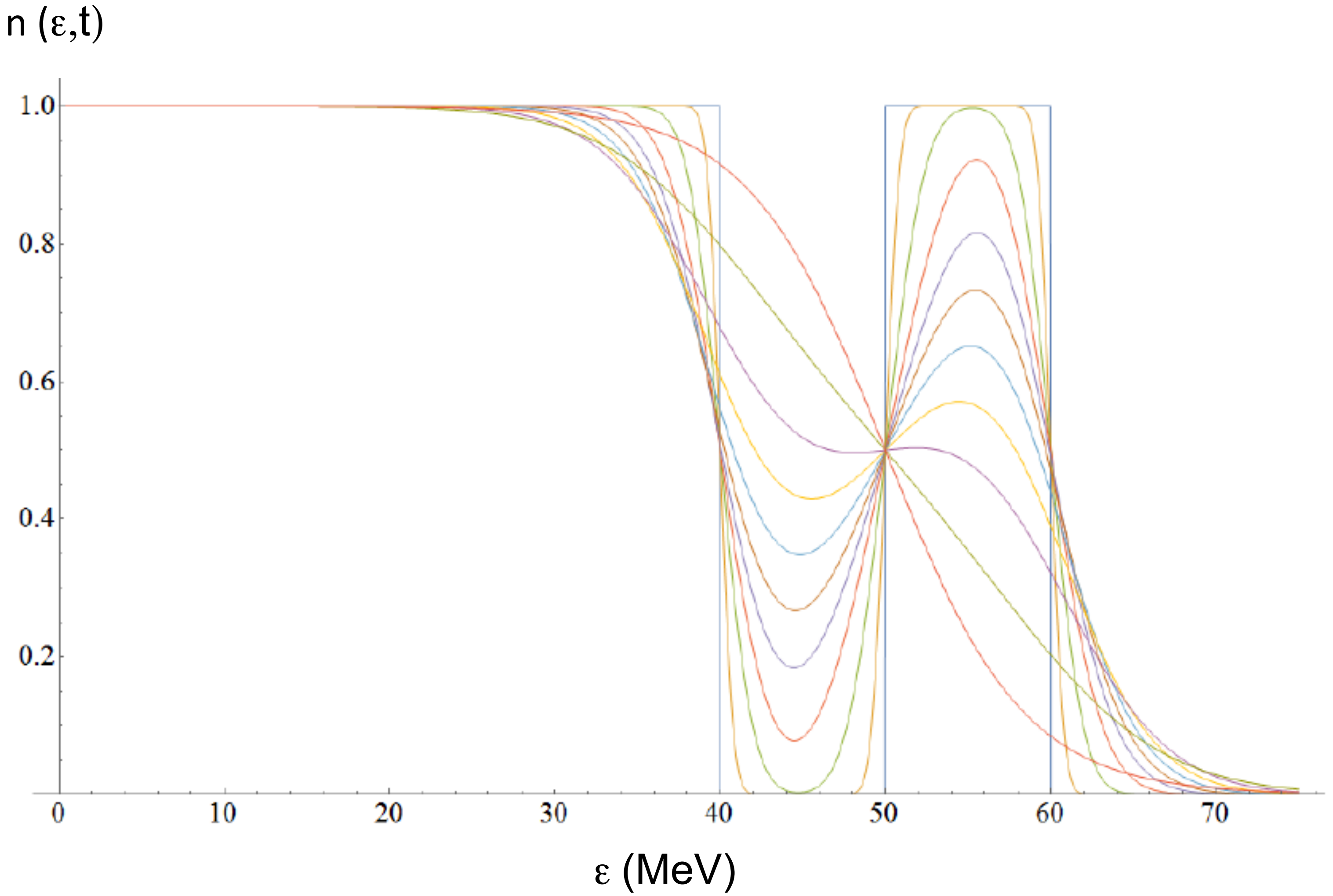


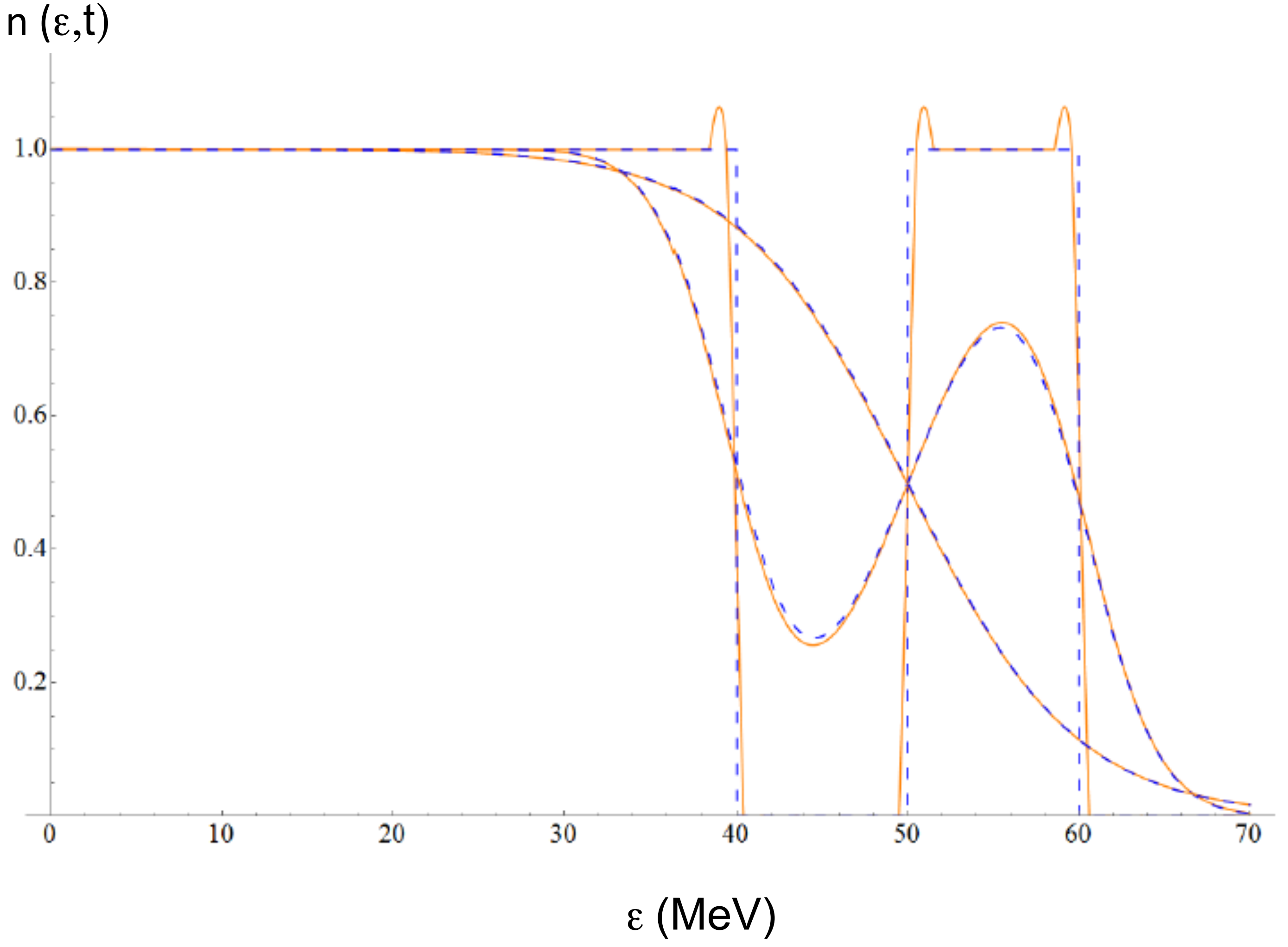
FIG. 2. Comparison of the results for the relaxation *Ansatz* (left-hand side, with $\tau_{equ} = 4D/v^2$) and the analytical solutions of Eq. (7) (right-hand side). Three different initial distributions n_0 are shown. The relaxation *Ansatz* causes a slower equilibration at short times.

Equilibration in a Fermi system (MeV energies)



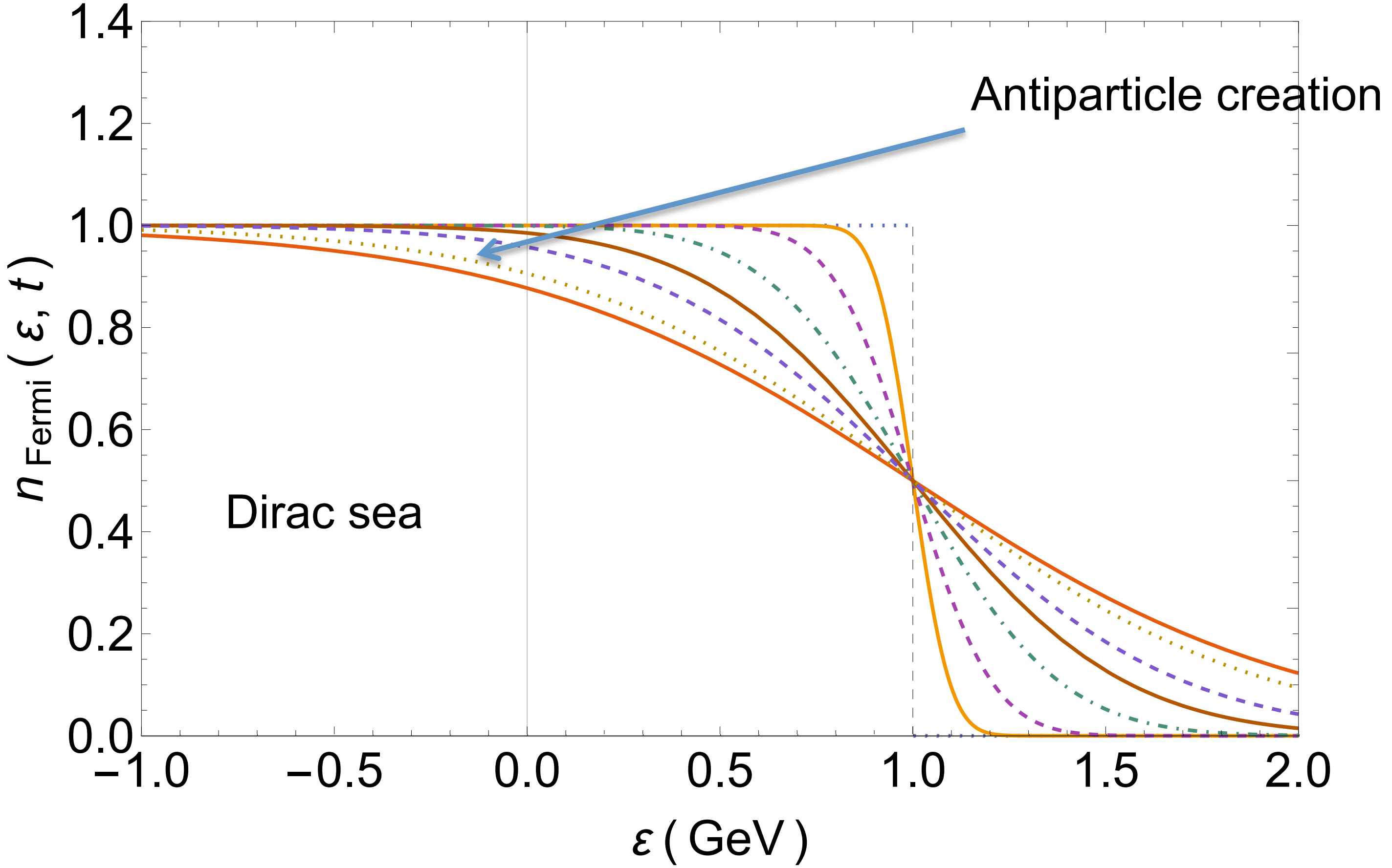
Analytical solutions recalculated by T. Bartsch, BSc student HD 2018

Equilibration in a Fermi system



Comparison of analytical (dashed) and numerical (solid) solutions recalculated by T. Bartsch, BSc student HD 2018

The fermionic solution at LHC energies with antiparticle creation



T=508 MeV

(Analytical solutions, agree with the numerical results)

3. An analytical model for equilibration in a Bose system

3.1 Relaxation ansatz

The system relaxes linearly from the initial nonequilibrium distribution $n_i(\varepsilon)$ towards the Bose-Einstein distribution

$$n_{\mathrm{eq}}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T} - 1}$$

according to

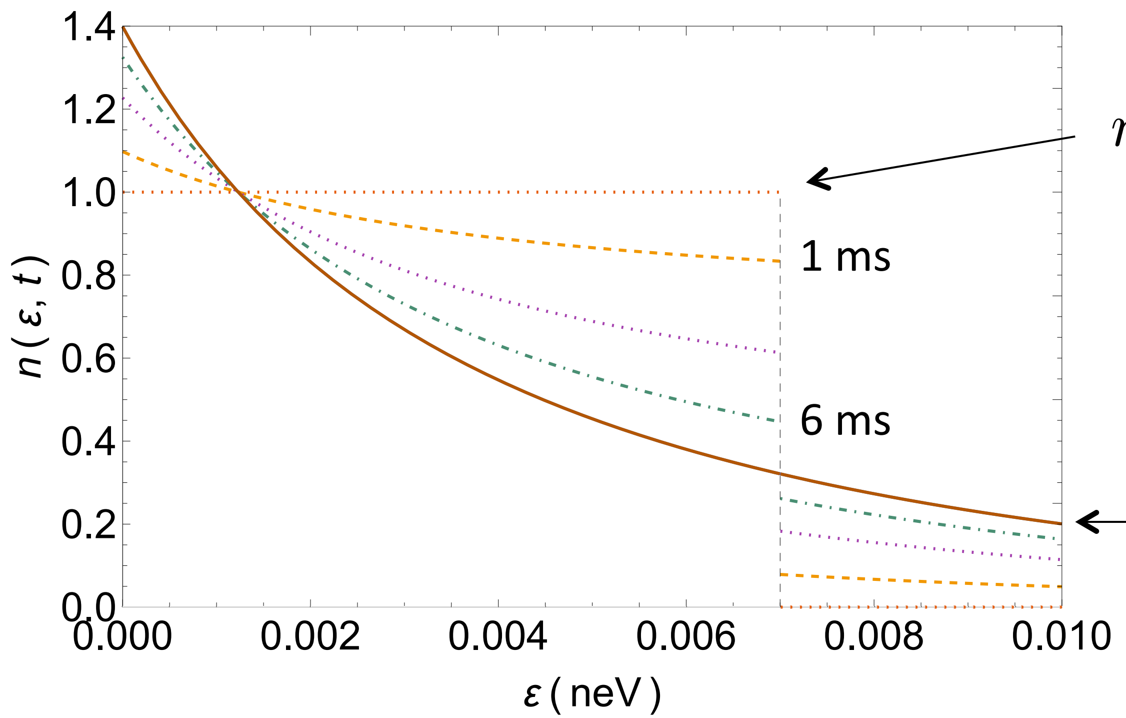
$$\partial\, n_{\mathrm{rel}}/\partial t = (n_{\mathrm{eq}} - n_{\mathrm{rel}})/\tau_{\mathrm{eq}}$$

with the solution

$$n_{\mathrm{rel}}(\epsilon, t) = n_{\mathrm{i}}(\epsilon)\, e^{-t/\tau_{\mathrm{eq}}} + n_{\mathrm{eq}}(\epsilon)(1 - e^{-t/\tau_{\mathrm{eq}}})$$

Apply this to a cold quantum gas (CQG) with a schematic θ-function initial condition in the peV-energy region

Linear relaxation ansatz



$$n_i(\epsilon) = N_i \theta(1 - \epsilon/\epsilon_t)$$

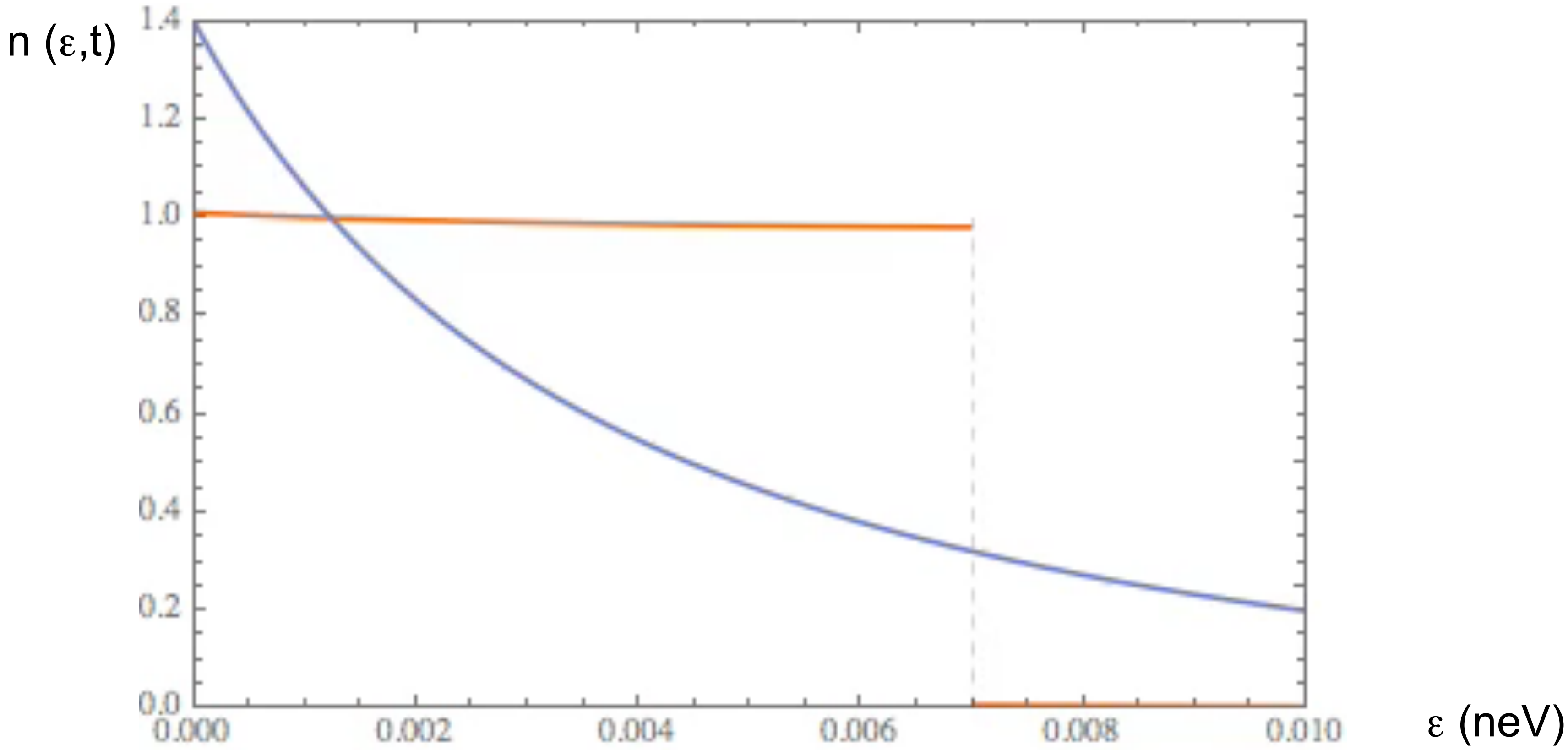
$$n_{\mathrm{eq}}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T} - 1}$$

T= 8 peV ≈ 90 nK < T_c

μ = - 4.3 peV

Bosons equilibrate quickly towards a Bose-Einstein distribution.
(τ_{eq} ~ 9x faster for bosons than for fermions)

Video: Linear relaxation ansatz



T= 8 peV ≈ 90 nK < T_c, τ_{eq}= 3.6 ms

μ = - 4.3 peV

- The time evolution is discontinuous at ε_t
- BEC formation not included
- The evolution is linear

3.2 Derivation of the nonlinear equation

Boltzmann collision term for bosons

$$\frac{\partial n_1}{\partial t} = \sum_{\epsilon_2,\epsilon_3,\epsilon_4}^{\infty} \langle V_{1234}^2 \rangle \, G\left(\epsilon_1+\epsilon_2, \epsilon_3+\epsilon_4\right) \times \left[(1+n_1)(1+n_2)\, n_3\, n_4 - (1+n_3)(1+n_4)\, n_1\, n_2\right]$$

$\langle V_{1234}^2 \rangle$ second moment of the interaction

$G\left(\epsilon_1+\epsilon_2, \epsilon_3+\epsilon_4\right)$ energy-conserving function

$\rightarrow \pi\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)$ in infinite systems

$n_j \equiv n\left(\epsilon_j, t\right)$ occupation number

The Bose-Einstein distribution is a stationary solution

$$n_{\mathrm{eq}}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T} - 1}$$

Derivation of the nonlinear equation

Write the collision term in form of a Master equation (ME) with gain- and loss term

$$\boxed{\frac{\partial n_1}{\partial t} = (1+n_1)\sum_{\epsilon_4} W_{4\to 1}\, n_4 - n_1 \sum_{\epsilon_4} W_{1\to 4}(1+n_4)}$$

with the transition probability ($W_{1\to 4}$ accordingly)

$$W_{4\to 1} = \sum_{\epsilon_2,\epsilon_3} \langle V_{1234}^2\rangle\, G\left(\epsilon_1+\epsilon_2, \epsilon_3+\epsilon_4\right)\left(1+n_2\right) n_3$$

Introduce the density of states $g_j = g(\varepsilon_j)$

$$W_{4\to 1} = W_{41} g_1, W_{1\to 4} = W_{14} g_4$$

$$W_{14} = W_{41} = W\left[\tfrac{1}{2}(\epsilon_4+\epsilon_1), |\epsilon_4-\epsilon_1|\right]$$

W is peaked at $\varepsilon_1 = \varepsilon_4$. Obtain an approximation to the ME through a Taylor expansion of n_4 and $g_4 n_4$ around $\varepsilon_1 = \varepsilon_4$ to second order.

Introduce transport coefficients via moments of the transition probability ($x=\varepsilon_4-\varepsilon_1$)

$$D = \frac{1}{2}\, g_1 \int_0^{\infty} W(\epsilon_1, x)\, x^2 dx, \quad v = g_1^{-1} \frac{d}{d\epsilon_1}(g_1 D)$$

and arrive at the nonlinear partial differential equation for the distribution of the occupation numbers $n \equiv n_{\rm th}(\epsilon, t) \equiv n_{\rm th}(\epsilon_1, t)$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \epsilon}\left[v\, n\, (1+n) - n^2 \frac{\partial D}{\partial \epsilon}\right] + \frac{\partial^2}{\partial \epsilon^2}\left[D\, n\right].$$

Dissipative effects are expressed through the drift term $v(\epsilon, t)$, diffusive effects through the diffusion term $D(\epsilon, t)$.

In the limit of constant transport coefficients, the nonlinear boson diffusion equation for the occupation-number distribution becomes

$$\boxed{\frac{\partial n}{\partial t} = -v\, \frac{\partial}{\partial \epsilon}\left[n\,(1+n)\right] + D\, \frac{\partial^2 n}{\partial \epsilon^2}}$$

The Bose-Einstein distribution $n_{eq}(\varepsilon)$ is a stationary solution of this equation with the equilibrium temperature

$$T = -D/v \text{ with } v < 0$$

(the drift is towards the infrared region).

For fixed equilibrium temperature T, the nonlinear evolution pushes a certain fraction of particles from the thermal cloud into the condensate, provided T is below T_c.

The nonlinear boson equation can also be written in the form of a continuity equation

$$\frac{\partial n}{\partial t} + \frac{1}{g(\epsilon)} \frac{\partial j}{\partial \epsilon} = 0$$

with the probability current

$$\boxed{j(\epsilon, t) = g(\epsilon) \left[v\, n\, (1 + n) - D\, \frac{\partial n}{\partial \epsilon} \right].}$$

At $\varepsilon = 0$, this corresponds to the flow of occupation probability from the thermal cloud into the condensate if the sign of the current is negative, and from the condensate into the thermal cloud if the sign is positive.

The stationary state – that replaces the thermal equilibrium solution – is reached for $t = \tau_{stat}$, which can be computed from the condition

$$v\, n(0, \tau_{\mathrm{stat}})\, [1 + n(0, \tau_{\mathrm{stat}})] = D\, \frac{\partial n(0, \tau_{\mathrm{stat}})}{\partial \epsilon}\,.$$

Overall **particle number** is conserved, if both the particles in the thermal cloud plus the ones in the condensed state are considered

$$N_{\mathrm{tot}} = N_{\mathrm{th}}(t) + N_{\mathrm{c}}(t)$$

with the time-dependent particle number in the thermal cloud

$$N_{\mathrm{th}}(t) = \int_0^{\infty} n\,(\epsilon, t)\, g\,(\epsilon)\, d\epsilon$$

and the density of states

$$g\,(\epsilon) = (2m)^{3/2} V\, \sqrt{\epsilon}\,/(4\pi^2)$$

3.3 Solution of the nonlinear equation

The transformation

$$n(\epsilon,t) = -\frac{D}{vP(\epsilon,t)}\frac{\partial P(\epsilon,t)}{\partial \epsilon}$$

reduces the nonlinear boson equation to a linear diffusion equation for P(ε,t)

$$P_t = -vP_\epsilon + DP_{\epsilon\epsilon}$$

Alternatively, the linear transformation

$$\boxed{n(\epsilon,t) = \frac{1}{2v}w(\epsilon,t) - \frac{1}{2}}$$

yields the nonlinear Burgers' equation for w(ε,t)

$$w_t + ww_\epsilon = Dw_{\epsilon\epsilon}.$$

It can be solved using Hopf's transformation

$$w(\epsilon,t) = -2D\phi_\epsilon/\phi$$

This reduces Burgers‘ equation to the heat equation

$$\phi_t = D\phi_{\epsilon\epsilon}.$$

Solving it and transforming back results in the final solution

$$\boxed{n(\epsilon,t) = \frac{\int_{-\infty}^{+\infty} \left(\frac{\epsilon - x}{2vt} - \frac{1}{2}\right) F(\epsilon - x, t)\, G(x)\, dx}{\int_{-\infty}^{+\infty} F(\epsilon - x, t)\, G(x)\, dx}}$$

with a gaussian part that arises from the heat equation

$$F(\epsilon - x, t) = \exp\left[-\frac{(\epsilon - x)^2}{4Dt}\right]$$

and an exponential function that contains an integral over the initial distribution $n_i(y)$

$$G(x) = \exp\left[-\frac{1}{2D}\left(vx + 2v\int_0^x n_{\mathrm{i}}(y)\, dy\right)\right]$$

3.4 Application to a cold quantum gas:

Exact analytical solution for θ-function initial condition

$$\boxed{n\,(\epsilon,t) = \frac{1}{2v} \times \left[\frac{n_a(\epsilon,t)+n_b(\epsilon,t)+n_c(\epsilon,t)}{n_d(\epsilon,t)+n_e(\epsilon,t)+n_f(\epsilon,t)}\right] - \frac{1}{2}}$$

$$n_a(\epsilon,t) = \exp\left[\frac{1}{2D}(v^2t/2 - v\epsilon)\right] \times \left[v\sqrt{\pi D t}\left[1+\mathrm{erf}\,(u_0(\epsilon,t))\right] + 2D\exp[-(u_0(\epsilon,t))^2]\right],$$

$$n_b(\epsilon,t) = \exp\left[\frac{1}{2D}(9v^2t/2 - 3v\epsilon)\right] \times \left[3v\sqrt{\pi D t}\left[\mathrm{erf}\,(u_2(\epsilon,t)) - \mathrm{erf}\,(u_1(\epsilon,t))\right]\right.$$
$$\left. + 2D\left[\exp[-(u_2(\epsilon,t))^2] - \exp[-(u_1(\epsilon,t))^2]\right]\right],$$

$$n_c(\epsilon,t) = \exp\left[\frac{1}{2D}(v^2t/2 - v\epsilon - 2v\,\epsilon_t)\right] \times \left[v\sqrt{\pi D t}\left[1-\mathrm{erf}\,(u_3(\epsilon,t))\right]\right.$$
$$\left. - 2D\exp[-(u_3(\epsilon,t))^2]\right],$$

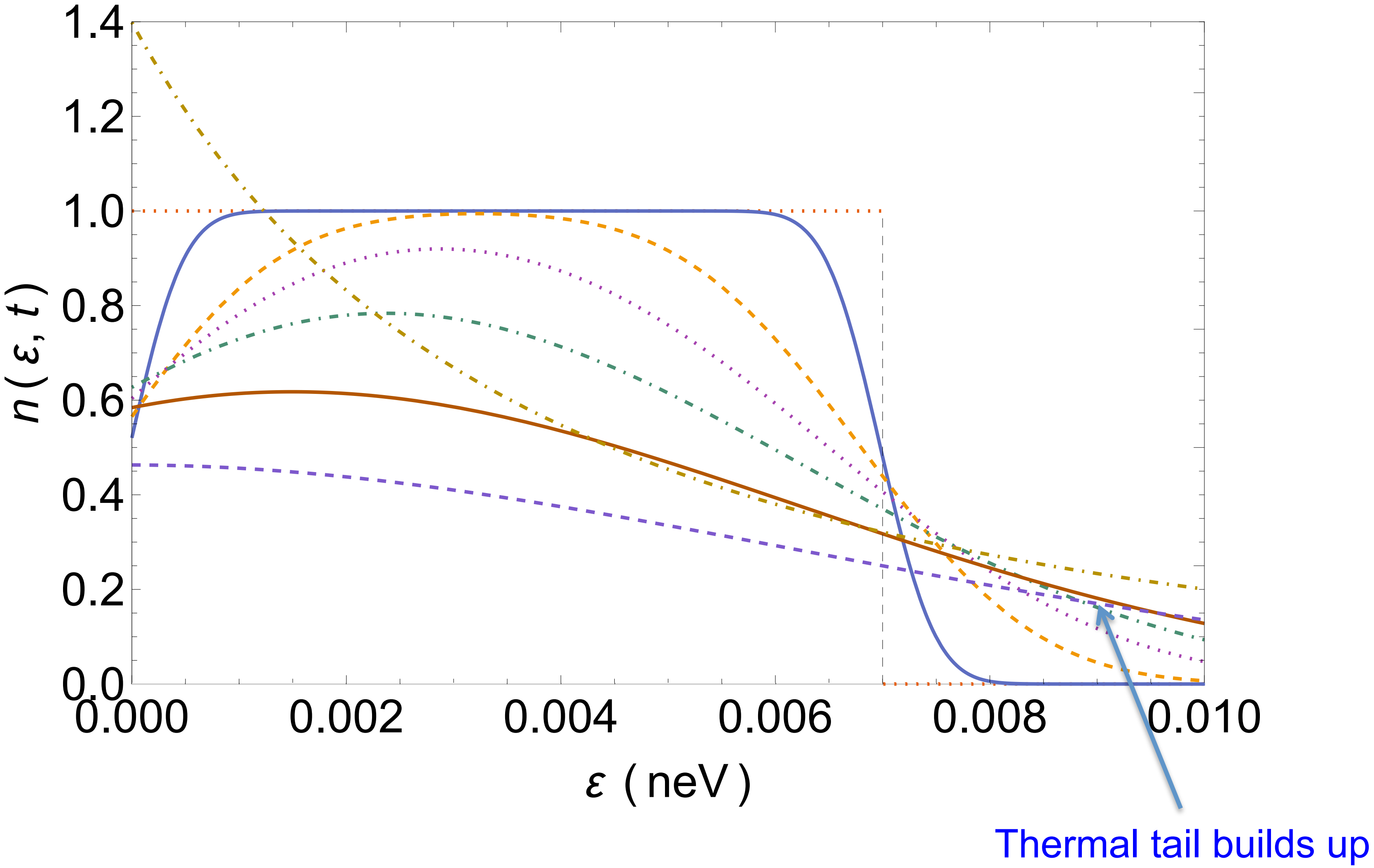
$$
\begin{aligned}
n_d(\epsilon,t) &= \sqrt{\pi D t}\,\exp\Big[\frac{1}{2D}(v^2 t/2 - v\epsilon)\Big] \times \Big[1 + \mathrm{erf}\,(u_0(\epsilon,t))\Big]\,,\\
n_e(\epsilon,t) &= \sqrt{\pi D t}\,\exp\Big[\frac{1}{2D}(9v^2 t/2 - 3v\epsilon)\Big] \times \Big[\mathrm{erf}\,(u_2(\epsilon,t)) - \mathrm{erf}\,(u_1(\epsilon,t))\Big]\,,\\
n_f(\epsilon,t) &= \sqrt{\pi D t}\,\exp\Big[\frac{1}{2D}(v^2 t/2 - v\epsilon - 2v\,\epsilon_t)\Big] \times \Big[1 - \mathrm{erf}\,(u_3(\epsilon,t))\Big]\,.
\end{aligned}
$$

with the auxiliary functions

$$
\begin{aligned}
u_0(\epsilon,t) &= \frac{1}{2\sqrt{Dt}}(-\epsilon + vt)\,,\\
u_1(\epsilon,t) &= \frac{1}{2\sqrt{Dt}}(-\epsilon + 3vt)\,,\\
u_2(\epsilon,t) &= \frac{1}{2\sqrt{Dt}}(\epsilon_t - \epsilon + 3vt)\,,\\
u_3(\epsilon,t) &= \frac{1}{2\sqrt{Dt}}(\epsilon_t - \epsilon + vt)\,.
\end{aligned}
$$

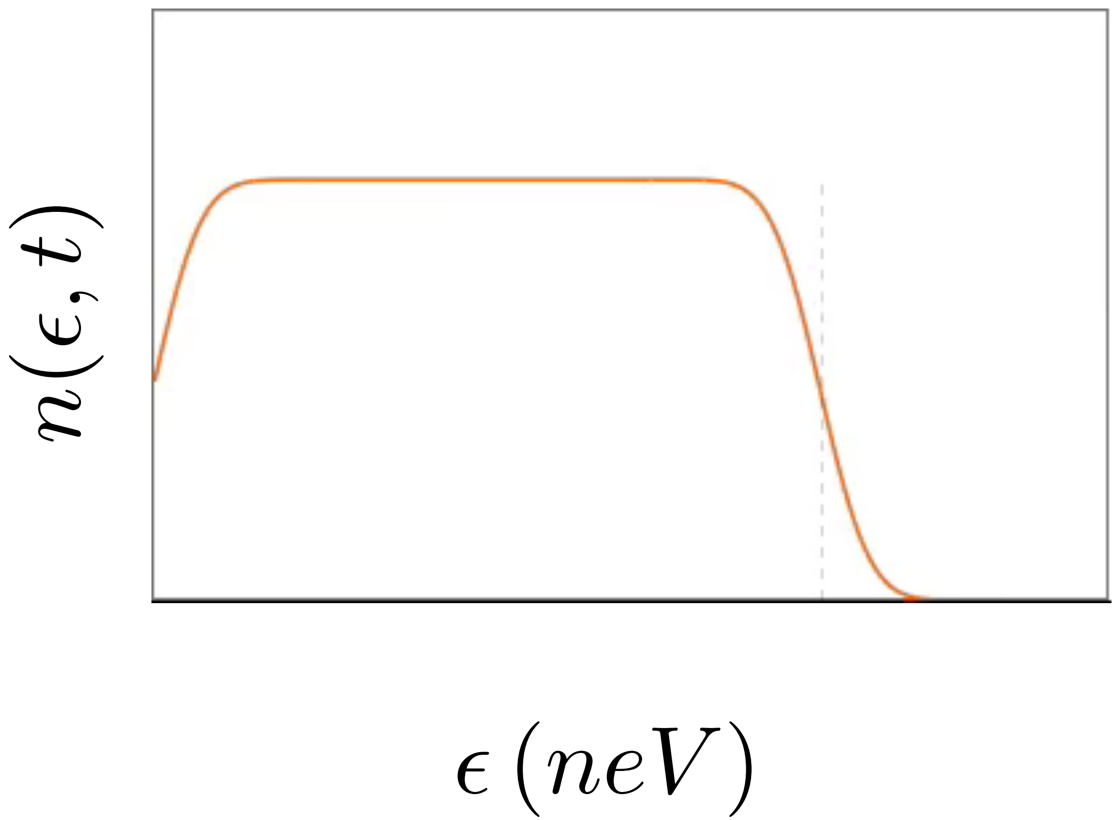
The analytical result agrees with the numerical solution.

Nonlinear equilibration of a bosonic CQG for θ-function initial condition



$t = 0.01 - 2$ ms, $\tau_{eq} = 4D/(9v^2) = 3.6$ ms, $T = 8$ peV ≈ 90 nK

Video: Nonlinear time evolution of the boson distribution



Equilibration time for bosons vs. fermions

An explicit expression for the bosonic equilibration time follows from an asymptotic expansion of the error functions occuring in the solutions

$$\mathrm{erf}\,(z_{\mathrm{b}}) \simeq 1 - \frac{1}{\sqrt{\pi}\, z_{\mathrm{b}}} \exp[-z_{\mathrm{b}}^2] + \exp(-z_{\mathrm{b}}^2)\, \mathcal{O}\Big(\frac{1}{z_{\mathrm{b}}^3}\Big)$$

with argument z_b at the boundary $x_b = \varepsilon_t$

$$z_{\mathrm{b}} = \frac{1}{2\sqrt{Dt}} [x_{\mathrm{b}} - \epsilon + (1 + 2N_{\mathrm{i}})\, vt\,] \,.$$

Deviations from the thermal solution thus scale with

$$\exp[-(1 + 2N_{\mathrm{i}})^2 v^2 t/(4D)] \equiv \exp[-t/\tau_{\mathrm{eq}}]$$

and the equilibration time in a Bose system becomes for $N_i = 1$

$$\boxed{\tau_{\mathrm{eq}}^{\mathrm{Bose}} = 4D/(9v^2) = \tau_{\mathrm{eq}}^{\mathrm{Fermi}}/9}$$

4. Application to CQG: Evaporative cooling

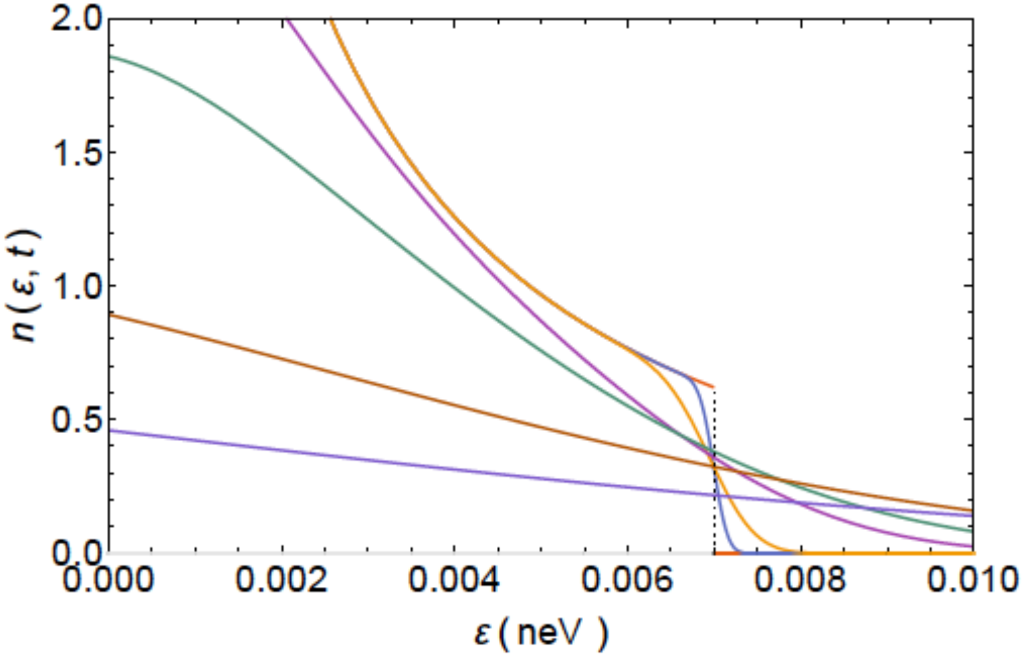


Figure 1: (color online) Equilibration of a finite Bose system based on the nonlinear evolution according to Eq. (7) starting from a truncated equilibrium distribution as in evaporative cooling Eq. (19), upper curve with cutoff at $\epsilon_i = 7\,\text{peV}$. The transport coefficients are $D = 8 \times 10^{-3}\,\text{neV}^2\,\text{s}^{-1}$, $v = -1\,\text{neV}\,\text{s}^{-1}$. The temperature $T = -D/v = 8 \times 10^{-3}\,\text{neV} \simeq 93\,\text{nK}$ is below the critical value for ^{87}Rb. The time sequence is $0.001, 0.01, 0.2, 0.5, 2$ and 5 ms (top to bottom) with the equilibration time $\tau_{\text{eq}} = 4D/(9v^2) \simeq 3.6\,\text{ms}$. The nonequilibrium occupation drops below the thermal equilibrium values because the particles are redistributed into the BEC ground state in the IR, and into a new UV thermal tail.

Integrands for an initial truncated BE distribution

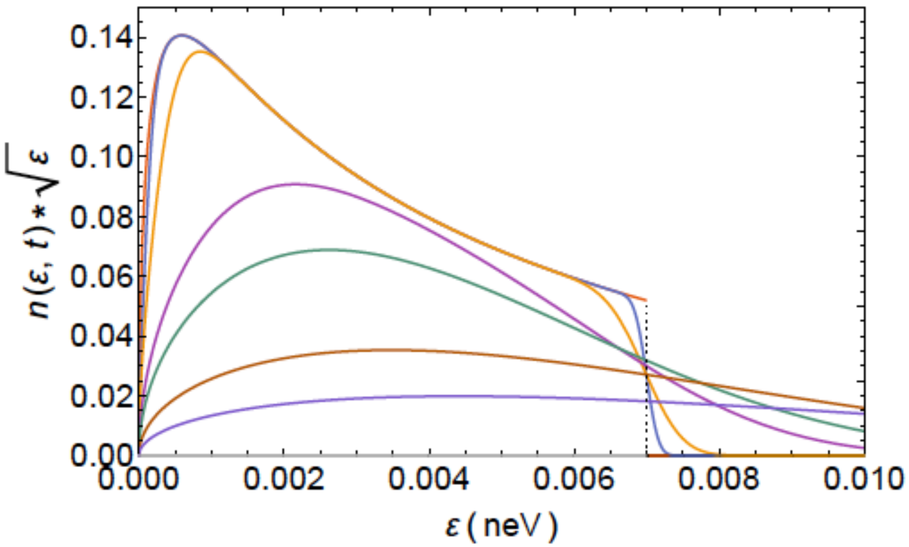


Figure 2: (color online) Integrands $n(\epsilon, t)\sqrt{\epsilon}$ for the nonlinear evolution according to Eq. (7) with initial condition Eq. (19), upper curve with cutoff at $\epsilon_i = 7$ peV. The integrands are shown at six values of time t from 0.001 ms to 5 ms, top to bottom, as in Fig. 1. The integrated particle number Eq. (20) in the nonequilibrium thermal cloud is not conserved during the time evolution since particles in the IR move into the $\epsilon = 0$ condensate. With increasing time a new nonequilibrium thermal tail develops in the UV.

- The particle content in the thermal cloud is reduced with time because particles move into the condensed state.
- The discontinuity at $\varepsilon = \varepsilon_t$ disappears and a thermal tail develops within the equilibration time τ_{eq} .
- The time evolution is nonlinear

5. Summary and Conclusion

- From the bosonic Boltzmann collision term a nonlinear partial differential equation for the time-dependent occupation-number distribution in a finite Bose system has been derived.

- The resulting nonlinear bosonic diffusion equation has been solved analytically.

- The solution has been applied to a cold quantum gas with a schematic initial distribution, and to evaporative cooling with a truncated equilibrium distribution. The flow from the thermal cloud into the condensate has been determined.

- The equilibration time in a Bose system has been calculated analytically within the nonlinear model, and is found to be ~ one order of magnitude shorter than for fermions due to the statistical properties.

Thank you for your attention !