Three-body unitarity in the Finite Volume

Michael Döring
The George Washington University

Collaboration:
3-body: M. Mai, J-Y. Pang, B. Hu, H.W. Hammer, A. Rusetsky, A. Szczepaniak, A. Pilloni
2-body: M. Mai, R. Molina, B. Hu, D. Guo, A. Alexandru

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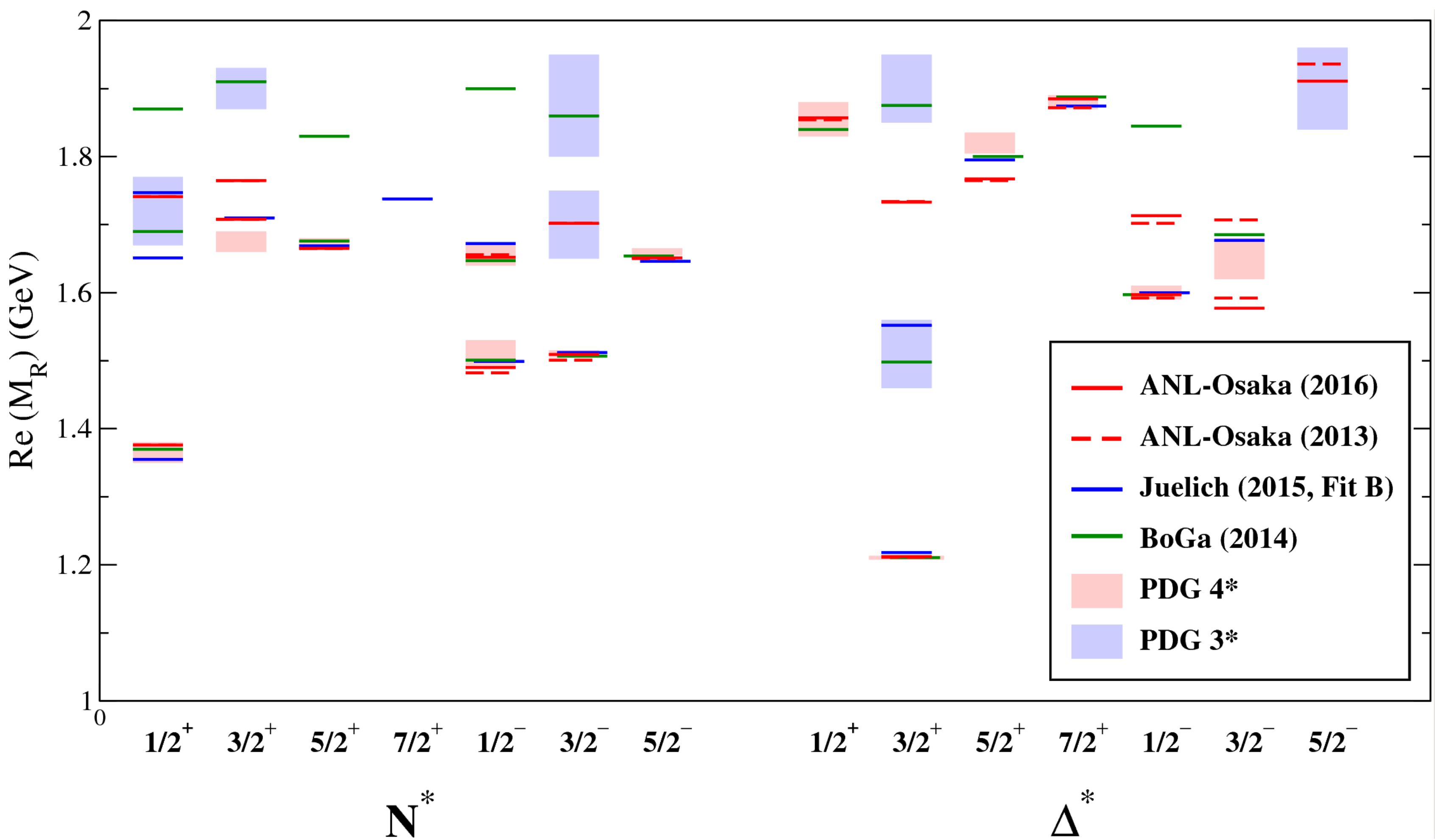
NSF PIF 1415459 & CAREER PHY-1452055

[Many slides from Maxim Mai]

- **QCD** at low energies → *mass generation & confinement*
- Non-perturbative dynamics → rich spectrum of excited states

Q1: how many are there? (missing resonance problem)

Q2: what are they? (2-quark/3-quark, hadron molecules, …)



[slide: ANL/Osaka Kamano@N*2017]

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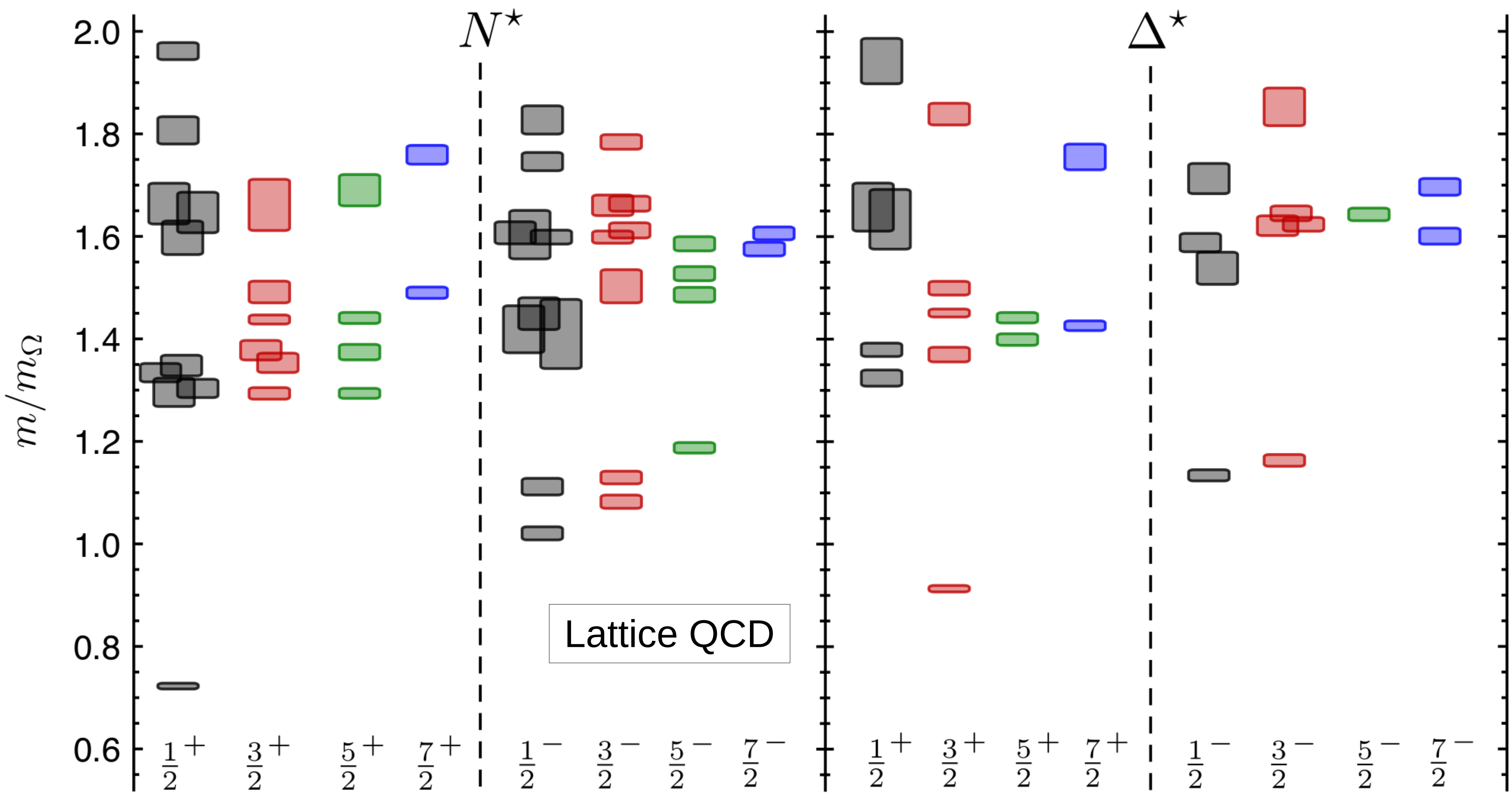

$M_\pi = 396$ MeV

Edwards et al. (2011)

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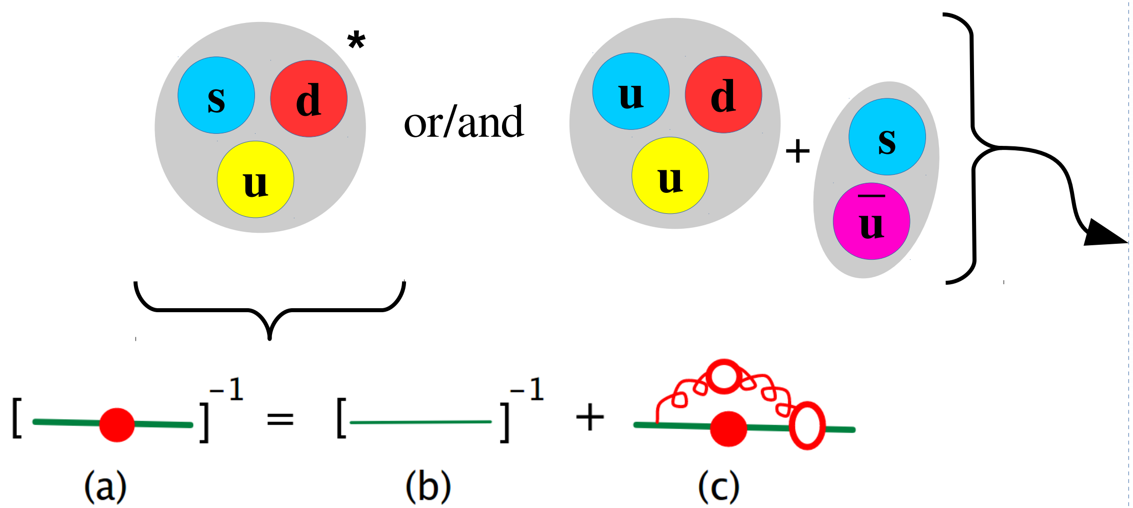
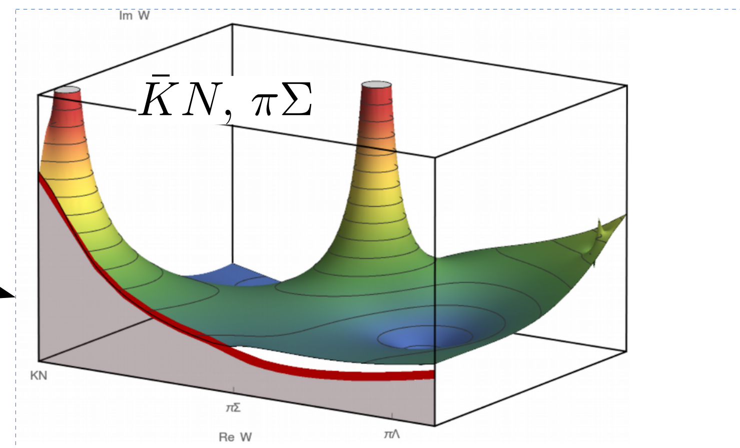
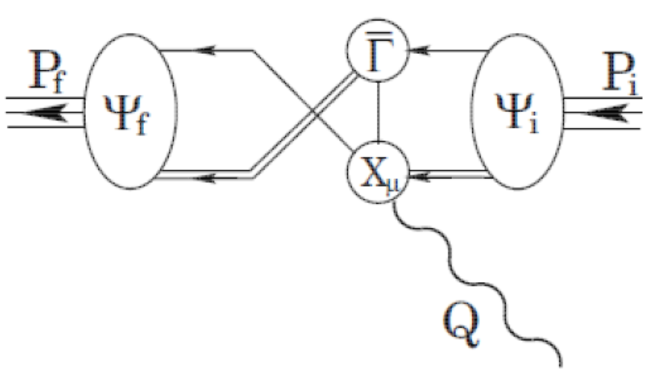


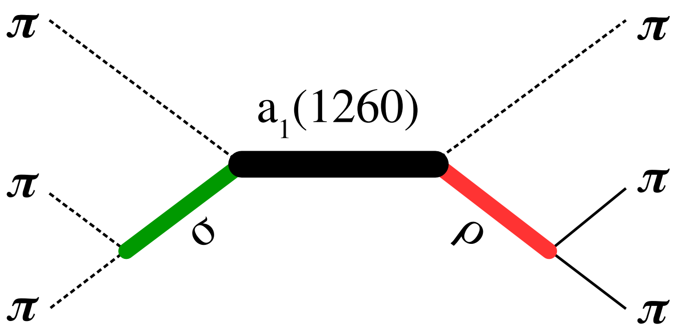


Oset, Ramos (1998)
Mai, Meißner (2015)
Cieply, MM, Meissner, Smejkal (2016)



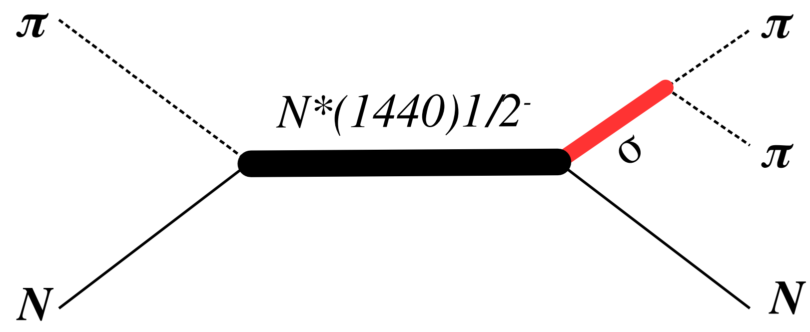
DSE (Wilson, Cloet, Chang, Roberts)

- Dynamics is important! Many states have dominant 3-body content



- important channel in GlueX @ JLab

- Finite volume spectrum from lattice QCD
Lang, Leskovec, Mohler, Prelovsek (2014)



- Roper is debated for ~50 years
- first Lattice QCD results:
  - Detection on lattice notoriously difficult
- 1st simulation w. meson-baryon operators:
- Finite volume spectrum **Lang et al. (2017)**

- Universal understanding of Lattice QCD or experimental searches (BESIII, COMPASS, GlueX)

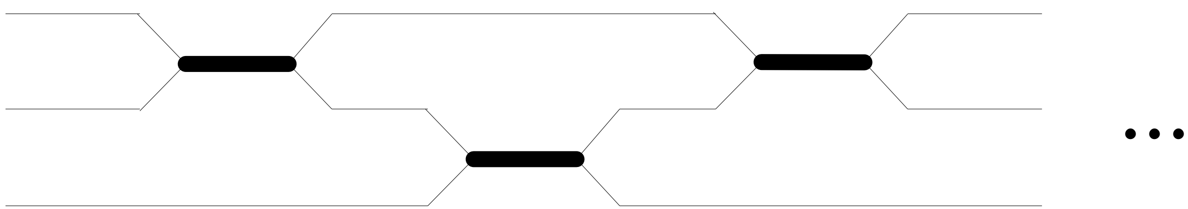
  → theory of 3-body scattering problem

- **Available tools:**
  - *Faddeev equations (F.E.)* **Faddeev(1959)**
  - F.E. in fixed-center approximation **Brueckner(1953)**

    → usefull for ***πd***, ***Kd*** … systems **Baru et al(2011) Mai et al. (2015)**

  - F.E. in isobar formulation **Omnes(1964) Aaron(1967)**

    → re-parametrization of two-body amplitude **Bedaque(1999)**



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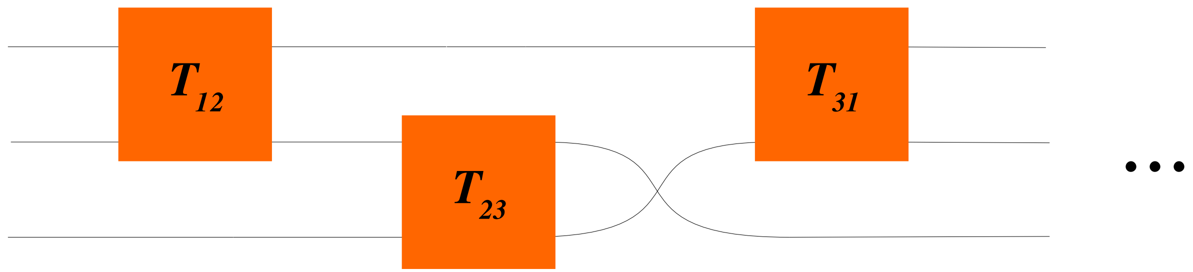


FADDEEV EQUATIONS WITH ISOBARS

M. Mai, Hu, M. D., Pilloni, Szczepaniak

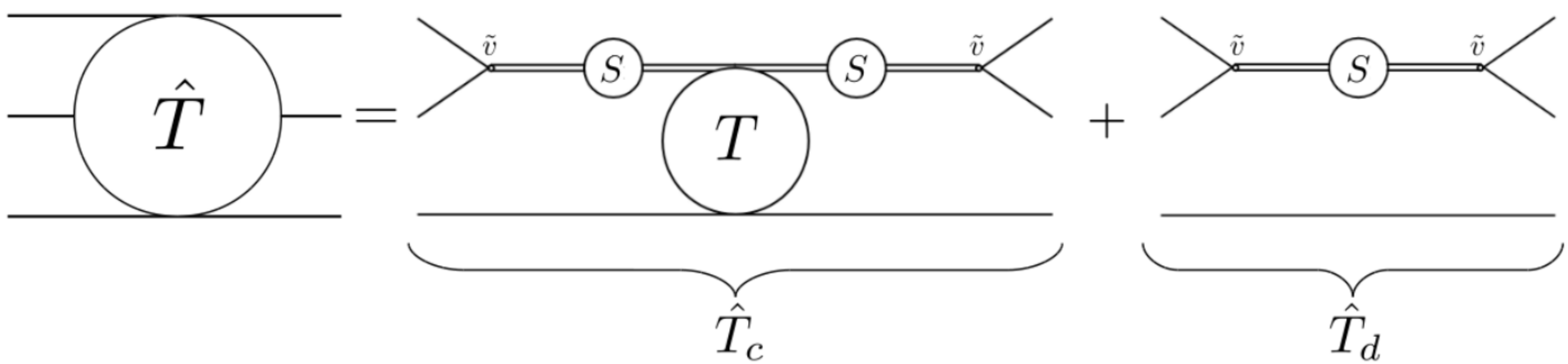
Eur.Phys.J. A53 (2017) 177

FE in isobar parametrization

Original study by Amado/Aaron/Young **AAY(1968)**

- 3-dimensional integral equation from unitarity constraint & BSE ansatz
- valid below break-up energies ($E < 3m$) & analyticity constraints unclear

One has to begin with asymptotic states



- v a general function <u>without cuts in the phys. region</u>
- two-body interaction is parametrized by an *"isobar"*

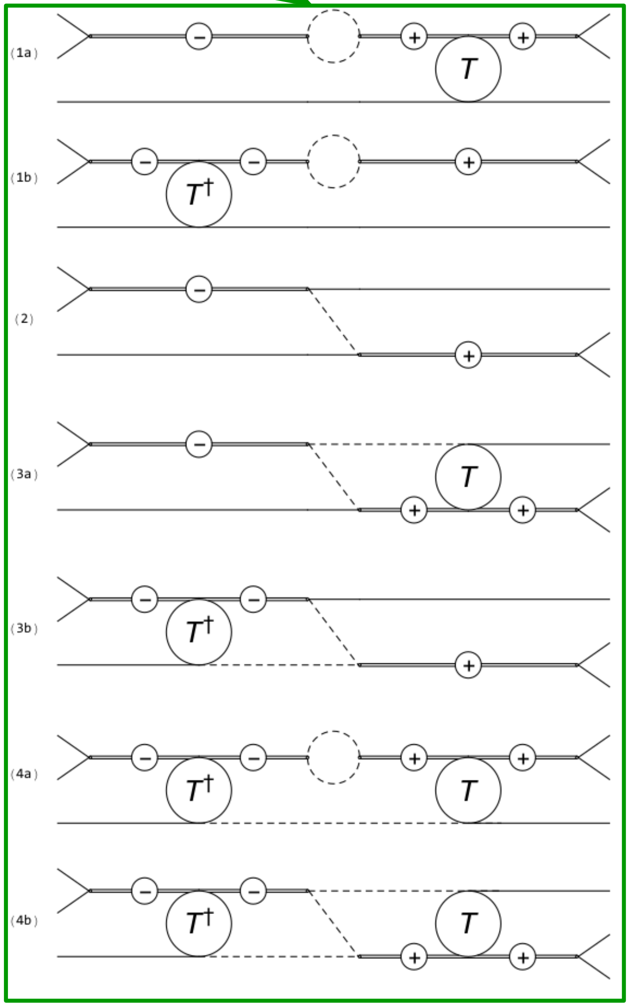
= has definite QN and correct r.h.-singularities w.r.t invariant mass

- S and T are yet unknown functions

Unitarity & Matching

3-body unitarity (normalization condition ↔ phase space integral)

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i \int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



Unitarity & Matching

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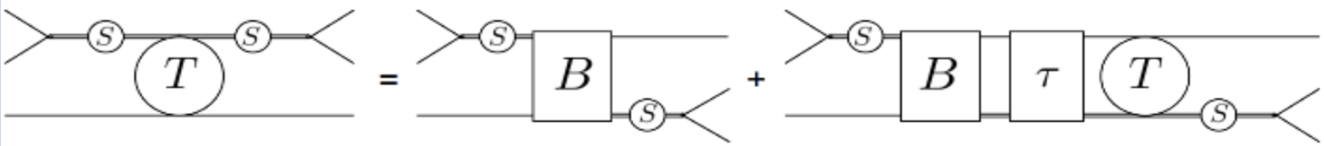
$$\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle \quad = \quad i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle$$

General ansatz for the Isobar-spectator interaction

→ B & τ are unknown!!!



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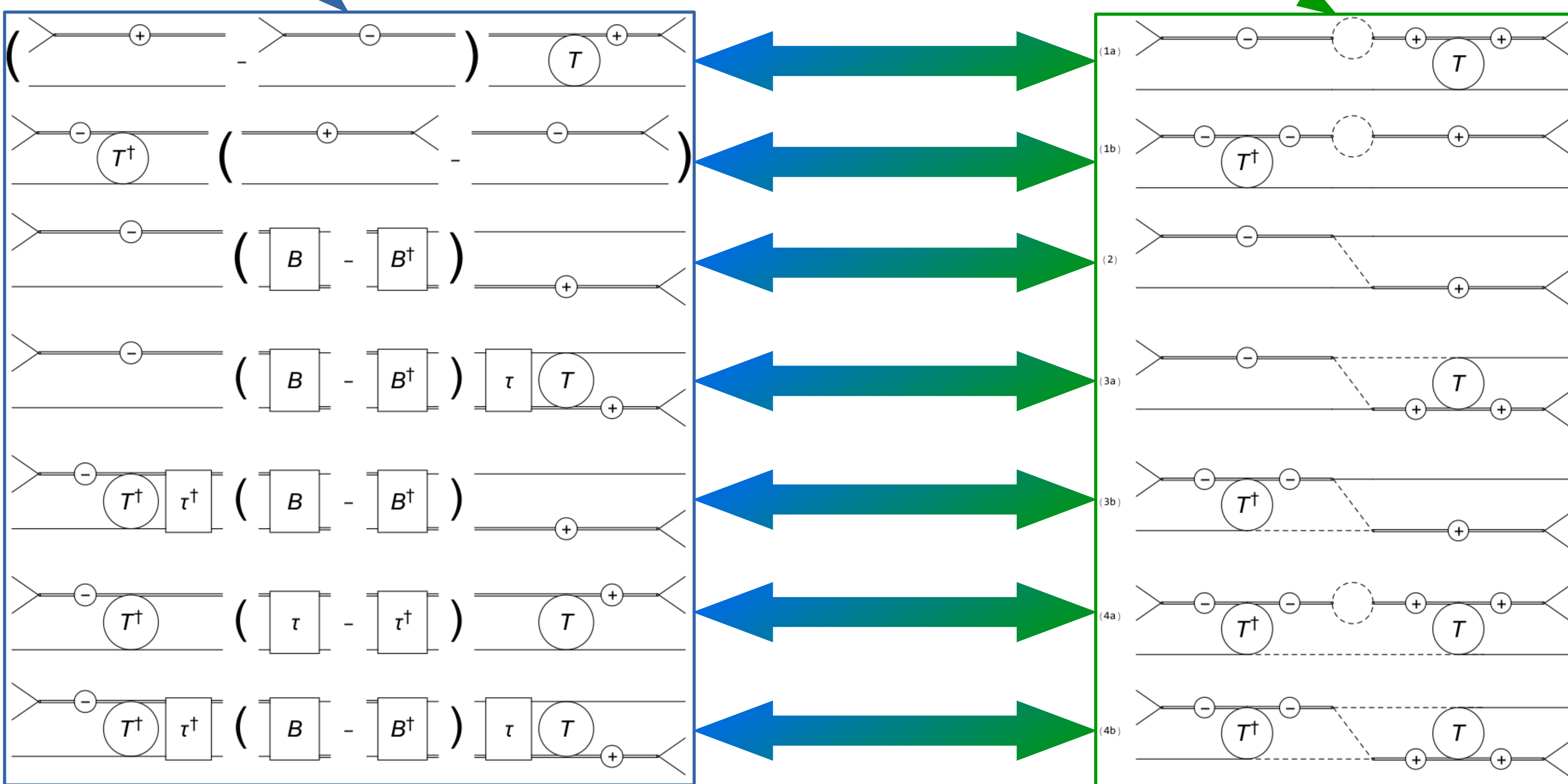
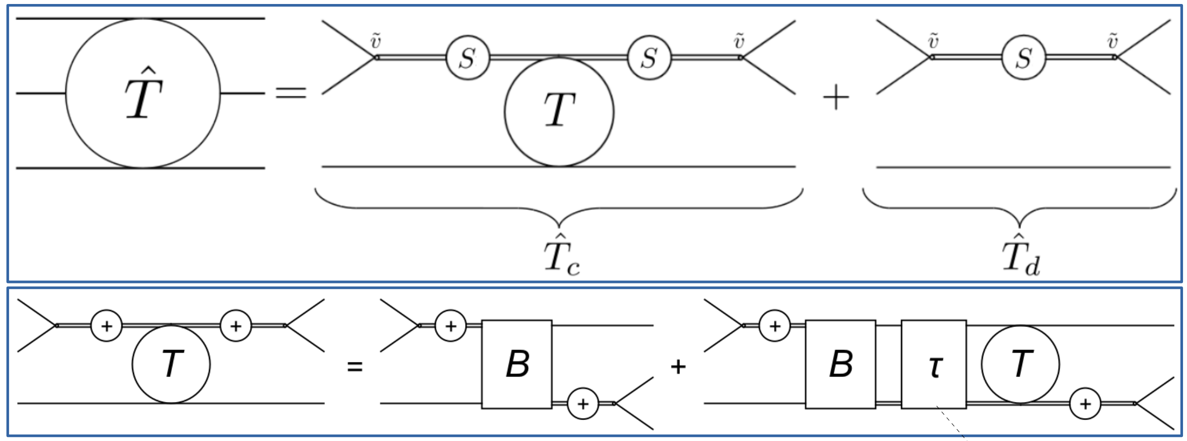


SCATTERING AMPLITUDE

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***v=λ*** (full relations available)

$$\tau(\sigma(k)) = (2\pi)\delta^+(k^2 - m^2)S(\sigma(k))$$

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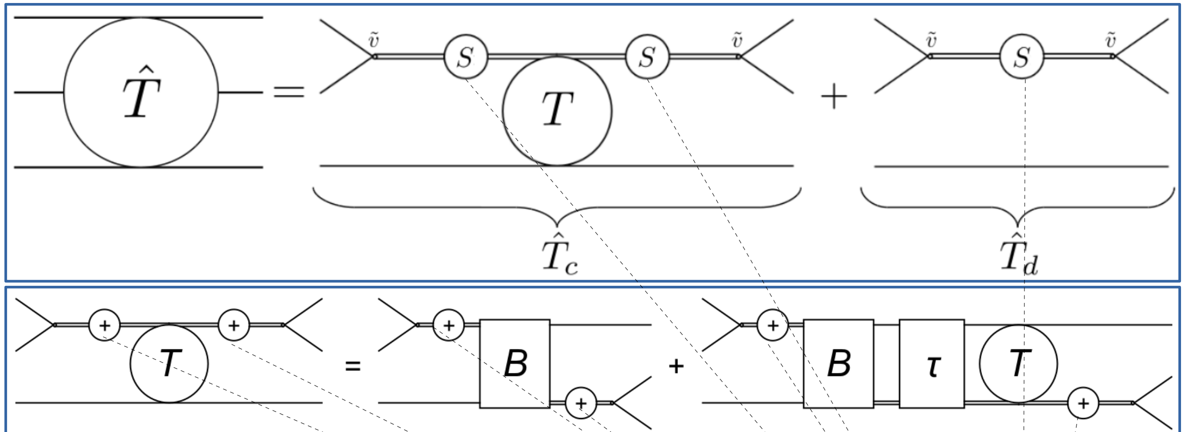

– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***υ=λ*** (full relations available)

$$\mathrm{Disc}\,\frac{1}{S} = -\frac{i}{8\pi}\frac{K_{\mathrm{cm}}}{\sqrt{\sigma(k)}}\lambda^2$$

- twice subtracted dispersion relation in invariant mass - ***σ(k)***

$$-\frac{1}{S} = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3}\int d^3\boldsymbol{\ell}\,\frac{\lambda^2}{2E_\ell(\sigma(k)-4E_\ell^2+i\epsilon)}$$

- in the rest-frame of isobar (*Lorentz invariance!*)

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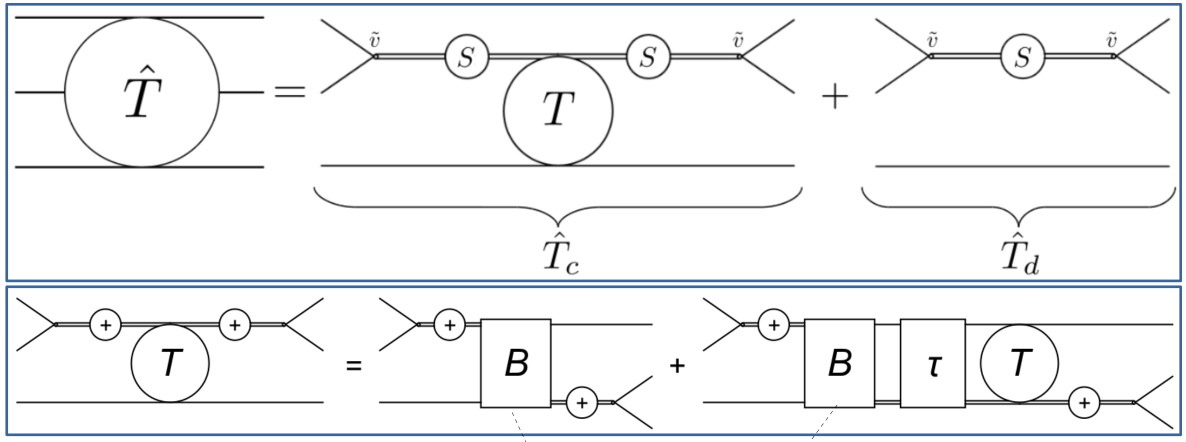


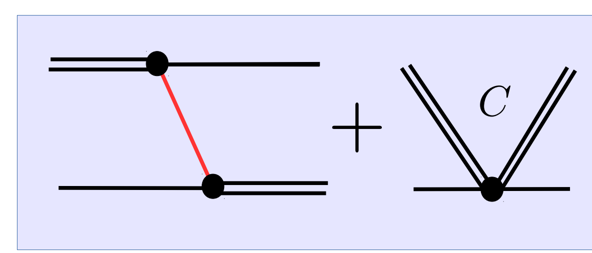
– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***υ=λ*** (full relations available)

$$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2+\mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$



- one-**π** exchange in TOPT → *RESULT !*

The Power of Unitarity

How general is the amplitude?
Are there other interactions/topologies not contained?

Completely general 3 → 3 amplitude up to **practical** approximations

Finite number of partial waves

Increase # according to availability of data; natural ordering scheme from centrifugal barrier and or input from PDG

"Blindfolded" PWA through model selection techniques **(Landay, M.D. *et al.*, 2017)**

Energy/momentum dependence from 3-body interactions unknown → model **polynomial** dependence

Constraints from known centrifugal barriers **(Ceci, M.D., Hanhart *et al.*, 2011)** and/or low-energy chiral dynamics **(e.g., Siemens *et al.*, 2014)**

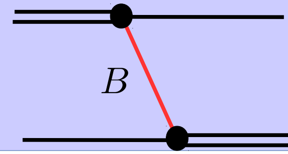
Applies to infinite and finite volume

Let's try to "disprove" the scheme

Diagrammatic "riddles"

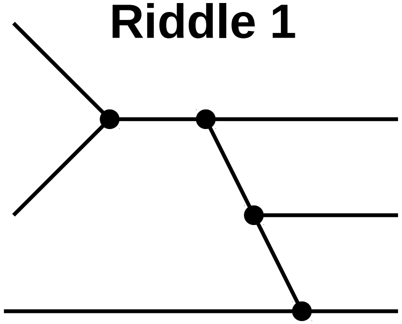
Doomed to fail because one cannot cheat unitarity (?)

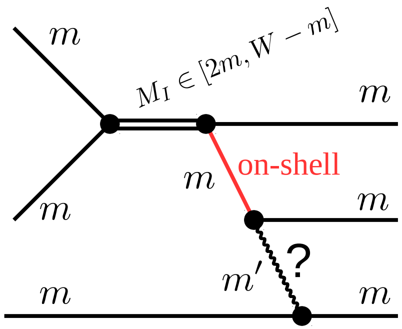
The Power of Unitarity

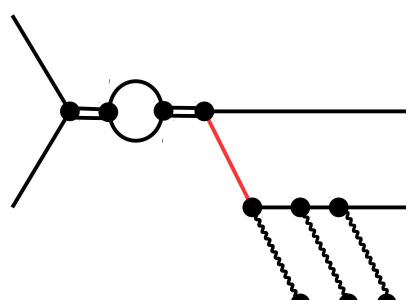
Question: Does 

provide full imaginary part of all possible $3 \rightarrow 3$ transitions?

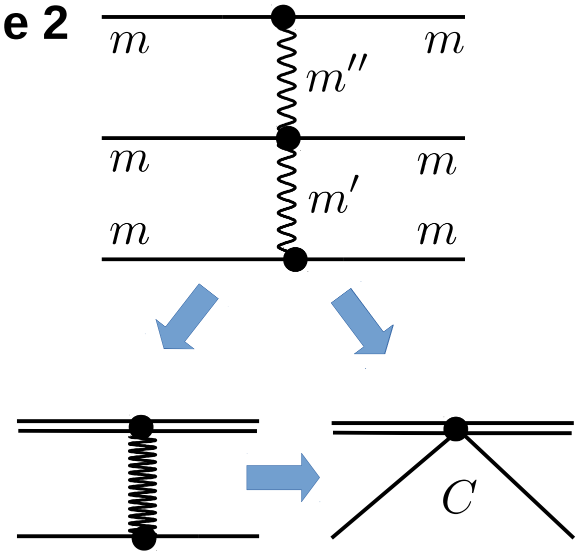
Riddle 1

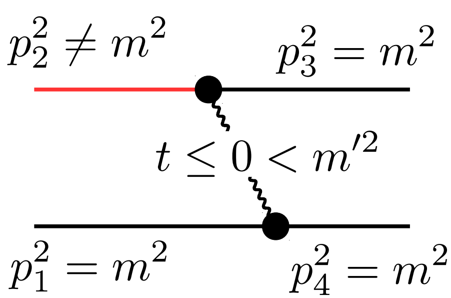


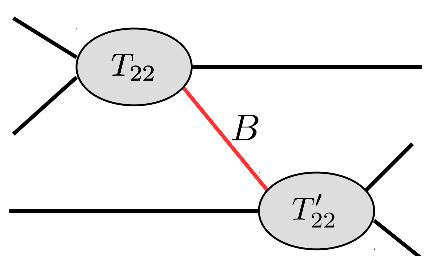




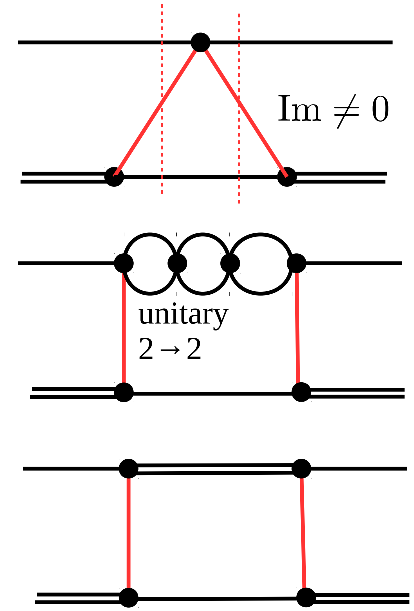
Riddle 2



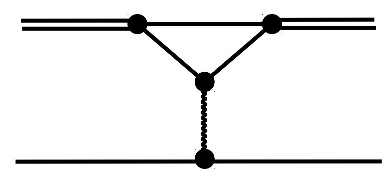




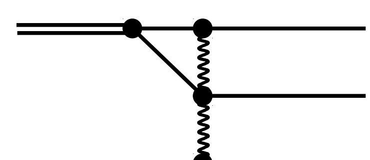
Riddle 3



Riddle 4



Riddle 5



SCATTERING AMPLITUDE

External on-shell 2-body interaction

Recasting in on-shell 2→2 amplitudes + real 3-body forces

$$\langle q_1, q_2, q_3|\hat{T}_c(s)|p_1, p_2, p_3\rangle =$$
$$\frac{1}{3!}\sum_{n=1}^{3}\sum_{m=1}^{3} T_{22}(\sigma(q_n))\langle q_n|T(s)|p_m\rangle T_{22}(\sigma(p_m))$$

with

Real three-body force

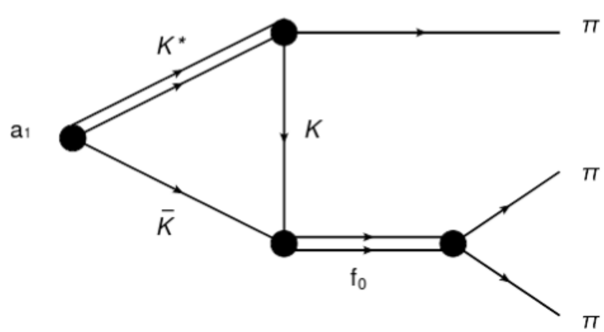
Exchange force

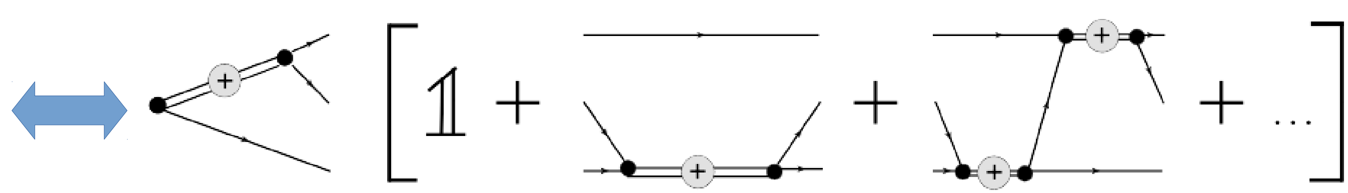
On-shell 2→ 2 interaction (even within integral)

$$\langle q|T(s)|p\rangle = \langle q|C(s)|p\rangle + \frac{1}{m^2-(P-p-q)^2-i\epsilon}$$
$$-\int\frac{\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3}\frac{1}{2E_\ell}T_{22}(\sigma(\ell))\left(\langle\ell|C(s)|p\rangle + \frac{1}{m^2-(P-p-\ell)^2-i\epsilon}\right)\langle\ell|T(s)|p\rangle$$

Future steps: Production reactions in coupled channels: generalize

Mikhasenko et al. (2015)
Aceti et al. (2016)
...

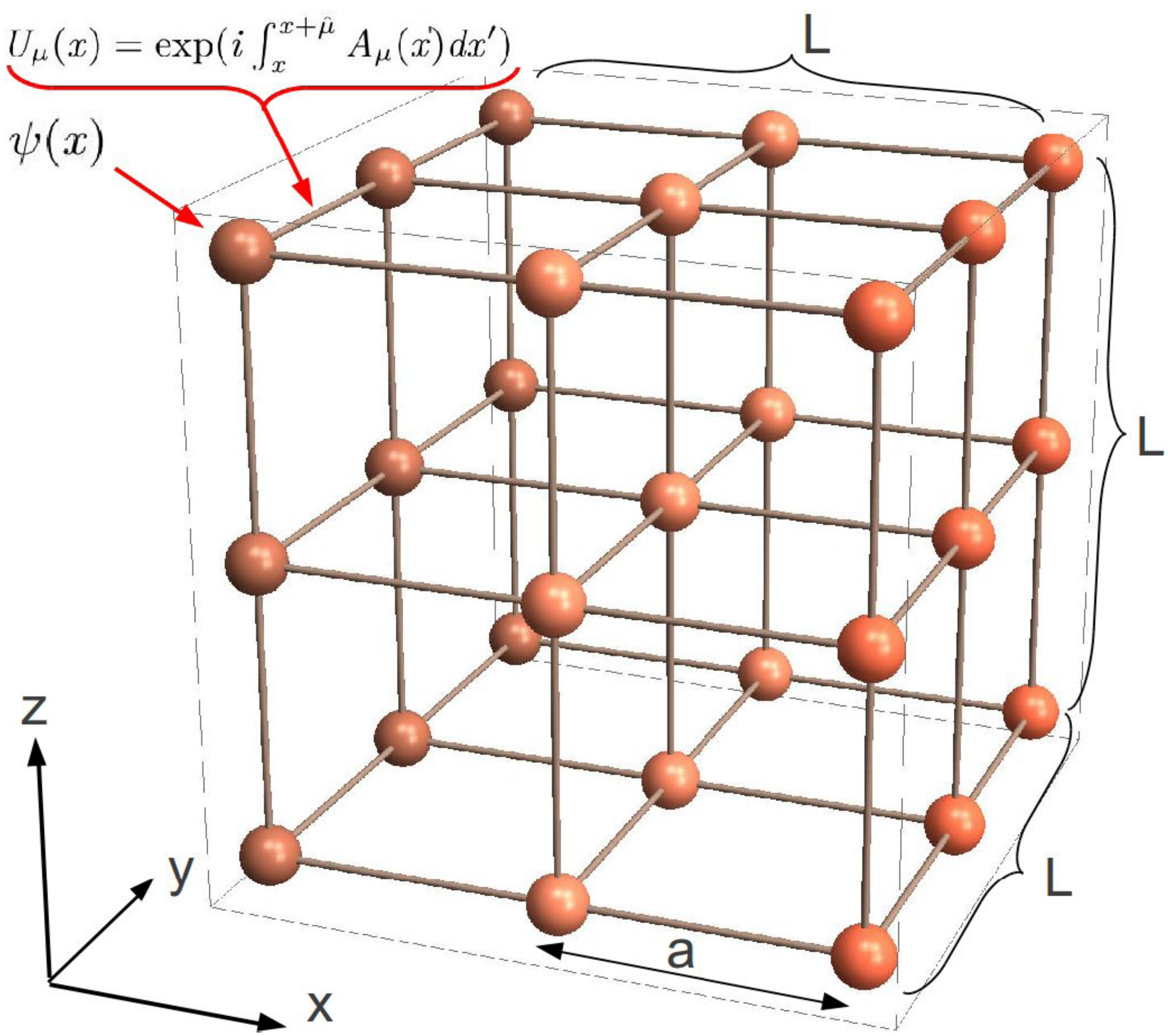




Two-body scattering on lattice

Input for 3-body

The cubic lattice



- Side length L, periodic boundary conditions $\Psi(\vec{x}) \overset{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L)$
  $\rightarrow$ finite volume effects
  $\rightarrow$ Infinite volume $L \rightarrow \infty$ extrapolation
- Lattice spacing a
  $\rightarrow$ finite size effects
  Modern lattice calculations: $a \simeq 0.07$ fm $\rightarrow p \sim 2.8$ GeV
  $\rightarrow$ (much) larger than typical hadronic scales;

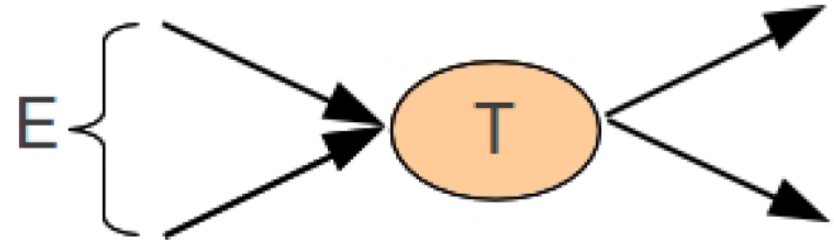
  not considered here.
- Unphysically large quark/hadron masses
  $\rightarrow$ (chiral) extrapolation required.

Two body scattering

In the infinite volume

- Unitarity of the scattering matrix S: $SS^\dagger = \mathbb{1}$ $\quad [S = \mathbb{1} - i\,\frac{p}{4\pi E}\,T]$.

$$\mathrm{Im}\,T^{-1}(E) = \sigma \equiv \frac{p}{8\pi\,E}$$



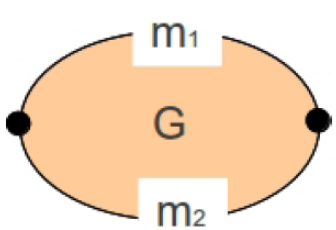
- $\rightarrow$ Generic (Lippman-Schwinger) equation for unitarizing the T-matrix:

$$T = V + V\,G\,T \qquad \mathrm{Im}\,G = -\sigma$$

V: (Pseudo)potential, σ: phase space.

- G: Green's function:

$$\begin{aligned} G &= \int \frac{d^3\vec{q}}{(2\pi)^3}\,\frac{f(|\vec{q}|)}{E^2 - (\omega_1+\omega_2)^2 + i\epsilon}, \\ \omega_{1,2}^2 &= m_{1,2}^2 + \vec{q}^{\;2} \end{aligned}$$



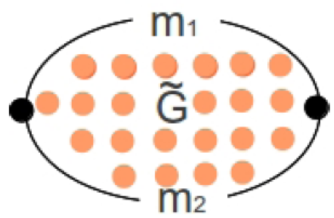
Discretization

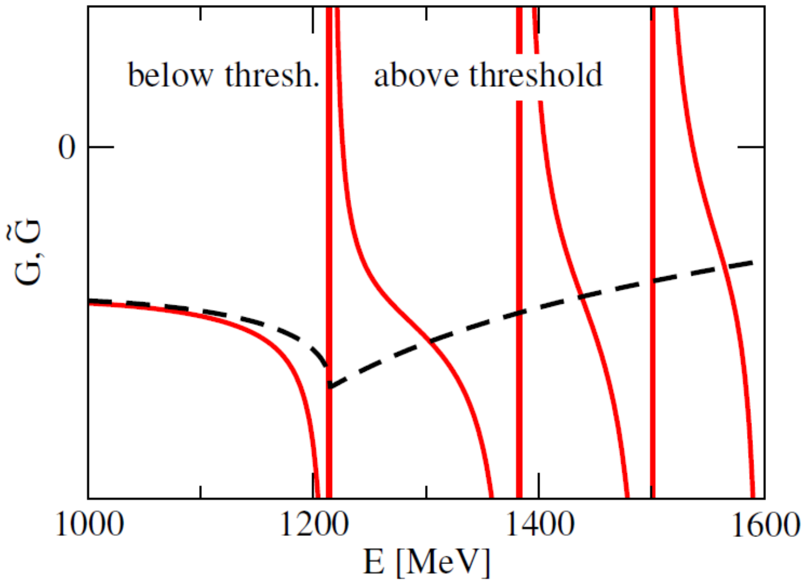
Discretized momenta in the finite volume with periodic boundary conditions

$$\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L) = \exp\left(i\, L\, q_i\right) \Psi(\vec{x}) \implies q_i = \frac{2\pi}{L}\, n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

$$\boxed{\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}\,|^2) \to \frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L}\,\vec{n}, \quad \vec{n} \in \mathbb{Z}^3}$$

$$G \to \tilde{G} = \frac{1}{L^3} \sum_{\vec{q}} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2}$$





- $E > m_1 + m_2$: $\tilde{G}$ has poles at *free energies in the box*, $E = \omega_1 + \omega_2$
- $E < m_1 + m_2$: $\tilde{G} \to G$ exponentially with L (regular summation theorem).

Finite → infinite volume: the Lüscher equation

Warning: rather crude re-derivation

- Measured eigenvalues of the Hamiltonian (tower of *lattice levels* $E(L)$)
  → Poles of scattering equation $\tilde{T}$ in the finite volume → determines V:

$$\tilde{T} = (1 - V\tilde{G})^{-1} V \;\rightarrow\; \quad V^{-1} - \tilde{G} \stackrel{!}{=} 0 \;\rightarrow\; V^{-1} = \tilde{G}$$

- The interaction V determines the T-matrix in the infinite volume limit:

$$T = \left(V^{-1} - G\right)^{-1} = \left(\tilde{G} - G\right)^{-1}$$

- Re-derivation of Lüscher's equation (T determines the phase shift δ):

$$\boxed{p \cot \delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)}$$

  - V and dependence on renormalization have disappeared (!)
  - p: c.m. momentum
  - E: scattering energy
  - $\tilde{G} - \mathrm{Re}G$: known kinematical function
    ($\simeq \mathcal{Z}_{00}$ up to exponentially suppressed contributions)
  - One phase at one energy.

From two to three particles in finite volume

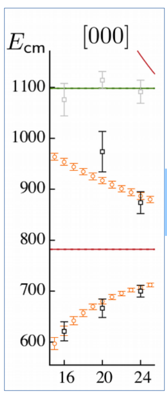
Finite-volume & chiral extrapolations

QCD calculations in finite volume

- unphysical pion mass
- (periodic) boundary conditions

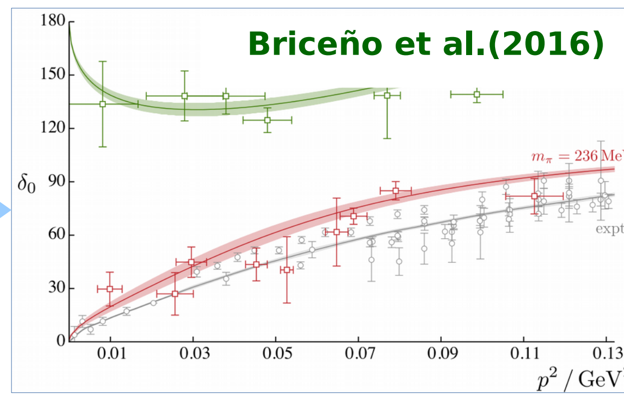
  → <u>discrete momenta</u> & <u>discrete spectrum</u>

Recipe for 2 → 2 scattering (e.g. *I=J=0* ππ scattering)

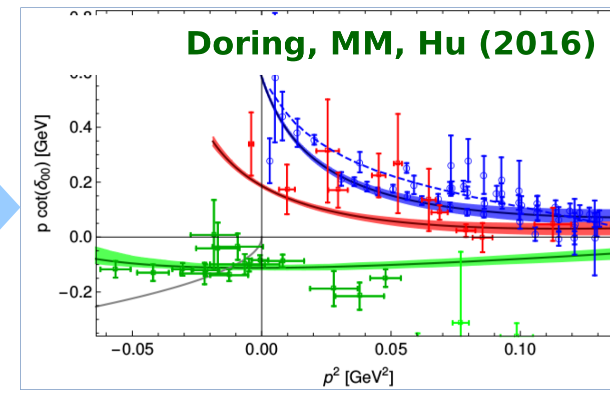


HSC(2016)

step 1



step 2



<u>LÜSCHER(1986)</u>

- 1 eigenenergy ↔ 1 phase-shift in infinite volume
- also with coupled channels **He et al. (2005)**

Doring et al.(2011) HSC (2015)...

<u>CHIRAL EXTRAPOLATIONS</u>

- M_π dependence from NLO ChPT (IAM)

Gasser, Leutwyler(1981)

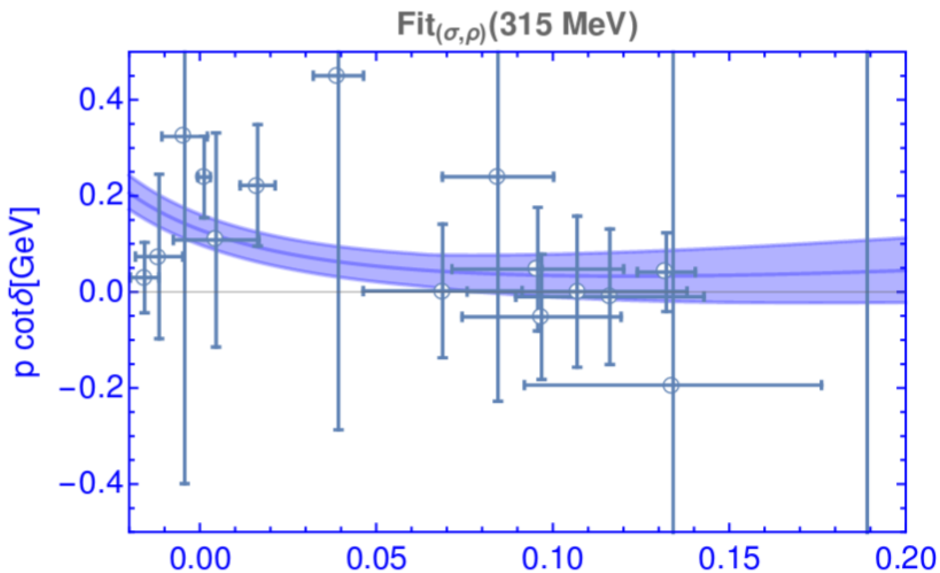
- Extrapolation in flavor

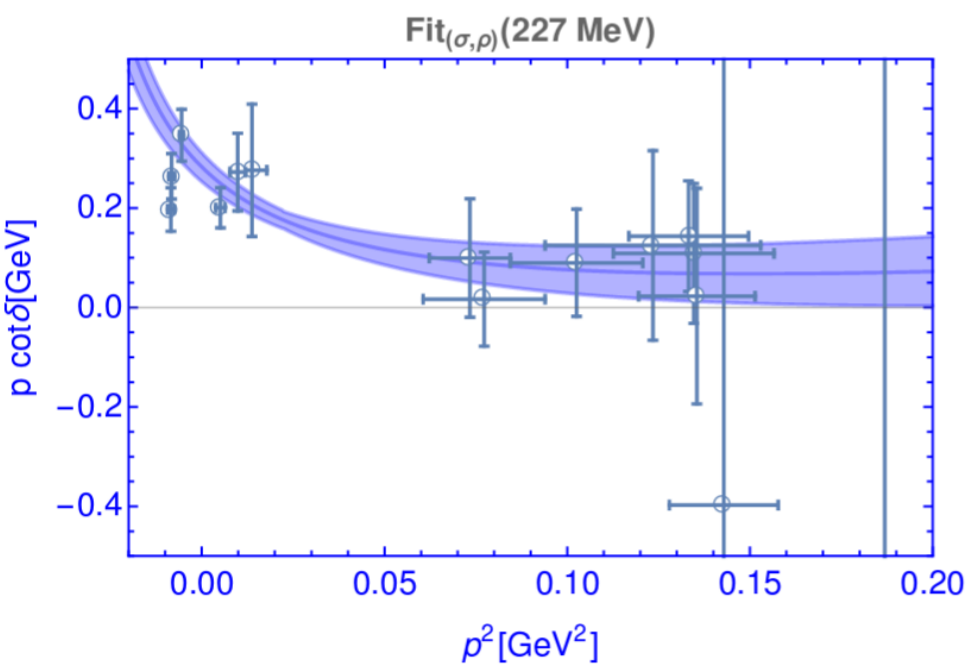
B. Hu, MD, R. Molina M. Mai et al. (2016)

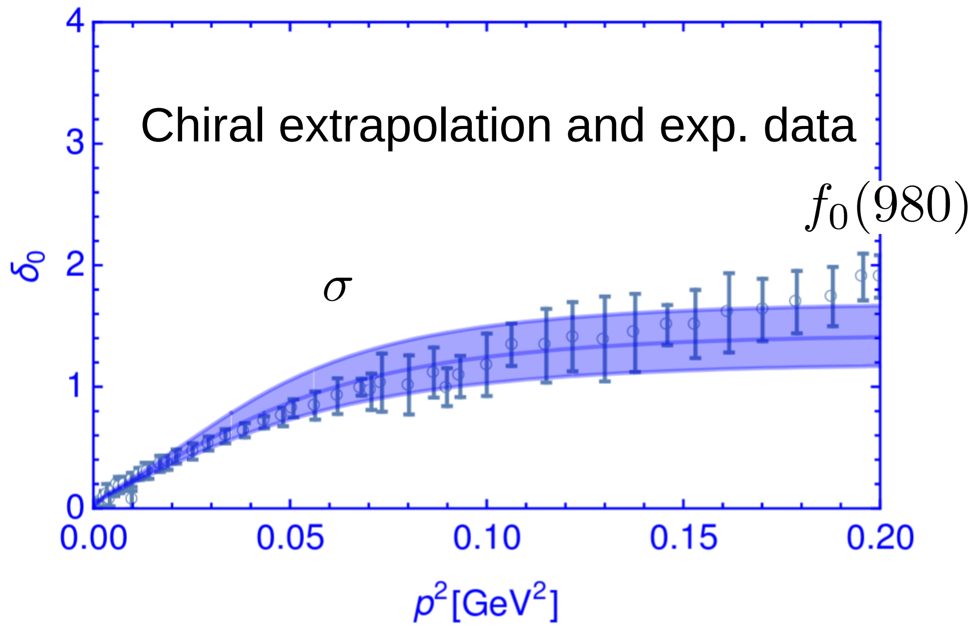
New: GWU lattice group calculates isoscalar

[Guo, Alexandru, Molina, M.D., M. Mai, preliminary]

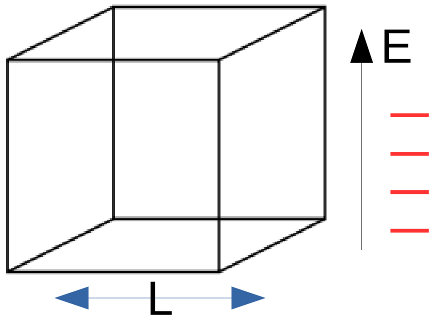
- nHYP-smeared clover fermions with mass-degenerate quark flavors (N_f = 2)
- M_π=227 MeV and 315 MeV
- 3 elongated boxes
- Large variational basis including several meson-meson operators
- Moving frames
- Unitarized Chiral Perturbation Theory fits for chiral extrapolation







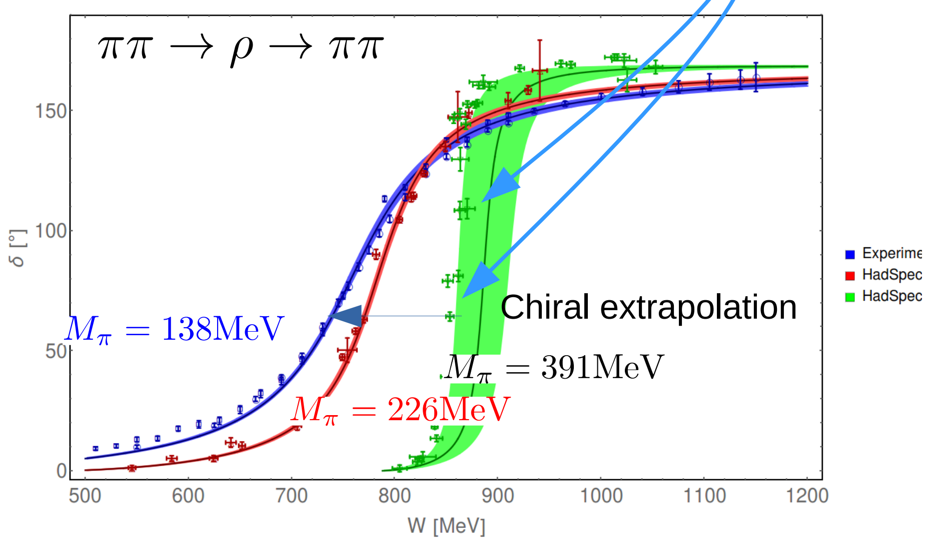
- Roper on lattice from BGR group [Lang et al., Phys.Rev. D95 (2017), 014510]

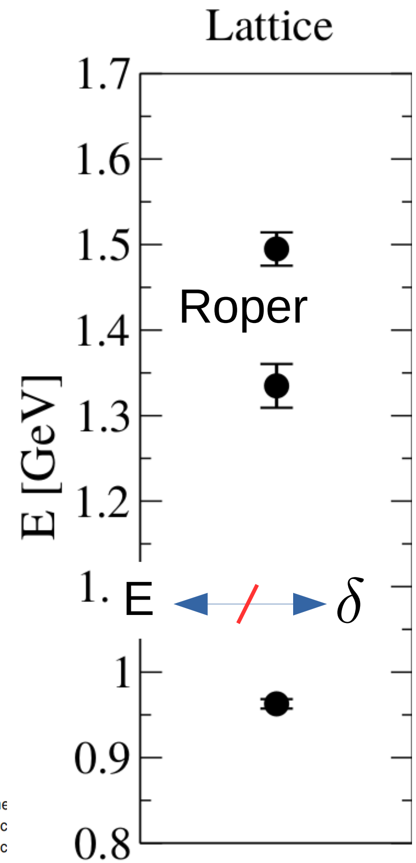


E ↔ δ

Lüscher

(Nucl. Phys. B, 1991)



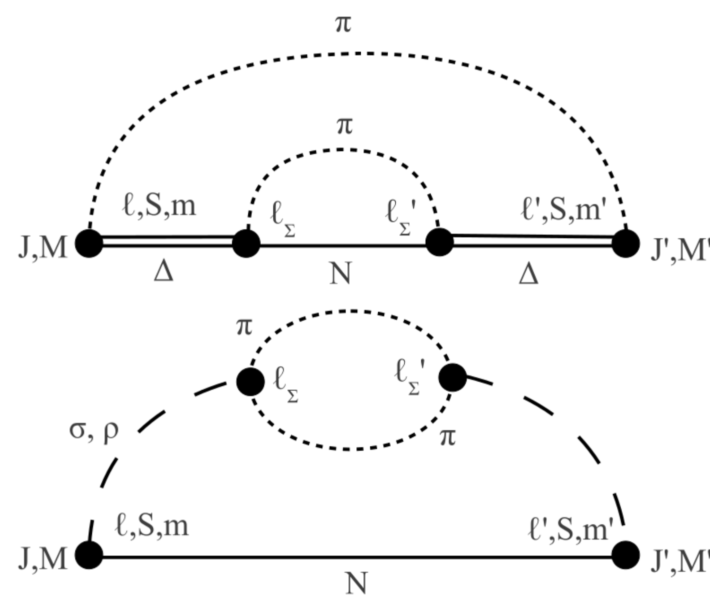


$M_\pi \approx 156\,\mathrm{MeV}$

Channels:

$\pi N, \eta N, \pi\pi N(\sigma N, \pi\Delta, \dots)$

Genuine three-body dynamics



Three-body methods:

- Briceño, Hansen, Sharpe PRD96 (2017)
- Hammer, Pang, Rusetsky JHEP (2017)
- ...

Data: HadronSpectrum (Dudek, PRD 2013,Briceño PRL 2016);
Analysis: M.D., B. Hu, M. Mai, arXiv 1610.10070
See also: Bolton, Briceno, Wilson, Phys.Lett. B757 (2016) 50

Large # of d.o.f. require efficient parametrizations

Example: The coupled-channel 2→2, 2→3, 3→3 meson-baryon system

μ	$J^P =$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^-$	$\frac{5}{2}^+$	$\frac{7}{2}^+$	$\frac{7}{2}^-$	$\frac{9}{2}^-$	$\frac{9}{2}^+$
1	πN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
2	$\rho N(S=1/2)$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
3	$\rho N(S=3/2, \lvert J-L\rvert=1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
4	$\rho N(S=3/2, \lvert J-L\rvert=3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
5	ηN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
6	$\pi\Delta(\lvert J-L\rvert=1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
7	$\pi\Delta(\lvert J-L\rvert=3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
8	σN	P_{11}	S_{11}	D_{13}	P_{13}	F_{15}	D_{15}	G_{17}	F_{17}	H_{19}	G_{19}
9	$K\Lambda$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
10	$K\Sigma$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}

including 3-body dynamics [Julich-Bonn; ANL-Osaka].

GOALS & CHALLENGES

Lüscher-like formalism in 3 → 3 case is under investigation

Polejaeva/Rusetsky (2012) Briceño/Hansen/Sharpe (2016)

Some challenges

- many systems involve (resonant) two-body sub-amplitudes (e.g. ***N*(1440)*** **→** ***Nσ*** **→** ***ππN***)
- multiple sources for singularities
  - → only some yield genuine 3-body dynamics
  - → cancellation mechanisms have to be visible
- extrapolations between different energies:
  - → 3 body scattering amplitude in infinite volume

Non-relativistic approaches based on dimer picture & effective field theory

Kreuzer, Griesshammer(2012) Hammer et al. (2016)

⟹ This work: **Quantization condition from 3-body unitarity**

THREE-BODY AMPLITUDE IN A BOX

M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]

DISCRETIZATION

Partial Waves in infinite volume

- separation of angular momentum → $Y_{lm}(\theta,\varphi)$
- reduces dimensionality of the problem

In finite volume this is different

- breakdown of spherical symmetry
- For a given "shell" (radius):

→ irreps of cubic group: A_1^+, E^+, etc..

→ finite number of basis vectors for each irrep

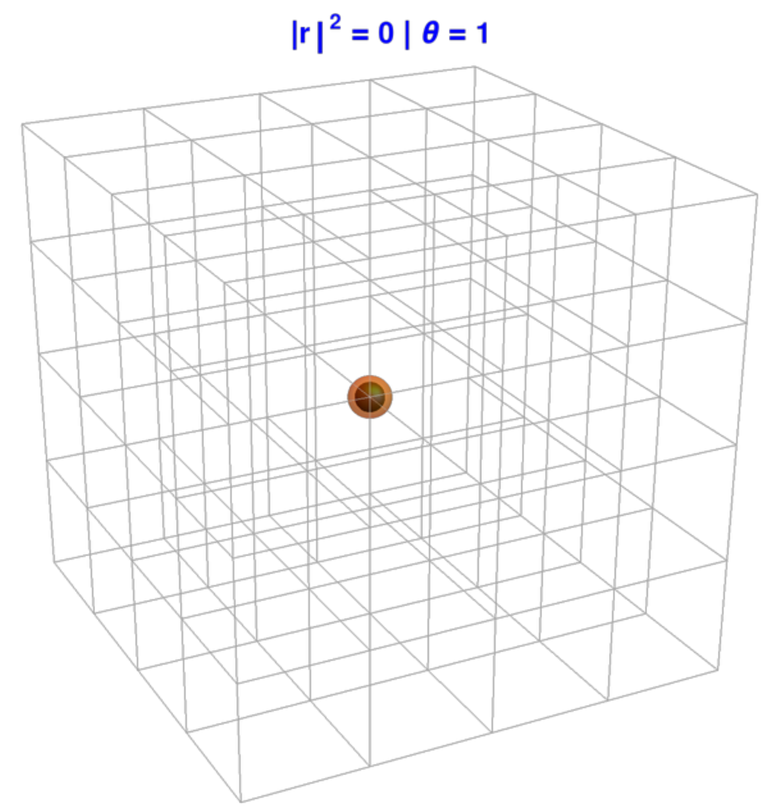
→ mapping to PWA not isomorph

Consider a world with one (s-wave) isobar

& project to A_1^+ (basis vector: $Y_{00}(\theta,\varphi)$)

Order momenta in shells

$$\boldsymbol{q}_{ni} = \frac{2\pi}{L}\boldsymbol{r}_i$$
$$\text{for } \{\boldsymbol{r}_i \in \mathbb{Z}^3 | \boldsymbol{r}_i^2 = n, i = 1, \ldots, \vartheta(n)\}$$



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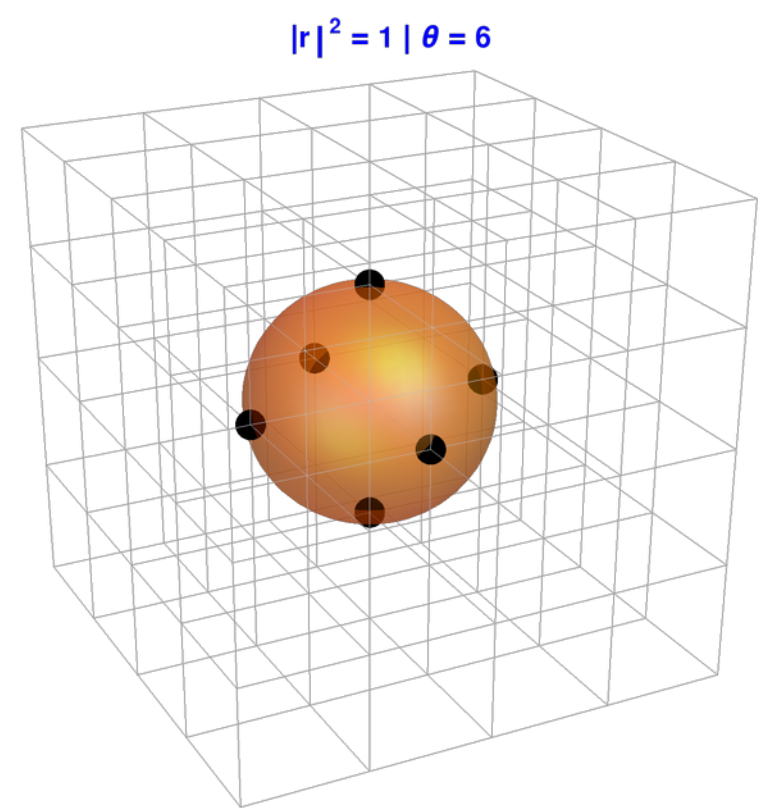

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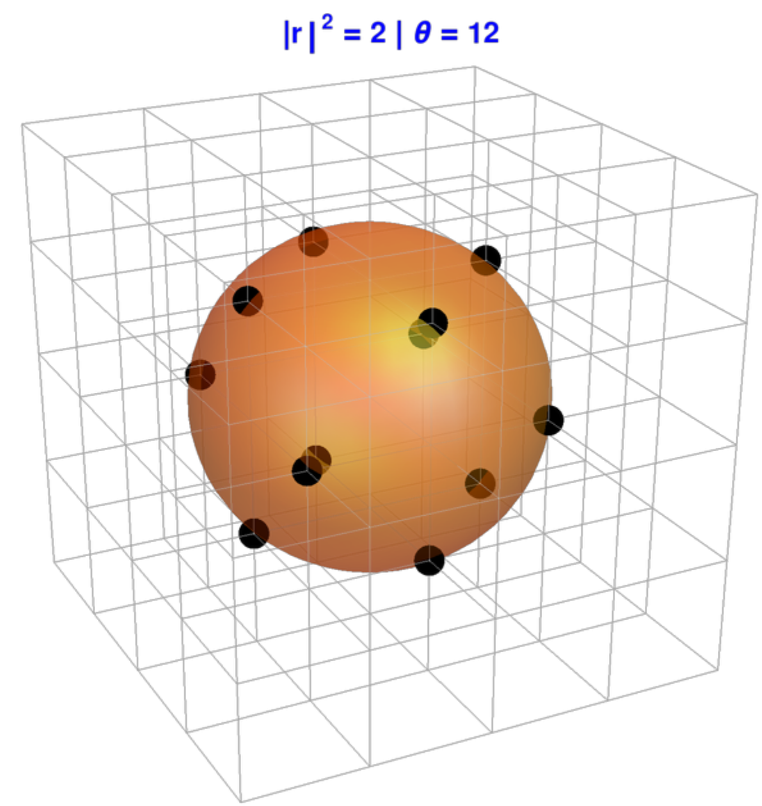

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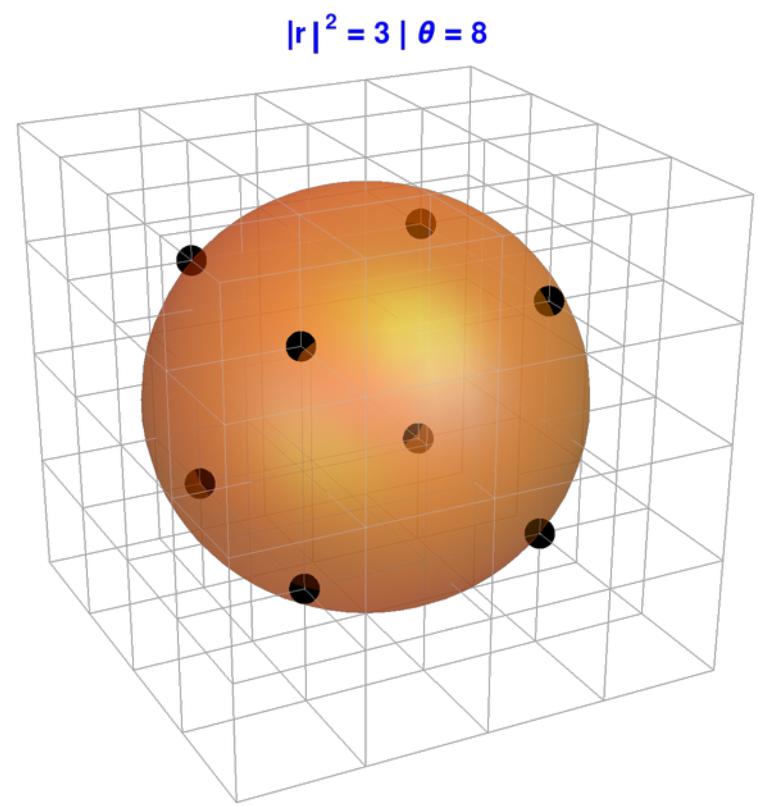

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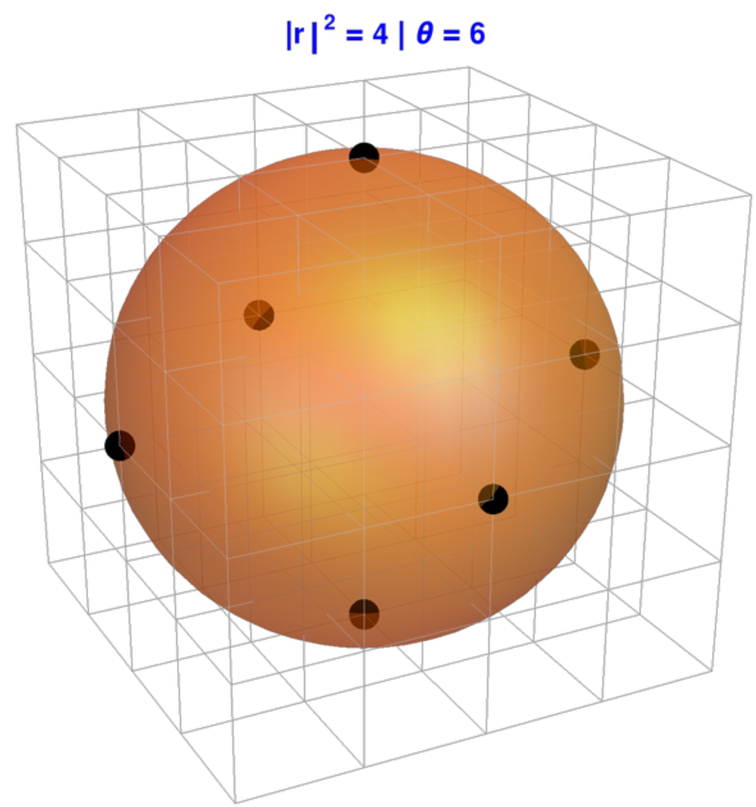
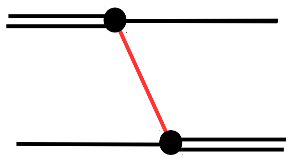


Three-body singularities in finite volume

[M.D., H.-W. Hammer, M. Mai, J.-Y. Pang, A. Rusetsky, J. Wu, in preparation]



is singular: **a genuine challenge in three-body physics**

- Compare to Lüscher: Regular summation theorem for regular 2 → 2 potentials
- Can we still preserve “orthogonality” of partial waves from infinite volume?
- Cubic symmetry instead of rotational symmetry
- Need to project interaction itself to the irreps of octohedral group
  → Talk by A. Rusetsky
- 2 methods available; both equivalent → Talk by J. Y. Pang
- Here: Expand a complex function on points of a shell
- Use cubic harmonics because they are orthogonal in the irreps
- Iterative scheme to determine cubic harmonics contributing to every shell
- Construct orthonormal basis functions w.r.t to scalar product

$$\langle f, g\rangle_s = \frac{4\pi}{\vartheta(s)} \sum_j^{\vartheta(s)} f(\hat{p}_j)^* g(\hat{p}_j)$$

- Orthonormal basis functions provided in supplemental material for easy implementation

$$f^s(\hat{p}_j) = \sqrt{4\pi} \sum_{\Gamma\alpha} \sum_a f_a^{\Gamma\alpha s} \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

$$f_a^{\Gamma\alpha s} = \frac{\sqrt{4\pi}}{\vartheta(s)} \sum_{j=1}^{\vartheta(s)} f^s(\hat{p}_j) \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

vs.

$$f(\mathbf{p}) = \sqrt{4\pi} \sum_{\ell m} Y_{\ell m}(\hat{p}) f_{\ell m}(p)$$

$$f_{\ell m}(p) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y_{\ell m}^*(\hat{p}) f(\mathbf{p})$$

DISCRETIZATION

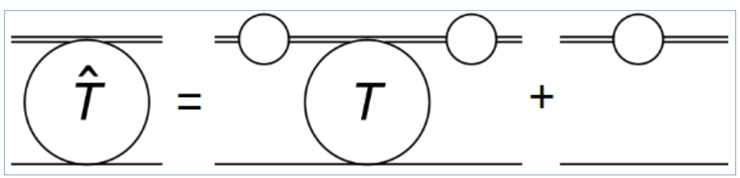
- Consider first 8 shells → ***Λ~1 GeV for L=3 fm***

  → no degeneracies like $\boldsymbol{9 = (\pm 3)^2 + 0^2 + 0^2 = (\pm 1)^2 + (\pm 2)^2 + (\pm 2)^2}$
- Replace integrals by sums

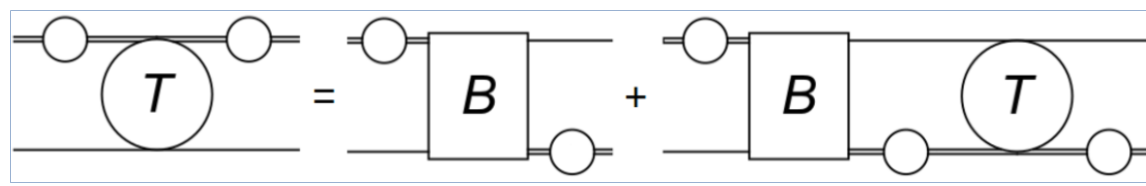
$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \to \frac{1}{L^3} \sum_{n \in set_8} \sum_{i=1}^{\vartheta(n)}$$

→ integration momenta in the isobar-propagator must be expressed by the 3-body cms momenta

Genuine 3-body eigenlevels = poles of $\check{T}(s)$ (v is cut-free)

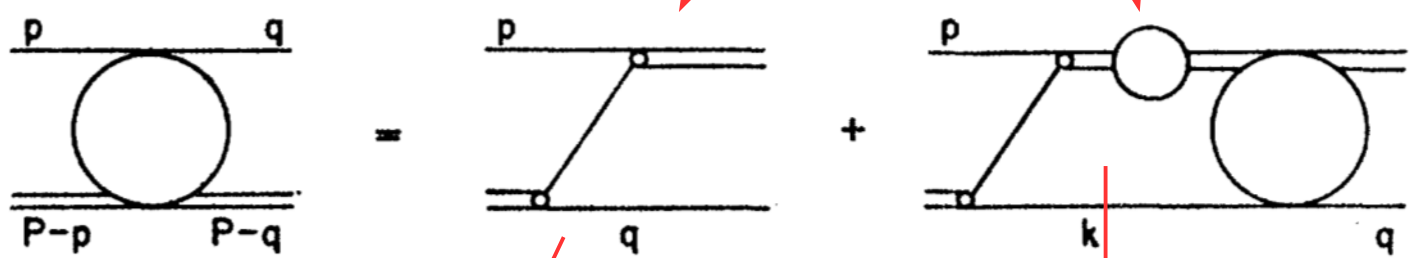


&



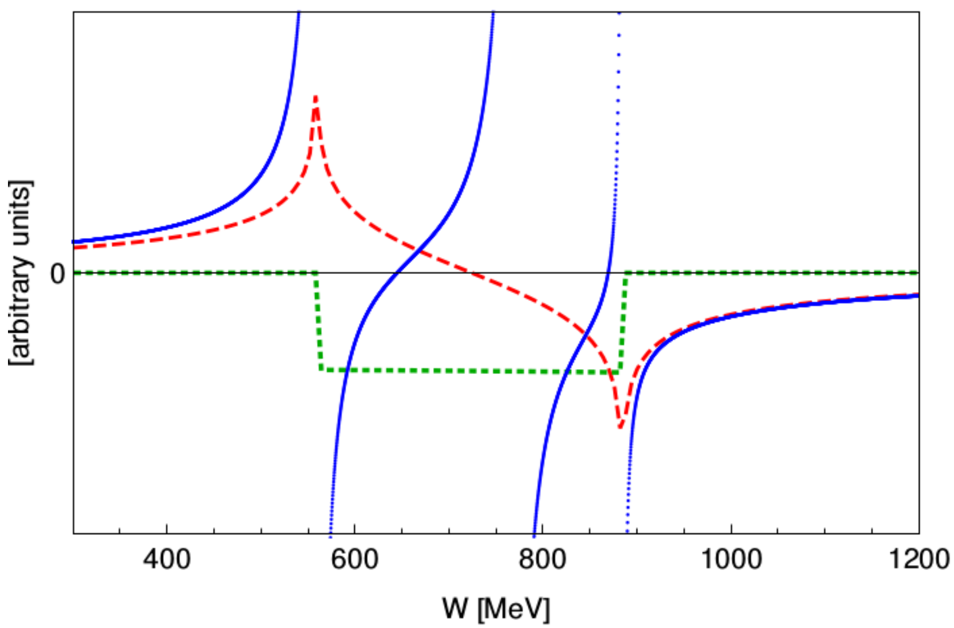
→ $\check{T}(s)$ is a matrix in $|q|, |p| = 0,1,2,3,4,5,6,8$

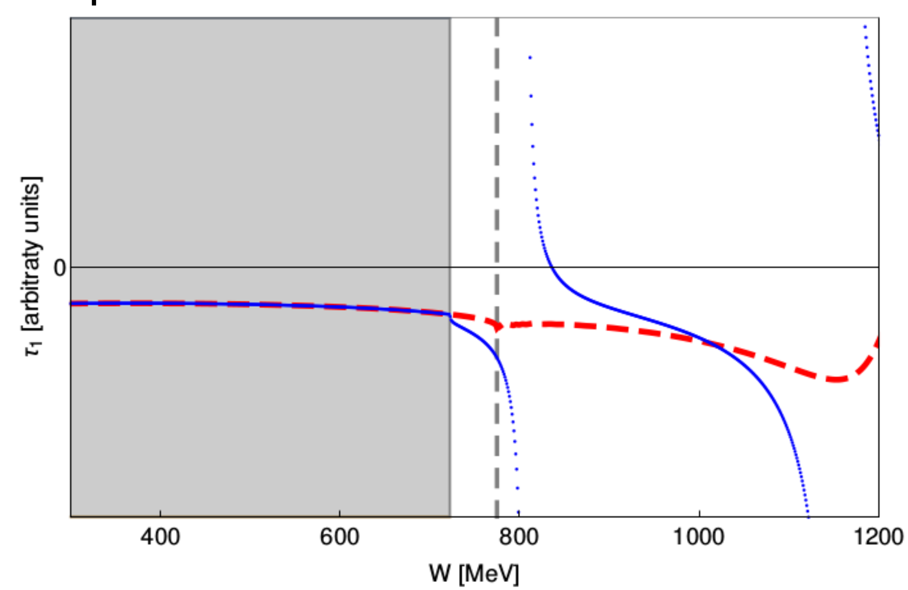
Power-law finite-volume effects dictated by three-body unitarity



Tower of boosted $2 \to 2$ amplitudes to implement 3-body quantization condition

S-wave infinite volume vs. A_1^+ finite volume

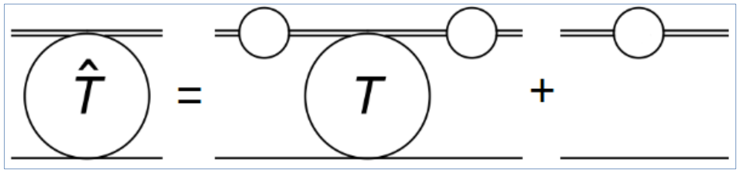




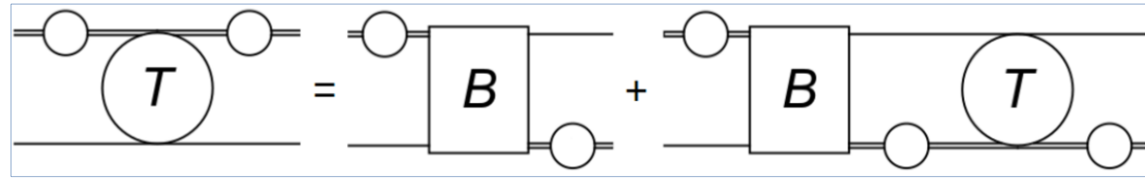
$$(W = \sqrt{s})$$

QUANTIZATION CONDITION

- Genuine 3-body eigenlevels = poles of *Ť(s)* (*υ* is cut-free)



&



→ ***Ť(s)*** is a matrix in **|*q*|, |*p*|=*0,1,2,3,4,5,6,8***

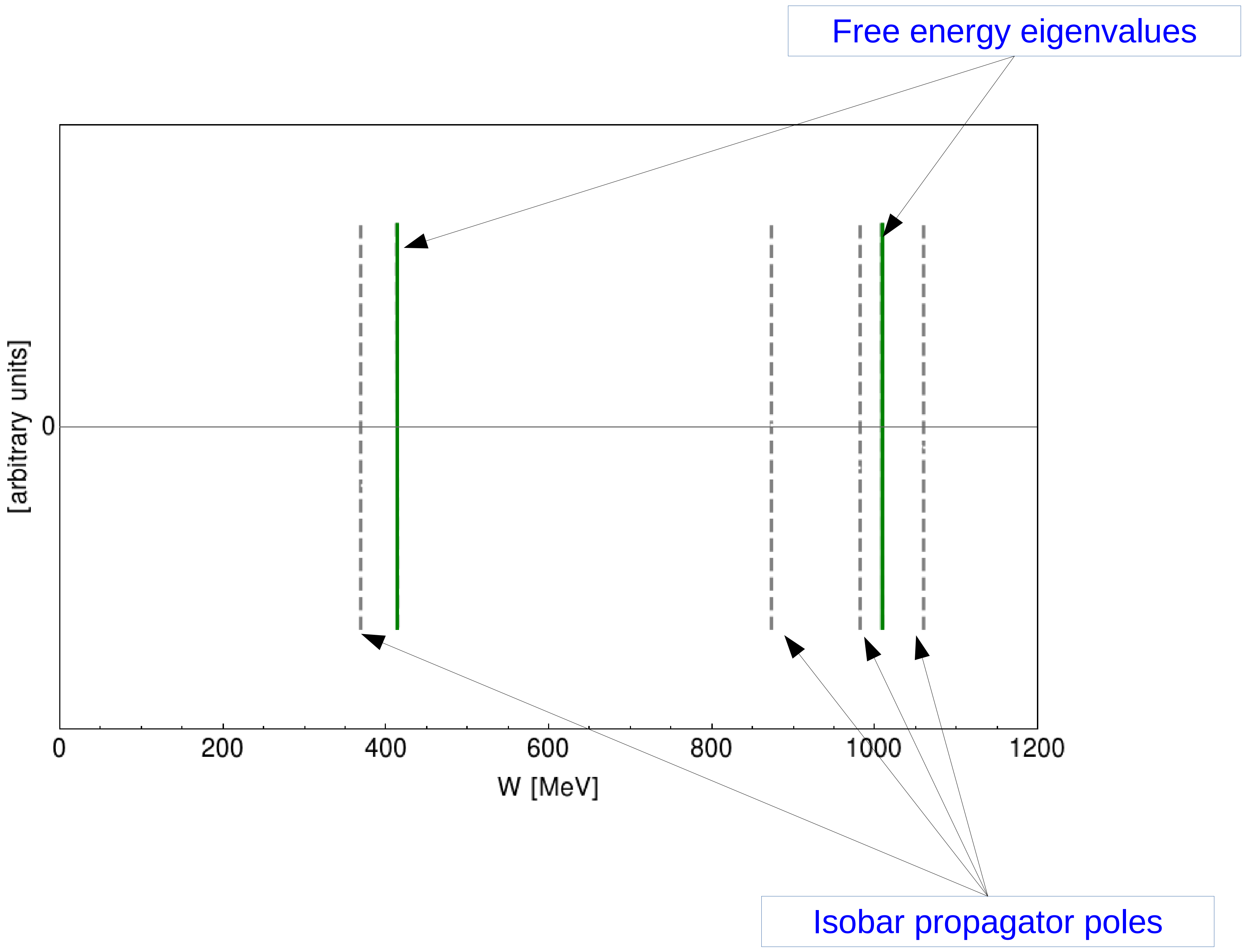
$$\hat{T}^{A_1^+}(s) = \left[X(s)B^{A_1^+}(s)X(s) + X(s)\tau(s)^{-1}\right]^{-1}$$

$$\text{for} \quad X(s) := \mathrm{Diag}_{n\in sets_8}\left(\frac{\vartheta(n)}{2E_n(s)L^3}\right)$$

→ ***Ť(W) = ∞ iff***

$$\boxed{\mathrm{Det}[B^{A_1^+}(s)X(s) + \tau(s)^{-1}] = 0\,.}$$

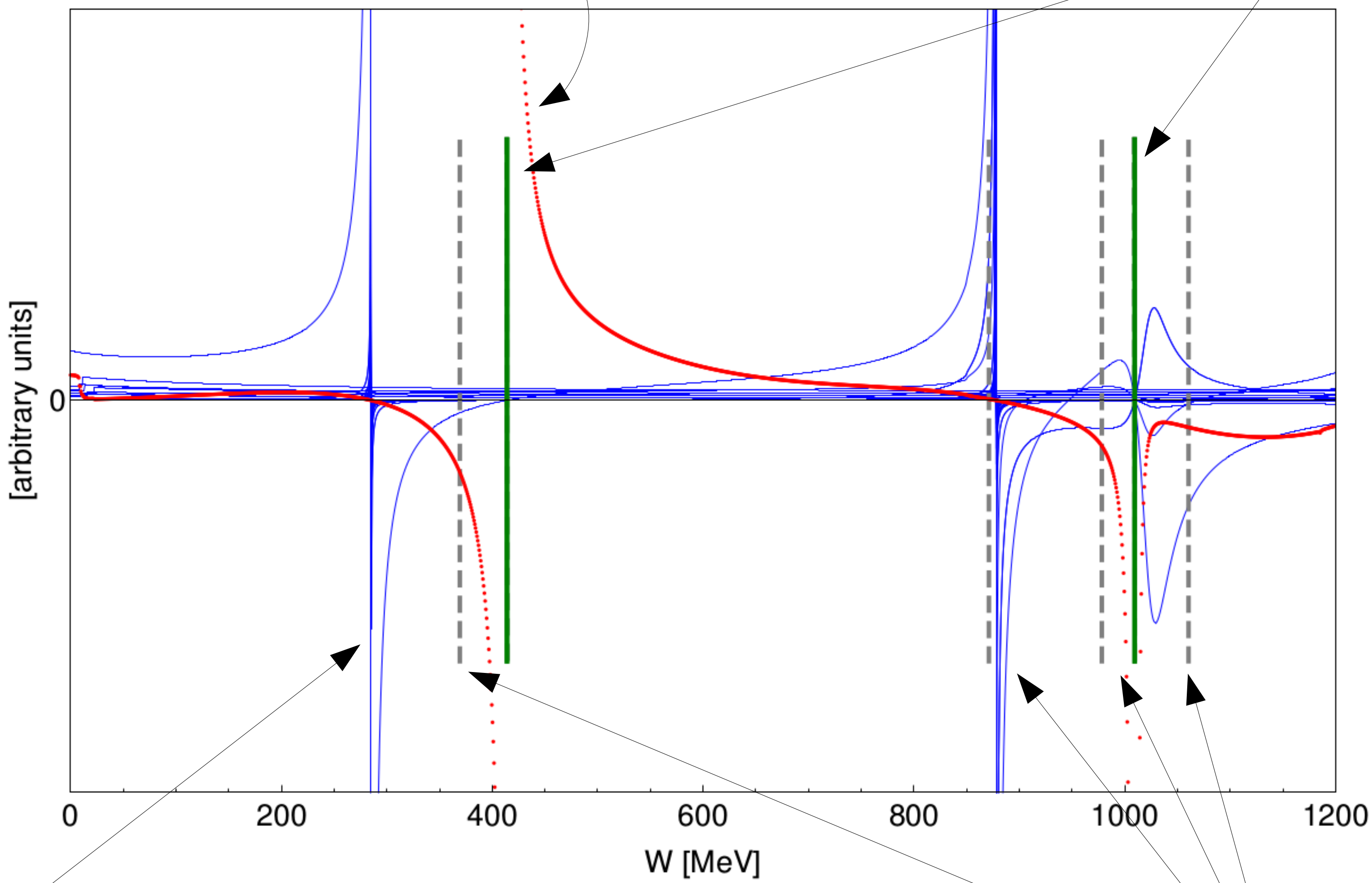
RESULTS (L=3 fm, M=138 MeV)



RESULTS (L=3 fm, M=138 MeV)

$\mathrm{Det}[B^{A_1^+}(s)X(s)+\tau(s)^{-1}$

Free energy eigenvalues



$\hat{T}^{A_1^+}(s)$

Isobar propagator poles

SUMMARY

3-body amplitude in infinite volume

- 3-body Unitarity dictates imaginary parts of the driving term & isobar propagator
- Result: 3-dim. relativistic integral equations

Finite volume investigation:

- Imaginary parts dictate leading finite-volume effects
- Discretization techniques
- Quantization condition

OUTLOOK

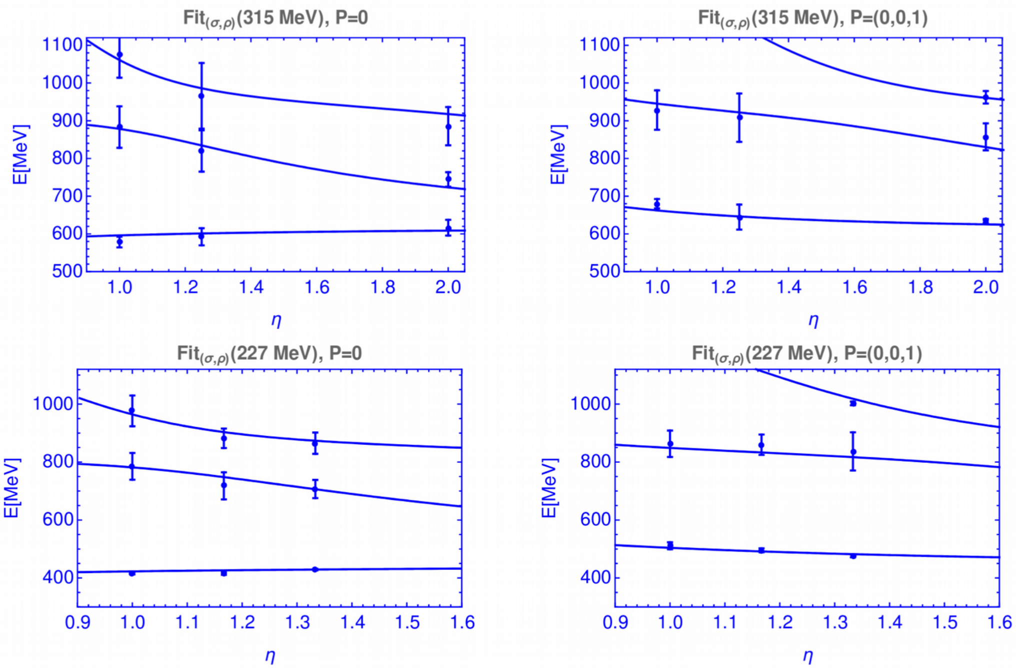
→ include angular momentum / isospin / multiple isobars

→ practical studies: a_1(1260), Roper...

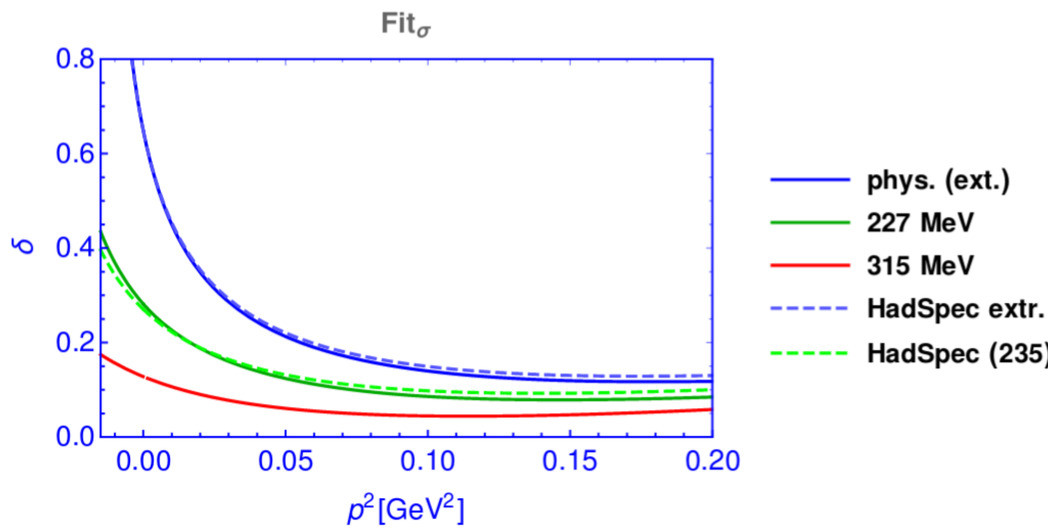
SPARES

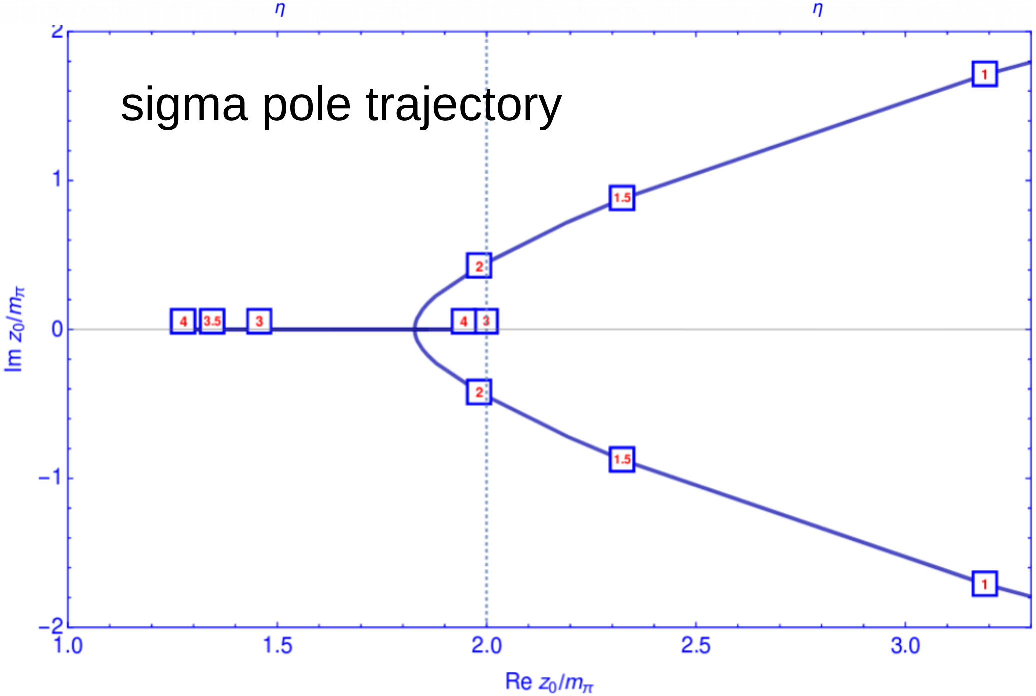
GWU lattice results: Chiral trajectory

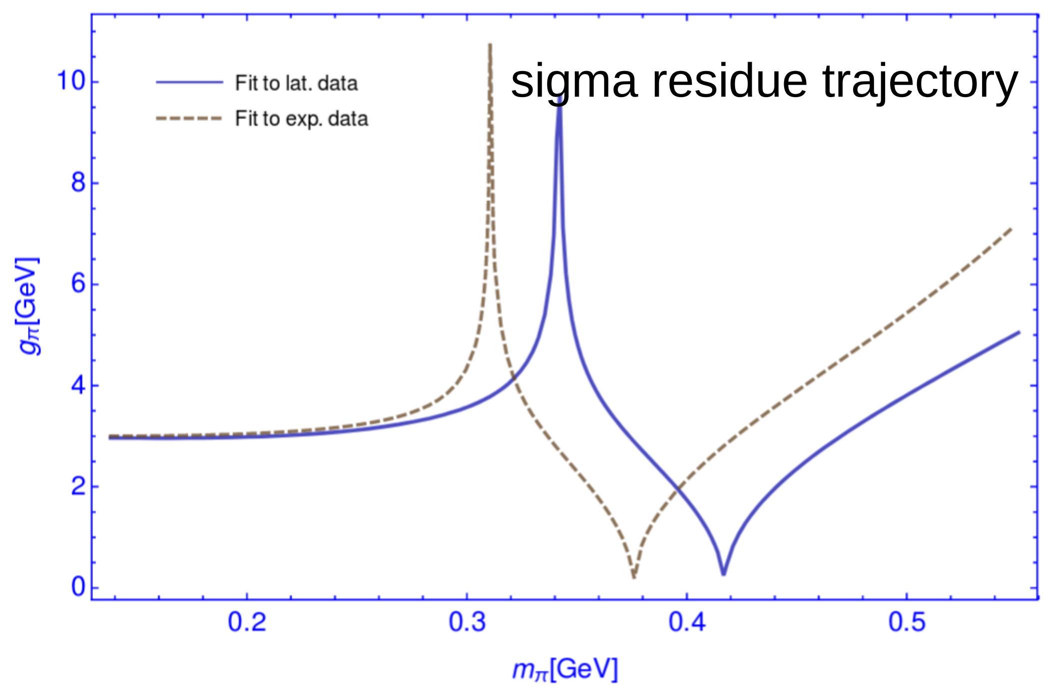
[Guo, Alexandru, Molina, M.D., Mai, preliminary]



Comparison with HadSpec [Briceno PRL 2016]





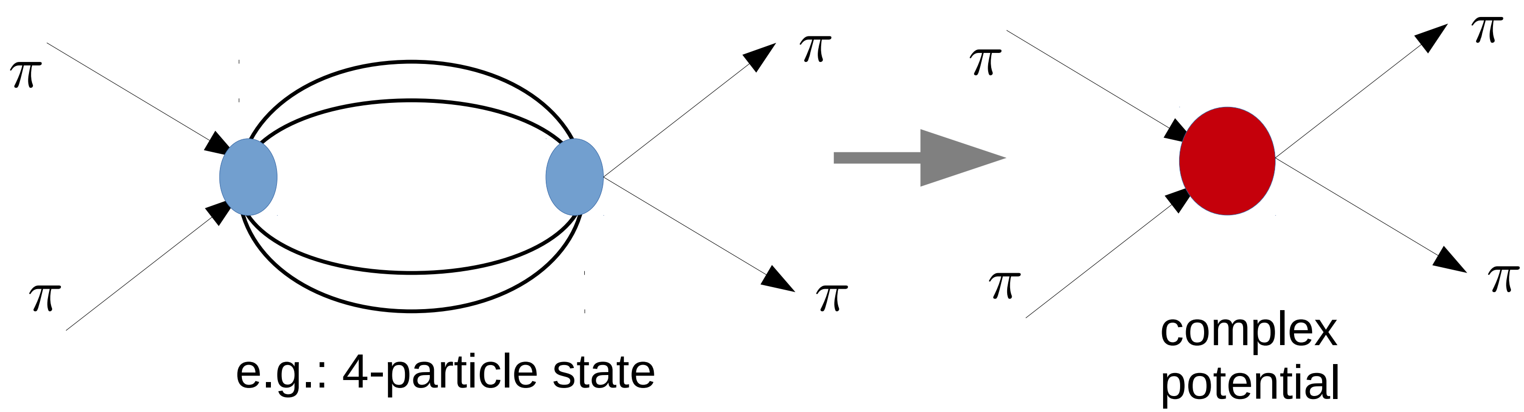


Effective method for multi-particle states

The Optical potential [D. Agadjanov, M.D., M. Mai, U.-G. Meißner, A. Rusetsky, JHEP (2016)]

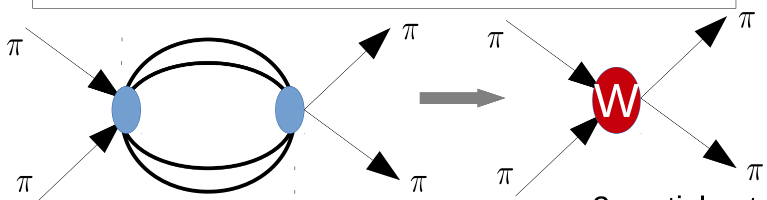
Optical potential in finite volume

- Finite-volume corrections for complex hadronic systems.
- Example: The optical potential on the lattice



- It is not always necessary to explicitly parameterize complicated intermediate states → Absorb all "uninteresting" dynamics in a complex-valued optical potential

Optical potential: The formal rewriting of a complicated scattering problem



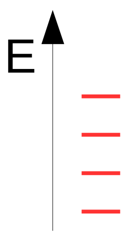
n-particle states
m coupled channel

2-particle state
1 channel
<u>BUT</u>
complex potential
with full dynamics

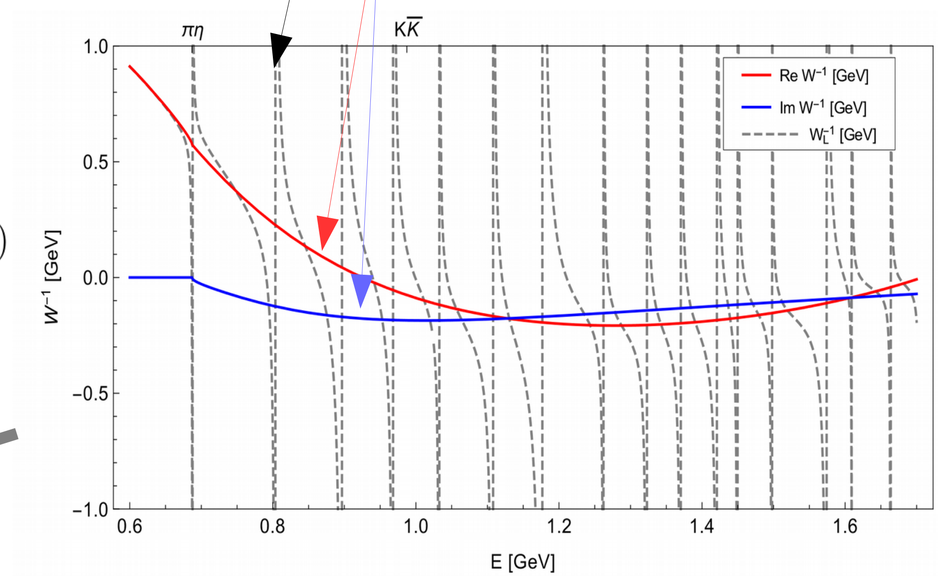
- Measured finite-volume optical potential
- Poles/functional form contain full multi-channel/ multi-particle dynamics
- How to efficiently measure this function → later

How to reconstruct true OP (complex) from finite volume OP (real)?

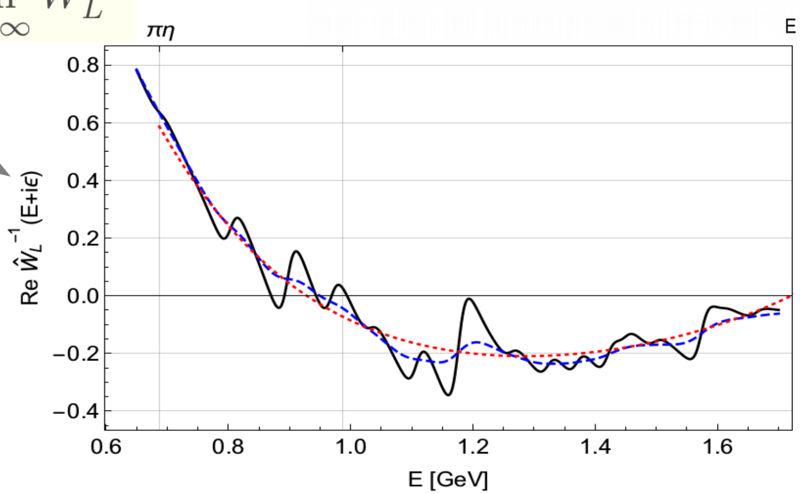
Lattice: measure eigenvalues, map to the optical potential



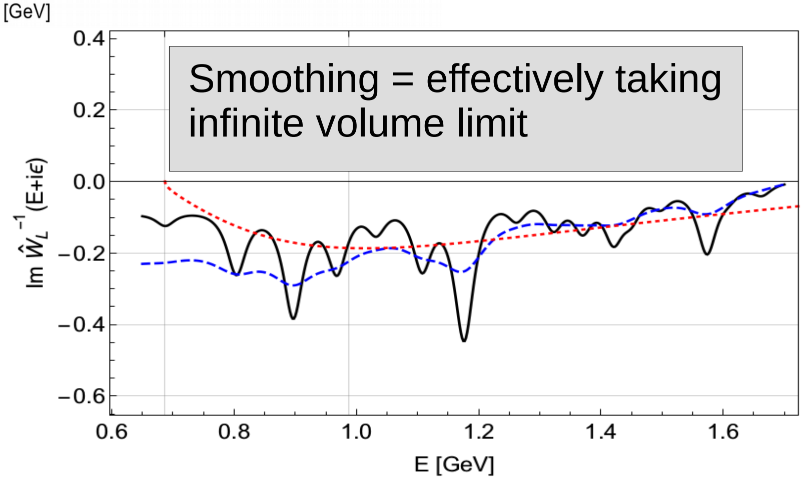
$$W_L^{-1}(E) \doteq \frac{2}{\sqrt{\pi}L} Z_{00}(1; q^2_{K\bar{K}})$$



$$W^{-1} = \lim_{\epsilon\to 0} \lim_{L\to\infty} W_L^{-1}$$



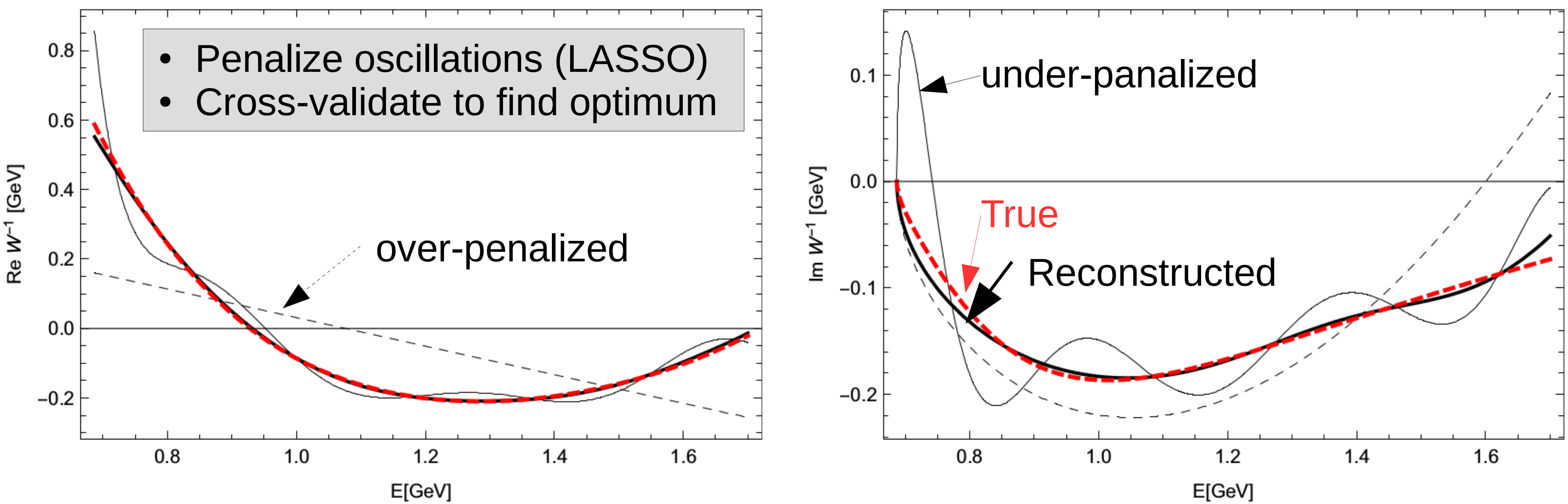
Smoothing = effectively taking infinite volume limit



Minimize: $$\chi^2 = \sum_{k=1}^{m} \frac{\left|\hat{W}^{-1}(E_k) - \hat{W}_L^{-1}(E_k)\right|^2}{\sigma_k^2} + P_i(a_j,\, b_j)$$

$$P_1(a_j,\, b_j) = \lambda^4 \int_{E_{\min}+i\varepsilon}^{E_{\max}+i\varepsilon} dE \left|\frac{\partial^2 \hat{W}^{-1}(E)}{\partial E^2}\right|$$

The reconstructed infinite-volume limit [LASSO + Cross Validation]

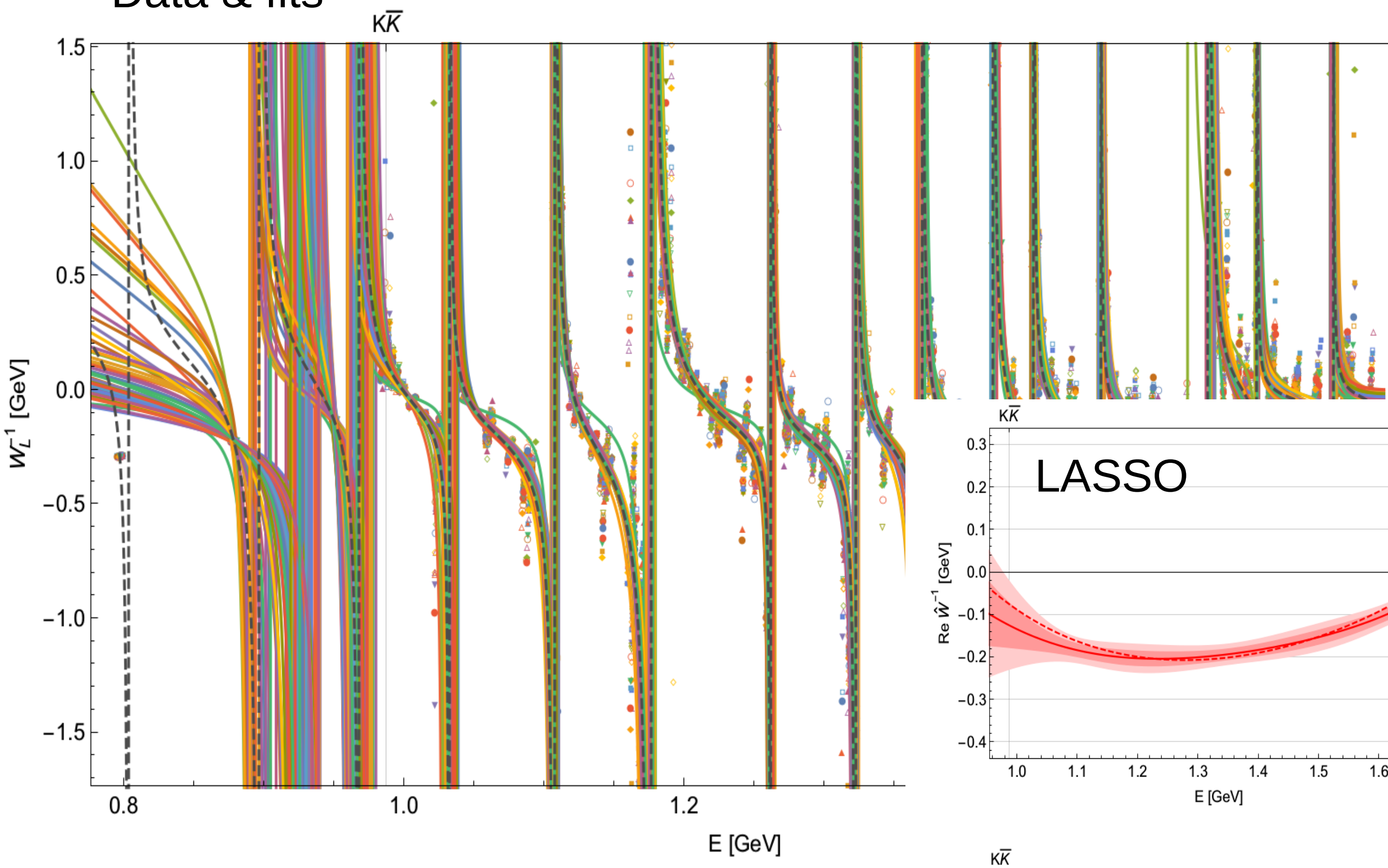


Correct Choice of penalization parameter λ through cross validation:

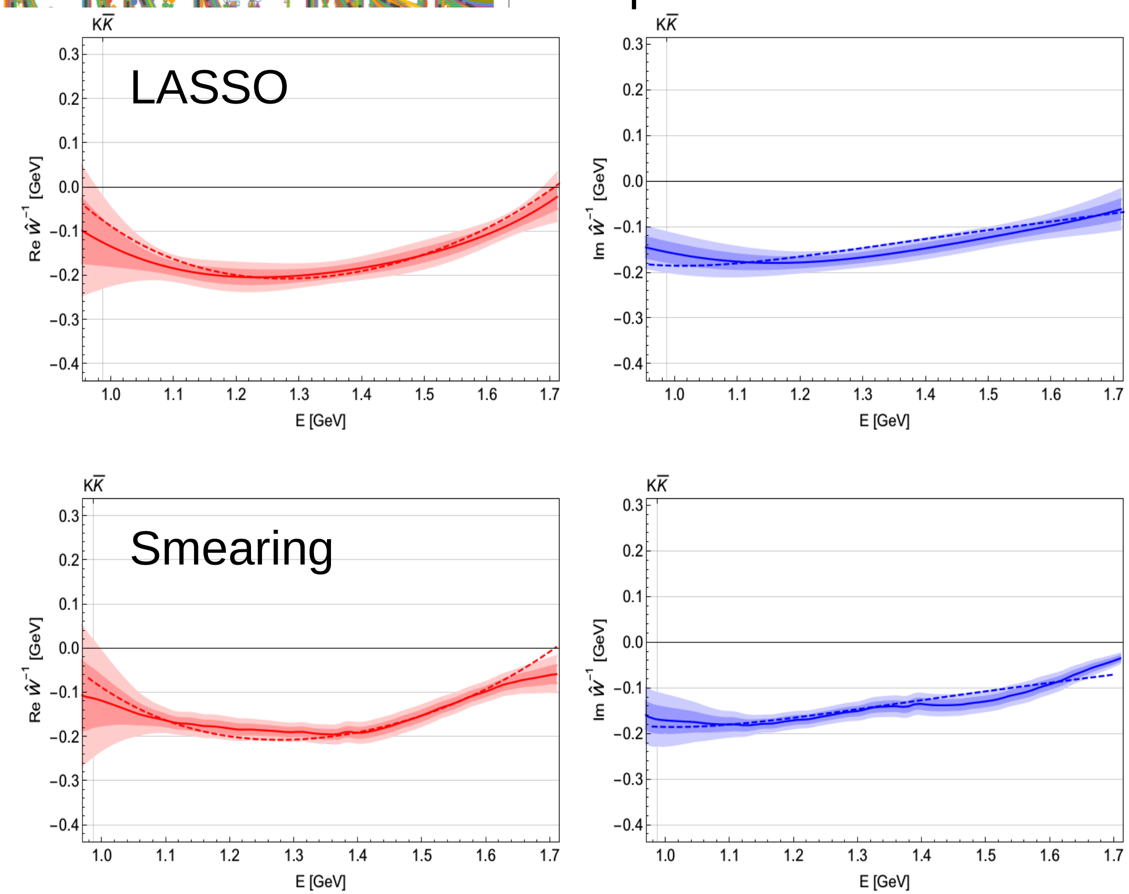
Fit at finite ϵ, validate at different ϵ' ($E \to E + i\epsilon$).

Numerical simulation

Data & fits



Reconstructed potential



Unitarity & Matching

- 3-body Unitarity (normalization condition ↔ phase space integral)

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$