Appendix B
Symmetries and group theory review

We give here a very short review of symmetries, and of the theory of Lie groups needed in their study. The treatment is not intended to be complete or rigorous, just to give a brief introduction for readers unfamiliar with the material.

B.1 Symmetry transformations as a group

A symmetry transformation is a transformation on the states of a theory $|\psi\rangle$ and the operators $\mathcal{O}$,

$$|\psi\rangle \;\to\; U|\psi\rangle\,, \qquad \mathcal{O} \;\to\; U\,\mathcal{O}\,U^* \tag{B.1}$$

which preserves "all physics." In particular, amplitudes must be preserved (up to a phase, a complication we ignore here and in the following)

$$\langle\psi_1|\,\psi_2\rangle \;\to\; \langle\psi_1|U^*U\,|\,\psi_2\rangle = \langle\psi_1|\,\psi_2\rangle \tag{B.2}$$

which shows that U must either be unitary or anti-unitary. We will only consider unitary symmetries here. (Here U^* is the Hermitian conjugate of U. We write $U^\dagger$ only if U is a matrix.)

A symmetry in which a local operator $\mathcal{O}(x)$ (an operator built out of fields at point x only) is transformed into a local operator at the same point, is called an *internal symmetry*. Such symmetries can be considered separately from *spacetime symmetries*, such as translations, rotations, and boosts. In fact, it is a theorem that the full group of symmetries is always a product of the internal symmetries and the spacetime symmetries. In this appendix we concentrate on internal symmetries; spacetime symmetries are discussed in the next appendix.

Because the states and operators under discussion may appear at different times, the symmetry operator must commute with time evolution,

$[H, U] = 0$. Similarly, it must commute with the momentum operators. This is summarized by saying that it must commute with the Lagrangian density, $[\mathcal{L}, U] = 0$. Therefore the symmetries of the theory can usually be identified by looking at the symmetries of the Lagrangian.

In a renormalizable theory, an operator $\mathcal{O}$ and its symmetry transform $U \mathcal{O} U^* = \mathcal{O}'$ must be of the same dimension. Since the fields are the operators of the smallest dimension, this means the fields transform linearly among themselves,

$$\varphi_a \;\to\; U\varphi_a U^* = M_{ab}^{-1}\varphi_b \tag{B.3}$$

where φ collectively symbolizes the fields of the theory and a, b are indices on the set of fields. Classically, the symmetries of a theory are the set of transformations on the fields of this form, under which $\mathcal{L}$ is unchanged. The relation between φ' and φ involves a matrix inverse essentially because φ annihilates a particle $|\varphi\rangle$, and must therefore have the inverse transformation properties of the particle.

Acting with two symmetry transformations successively, $|\psi\rangle \to U_1|\psi\rangle \to U_2U_1|\psi\rangle$, yields another symmetry transformation, namely, the one induced by the operator (U_2U_1). This defines a multiplication rule under which symmetry transformations form a group. In general the symmetry group can be factorized as a product of nonfactorizable subgroups, and it is sufficient to examine the behavior of the subgroups individually. In particle physics these subgroups are usually small discrete groups (which we will not discuss) and continuous (Lie) groups. The latter can generally be factorized into a product of simple Lie groups and $U(1)$ groups.

B.2 Lie groups and Lie algebras

A Lie group is a group which is also a manifold. In particular, there is a small neighborhood around the identity $\mathbf{1}$ which looks like a piece of $\mathbf{R}^n$, with n the dimension of the group. One can always choose a coordinate basis for this region; the coordinate unit vectors t_a are called the *Lie algebra* and an arbitrary element g of the group which is close to the identity can always be expanded in the coordinates,

$$g = \mathbf{1} + \mathrm{i}\omega^\alpha t_\alpha \tag{B.4}$$

with ω_α (infinitesimal) parameters. (The i is customary so that for groups of unitary matrices, the t_α are Hermitian.)

Now consider two elements of the group which are each close to the identity, say, $g_1 = \mathbf{1} + \mathrm{i}\omega_1^\alpha t_\alpha$ and $g_2 = \mathbf{1} + \mathrm{i}\omega_2^\alpha t_\alpha$. The multiplication rule to first order in these parameters is given by

$$g_1 g_2 = \mathbf{1} + \mathrm{i}(\omega_1^\alpha + \omega_2^\alpha)t_\alpha + O(\omega^2) \tag{B.5}$$

which is addition of the departures from the identity. At the next order, $g_1 g_2$ and $g_2 g_1$ can differ:

$$\begin{aligned} g_1 g_2 (g_2 g_1)^{-1} &= (\mathbf{1} + \mathrm{i}\omega_1^\alpha t_\alpha)\left(\mathbf{1} + \mathrm{i}\omega_2^\beta t_\beta\right)(\mathbf{1} - \mathrm{i}\omega_1^\gamma t_\gamma)(\mathbf{1} - \mathrm{i}\omega_2^\sigma t_\sigma) \\ &= \mathbf{1} - \omega_1^\alpha \omega_2^\beta \Big[t_\alpha \, , \, t_\beta\Big] \end{aligned} \tag{B.6}$$

Therefore, to determine the multiplication rule to second order we need to know the commutators of the Lie algebra elements. Since $g_1 g_2 g_1^{-1} g_2^{-1}$ is still close to the identity, it can still be expressed in terms of coefficients multiplying Lie algebra elements, so the commutator must also be a sum of elements of the Lie algebra:

$$\Big[t_\alpha \, , \, t_\beta\Big] = \mathrm{i} f_{\alpha\beta}^\gamma t_\gamma \tag{B.7}$$

The *structure constants*, $f_{\alpha\beta}^\gamma$ are real valued and explicitly antisymmetric in the last two indices, and are antisymmetric in all indices if the t_α are chosen orthonormal.

The Lie algebra elements and the structure constants together constitute the *Lie algebra* of the group. They turn out to be sufficient to determine the group almost uniquely.†

The groups of interest in particle physics are compact Lie groups. These can all be thought of as groups of matrices. Of particular interest is the group of $N \times N$ special (unit determinant) unitary matrices, $SU(N)$, which we describe in more detail in the next section.

B.3 Group representations

We saw in Eq. (B.3) that a symmetry transformation acts on a field operator like a matrix multiplication. Successive symmetry transformations act like a series of matrix multiplications, which gives us a condition on the matrices which can appear in Eq. (B.3). Namely, under successive transformations by two group elements,

$$U(g_2 g_1)\varphi_a U^*(g_2 g_1) \;\; = \;\; U(g_2)U(g_1)\varphi_a U^*(g_1)U^*(g_2)$$

† A Lie group can have several disconnected pieces; the Lie algebra specifies only the connected piece containing the identity. For simple compact Lie groups, the Lie algebra gives a unique simply connected group, and any other connected group with the same Lie algebra must be a quotient of this group over a discrete identification map. For instance, the Lie algebra of rotations gives the group $SU(2)$. The group $SO(3)$ has the same Lie algebra, but differs in that a rotation by 360°, represented in $SU(2)$ by diag$[-1, -1]$, is identified with the identity in $SO(3)$.

$$\begin{aligned} M^{-1}_{ab}(g_2 g_1)\varphi_b &= U(g_2) M^{-1}_{ab}(g_1)\varphi_b U^*(g_2) \\ M^{-1}_{ac}(g_2 g_1)\varphi_c &= M^{-1}_{ab}(g_1) M^{-1}_{bc}(g_2)\varphi_c \end{aligned} \tag{B.8}$$

Since this must hold for any field φ, the matrices themselves must be equal,

$$M^{-1}(g_2 g_1) = M^{-1}(g_1) M^{-1}(g_2) \quad \text{or} \quad M(g_2 g_1) = M(g_2) M(g_1) \tag{B.9}$$

Matrix multiplication must respect group element multiplication. A set of matrices associated with elements of a group which satisfy this condition are called a *representation* of the group. Since the matrices can be thought of as operating on column vectors, physicists often refer to column vectors (or fields) which are multiplied by such matrices as representations. More properly, one should say that such column vectors or fields are "acted on" or "transform under" the representation. To understand the ways symmetries can act on fields and field products we must understand representations and their tensor products.

In any representation, the identity element of the group must be mapped into the identity matrix $\mathbf{1}$. An element close to the identity must map into an element close to the identity matrix, so

$$M(\mathbf{1} + \mathrm{i}\omega^\alpha t_\alpha) = \mathbf{1} + \mathrm{i}\omega^\alpha T_\alpha \tag{B.10}$$

with T_α some matrices particular to the representation. (We use Greek letters to index the Lie algebra and Roman letters for matrix indices.) It then follows by considering products of such matrices and using Eq. (B.9) that the matrices T_α must satisfy a Lie algebra with the same structure constants as the t_α:

$$\Big[T_\alpha \,,\, T_\beta\Big] = \mathrm{i} f^\gamma_{\alpha\beta} T_\gamma \tag{B.11}$$

Furthermore, any set of matrices T_α which satisfy this identity can be exponentiated to give a representation. Frequently a basis of fields can be found under which the T_α are all block diagonal, in which case the representation is said to *reduce* into the blocks. A representation which cannot be block diagonalized by any basis change is called *irreducible*. The problem of classifying representations of a group G is the problem of finding all sets of matrices T_α which obey the same commutation relations as the Lie algebra of the group.

Every group has a representation, called the *singlet* or *trivial* representation, in which $M(g)$ is the 1×1 identity matrix for each g, and $T_\alpha = 0$ for all α. (Equation (B.9) is satisfied because $1 \times 1 = 1$, and Eq. (B.11) is automatically satisfied since both sides are zero.) Invariance of the Lagrangian under a symmetry is equivalent to the requirement that the Lagrangian transform under the singlet representation. Therefore it will be important to see how other representations can be combined together to give the singlet representation.

Every group also contains a representation called the *adjoint* representation, made up of $n \times n$ real matrices, with n the number of elements in the Lie algebra, and with T_α given by $(T_\alpha)_{bc} = -\mathrm{i} f^{\alpha}_{bc}$ (with b, c the matrix indices). For the case of an abelian group (a group where the f vanish) the adjoint representation is the same as the singlet representation. For the group of rotations, $SU(2)$, it is the spin-one representation.

Physicists generally refer to the several fields which transform together in an irreducible representation of the symmetry group as "a" field transforming under that representation. If two such fields ϕ, χ transform under two different representations with representation matrices M, N which are respectively $m \times m$ and $n \times n$, then the operator $\phi_a \chi_b$ transforms as

$$U(g^{-1})\phi_a \chi_b U^*(g^{-1}) = M_{ac}(g) N_{bd}(g) \phi_c \chi_d \tag{B.12}$$

The object $M_{ac}N_{bd}$ can be considered an $(mn) \times (mn)$ matrix obtained as the tensor product of the matrices M and N. So the product of two operators transforms under the tensor product of the representation matrices. In general such tensor products are reducible – for instance, in the familiar example of angular momentum $SU(2)$, two spin-half operators can combine into a spin-one or a spin-zero operator, because the tensor product $\frac{1}{2} \otimes \frac{1}{2}$ is reducible, $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

Representations and the rules for their tensor products are quite group dependent. We will quickly outline what happens for $U(1)$ and $SU(N)$, since these arise the most often in physics and in particular are the only groups needed in the standard model.

B.3.1 Representations of U(1)

The group $U(1)$ is the group of phase rotations. A generic element is $\mathrm{e}^{\mathrm{i}\theta}$ and the group is parametrized by θ. Any irreducible representation can be written as a 1×1 complex matrix (a complex number) and the representation is determined by a charge q, with the group element $g = \mathrm{e}^{\mathrm{i}\theta}$ being represented by $\mathrm{e}^{\mathrm{i}q\theta}$. The tensor product of two representations is just a representation with the sum of the charges. Therefore, the charge of a product of operators is the sum of their charges. For the Lagrangian to have a $U(1)$ symmetry, each term in the Lagrangian must have the charges of the fields add up to zero.

B.3.2 Representations of $SU(N)$

The group $SU(N)$ consists of complex $N \times N$ matrices U which are unitary, $U^\dagger = U^{-1}$, and satisfy $\det U = 1$. (The U in $SU(N)$ stands for unitary, the S for "special," meaning determinant 1.)

A generic element of $SU(N)$ can be written $U(\omega) = \exp(\mathrm{i}\omega_\alpha t_\alpha)$, with t_α a standard set of $N \times N$ complex matrices and ω_α parameters. Before imposing unitarity and determinant 1, there are $2N^2$ independent t_α. However, unitarity requires each t_α be Hermitian, eliminating half, and the unit determinant condition requires each t_α be traceless, eliminating one more possibility. Therefore there are $N^2 - 1$ independent elements t_α of the Lie algebra, which should be chosen to be orthogonal and to satisfy the same normalization condition. For $SU(2)$ they can be chosen to be half the Pauli matrices, Eq. (2.14). In this case the structure functions are

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = \mathrm{i} f_{kij} \frac{\tau_k}{2}, \qquad f_{kij} = \epsilon_{kij} \tag{B.13}$$

the totally antisymmetric tensor. For $SU(3)$ the Lie algebra elements can be chosen to be half the Gell-Mann matrices of Eq. (1.186). There is no simple expression for the resulting structure functions.

Besides the singlet representation, the smallest representation for $SU(N)$ is the $SU(N)$ matrices themselves, $M(U) = U$. This is called the *fundamental representation*.

$$\text{Fundamental representation:} \quad T_\alpha = t_\alpha \tag{B.14}$$

This is, for instance, the representation quarks transform under in QCD. It is customary to refer to the representations of $SU(N)$ according to the rank of the representation matrices, so the singlet representation is called the **1** representation and the fundamental representation is called the **N** representation.

Equally important is the *antifundamental representation*, $M(U) = U^*$, given by complex conjugating (but not transposing) the $SU(N)$ matrices.

$$\text{Antifundamental representation:} \quad T_\alpha = -t_\alpha^* \tag{B.15}$$

This is the representation which antiquarks transform under. To see that it is a valid representation, note that

$$\left[-t_\alpha^*\,,\, -t_\beta^*\right] = \left(\left[t_\alpha\,,\, t_\beta\right]\right)^* = (\mathrm{i} f_{\alpha\beta}^\gamma t_\gamma)^* = \mathrm{i} f_{\alpha\beta}^\gamma (-t_\gamma^*) \tag{B.16}$$

Therefore the $-t_\alpha^*$ obey the same commutation relations as the t_α. These matrices have rank N, but since the symbol **N** is taken, the representation is called the $\overline{\mathbf{N}}$ representation.

A field which transforms under the fundamental representation is left-multiplied by U, and can be thought of as a column vector. For the antifundamental representation, the column vector is left-multiplied by U^*, and is more conveniently thought of as a row vector right-multiplied by $U^\dagger = U^{-1}$. In general, if M is a representation of a group, M^* is also, and is called the *conjugate representation* to M. The contraction of a fundamental and an antifundamental field (or any operators in conjugate representations to each other) forms a singlet,

$$\text{for } \chi_a \overline{\mathbf{N}}, \quad \phi_a \mathbf{N}, \qquad \chi^T \phi \equiv \chi_a \phi_a \text{ is singlet } \mathbf{1} \tag{B.17}$$

At the same time, inserting the generators of the representation between them,

$$\text{for } \chi_a \overline{\mathbf{N}}, \quad \phi_a \mathbf{N}, \qquad \chi^T T_\alpha \phi \equiv \chi_a (T_\alpha)_{ab} \phi_b \;\text{ is adjoint} \tag{B.18}$$

Since there are $N^2 - 1$ elements in the adjoint representation and one in the singlet, this uses up the $N \times N = N^2$ objects in the tensor product of fundamental and antifundamental representations;

$$\overline{\mathbf{N}} \otimes \mathbf{N} = (\mathbf{N^2} - \mathbf{1}) \oplus \mathbf{1} \tag{B.19}$$

It is also important to know how multiple fundamental representations tensor together. Here it is important to know that the totally antisymmetric object $\epsilon_{ab\ldots}$, which contracts N fundamental indices, is an invariant. Using it to contract $N-1$ objects transforming in the fundamental representation gives an object transforming in the antifundamental representation. For $SU(2)$, this means that a single fundamental representation object can be "flipped" into an antifundamental representation object by ε, as we did with the Higgs field in Eq. (2.12). In $SU(2)$ the fundamental and antifundamental representations are equivalent and generally not distinguished from each other. In $SU(3)$, contracting two fundamental fields with the antisymmetric tensor, $\epsilon_{abc}\phi_b\psi_c$, produces an antifundamental object, and three gives a singlet. The other (symmetrized) linear combination of two fundamental fields has six components and is called the **6** representation:

$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6} \qquad \text{in } SU(3) \tag{B.20}$$

More generally one gets representations containing $N(N-1)/2$ and $N(N+1)/2$ elements. In general, $SU(N)$ groups have a large number of representations, all of which can be found by taking antisymmetrized and symmetrized combinations of fundamental representation objects. Further enumerating them is beyond the scope of this appendix.