Appendix C
Lorentz group and the Dirac algebra

This appendix provides a review and summary of the Lorentz group, its properties, and the properties of its infinitesimal generators. It then reviews representations of the Lorentz group and the Dirac algebra. This material is intended to supplement Chapter 1, for those students who are not as familiar with the Lorentz group and Dirac equation as they find they need to be.

C.1 Lorentz group

According to special relativity, physical laws are unchanged by a linear change of coordinates,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \xi^{\mu} \tag{C.1}$$

with Λ and ξ real, provided it leave unchanged the invariant separation between points,

$$(x-y)_{\mu}(x-y)^{\mu} = \eta_{\mu\nu}(x-y)^{\mu}(x-y)^{\nu} = -[(x-y)^0]^2 + [\vec{x}-\vec{y}]^2$$

This condition does not constrain ξ, since it cancels in the difference, but it imposes a constraint on Λ,

$$x_{\mu}x^{\mu} = x'_{\mu}x'^{\mu} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}x^{\alpha}\Lambda^{\nu}{}_{\beta}x^{\beta} \tag{C.2}$$

for all x^{μ}. A transformation of the form shown in Eq. (C.1) which satisfies Eq. (C.2) is called a Poincaré transformation. These transformations close and form a group, called the Poincaré group. The subset where Λ is the identity matrix and ξ is arbitrary is a subgroup called the group of translations. We assume that this group and its implications, such as conservation of energy and momentum, are familiar to the reader. Instead we concentrate on the subgroup in which $\xi = 0$, which is called the Lorentz group.

It is convenient to think of an element of the Lorentz group as a matrix which is operating on the coordinate x^μ. This is possible if we always write Λ with its first index raised and second index lowered, so it carries x^μ to x'^μ, both with raised index. Writing in this way, repeated Lorentz transformations are implemented via matrix multiplications of the respective Λ matrices:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \text{ and } x''^\mu = \Lambda'^\mu{}_\nu x'^\nu \;\Rightarrow\; x''^\mu = \Lambda'^\mu{}_\nu \Lambda^\nu{}_\alpha x^\alpha \equiv \Big[\Lambda'\Lambda\Big]^\mu{}_\alpha x^\alpha \quad \text{(C.3)}$$

We see from Eq. (C.2) that the condition on $\Lambda^\mu{}_\nu$ to be a Lorentz transformation is

$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu x^\mu x^\nu \quad \text{(C.4)}$$

for all x^μ. Since this must hold for all x^μ, we have

$$\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \eta_{\alpha\beta} \Lambda^\beta{}_\nu \quad \text{(C.5)}$$

or (writing $\eta_{\mu\nu}$ as η when using matrix notation)

$$\eta = \Lambda^T \eta\, \Lambda \quad \text{(C.6)}$$

The group of matrices Λ satisfying Eq. (C.6) is called $O(3,1)$, and is a Lie group. Therefore the same technology of Lie algebra generation may be applied to it as to the groups of the previous appendix.

As we will discuss momentarily, not all elements of $O(3,1)$ can be built infinitesimally from the identity. Those elements which can, form a subgroup written $SO(3,1)$, which we will now analyze. A Lorentz transformation $\Lambda^\mu{}_\nu$ which is infinitesimally close to the identity must be of the form

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \quad \text{(C.7)}$$

with $\omega^\mu{}_\nu$ a matrix of infinitesimal coefficients. The condition on $\omega^\mu{}_\nu$ for $\Lambda^\mu{}_\nu$ to be a valid Lorentz transformation is found by inserting Eq. (C.7) into Eq. (C.6) and expanding to linear order in ω:

$$\begin{aligned} \eta_{\mu\nu} &= \left(\delta^\alpha_\mu + \omega^\alpha{}_\mu\right) \eta_{\alpha\beta} \left(\delta^\beta_\nu + \omega^\beta{}_\nu\right) \\ &= \eta_{\mu\nu} + (\omega_{\nu\mu} + \omega_{\mu\nu}) + O(\omega^2) \\ 0 &= \omega_{\nu\mu} + \omega_{\mu\nu} \end{aligned} \quad \text{(C.8)}$$

That is, the condition on $\omega^\mu{}_\nu$ is that $\omega_{\mu\nu}$ be antisymmetric on its indices. The space of real antisymmetric 4×4 matrices is six-dimensional, so the Lorentz group is six-dimensional.

Now $\omega^\mu{}_\nu$ is related to $\omega_{\mu\nu}$ as $\omega^\mu{}_\nu = \eta^{\mu\alpha}\omega_{\alpha\nu}$. Since $\eta^{00} = -1$ and $\eta^{ii} = 1$ for $i = 1, 2, 3$, the sign of the space–time component of $\omega^\mu{}_\nu$ must be the same as the sign of the time–space component, while the space–space components must be antisymmetric. Thus, the most general form of $\omega^\mu{}_\nu$ is

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -r_3 & r_2 \\ b_2 & r_3 & 0 & -r_1 \\ b_3 & -r_2 & r_1 & 0 \end{pmatrix} \tag{C.9}$$

symmetric in the space–time entries and antisymmetric in the space–space entries. The b_1, b_2, b_3 entries respectively cause infinitesimal boosts in the $1, 2, 3$ directions; the r_1, r_2, r_3 entries cause infinitesimal rotations about the $1, 2, 3$ axes. A general element of $SO(3,1)$ can be written as an exponential of a *finite* $\omega^\mu{}_\nu$,

$$\Lambda^\mu{}_\nu = (\exp \omega)^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + \frac{1}{2}\omega^\mu{}_\alpha \omega^\alpha{}_\nu + \frac{1}{6}\omega^\mu{}_\alpha \omega^\alpha{}_\beta \omega^\beta{}_\nu + \cdots \tag{C.10}$$

When only the r_i are nonzero, this gives a rotation by angle $|\vec{r}|$ about the $\hat{r}$ axis. When only the b_i are nonzero, this gives a boost by velocity $\tanh|\vec{b}|$ along the $\hat{b}$ axis. When both $\vec{r}$ and $\vec{b}$ are nonzero the Lorentz transformation cannot be described either solely as a rotation or as a boost. Note that, while a rotation by angle $|\vec{r}| = 2\pi$ gives the identity Λ, no nonzero magnitude of boost $|\vec{b}|$ returns the identity. Hence the group $SO(3,1)$ is *noncompact*.

Now we argue that $O(3,1)$ has four disconnected pieces, one of which is $SO(3,1)$. To see this, take the determinant of Eq. (C.6):

$$\det \eta = \det \Lambda^T \eta\, \Lambda = \det \eta \times (\det \Lambda)^2 \tag{C.11}$$

Since η is nonsingular, we can divide by $\det \eta$:

$$(\det \Lambda)^2 = 1 \quad \Rightarrow \quad \det \Lambda = \pm 1 \tag{C.12}$$

The determinant must vary continuously within a path connected region of $O(3,1)$, but you cannot go continuously from 1 to -1, so any elements of $O(3,1)$ with $\det \Lambda = -1$ cannot be elements of the connected group $SO(3,1)$. An element of $O(3,1)$ with $\det \Lambda = 1$ is called *proper*, and an element with $\det \Lambda = -1$ is called *improper*.

Furthermore, if we write out the $\mu = 0$, $\nu = 0$ element of Eq. (C.6), it is

$$\begin{aligned} \eta_{00} &= \Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0 \\ -1 &= -\Lambda^0{}_0 \Lambda^0{}_0 + \sum_{i=1,2,3} \Lambda^i{}_0 \Lambda^i{}_0 \\ (\Lambda^0{}_0)^2 &= 1 + \sum_{i=1,2,3} (\Lambda^i{}_0)^2 \quad \geq 1 \end{aligned} \tag{C.13}$$

so the square of the time–time component of any Λ must always be at least

1, and ${\Lambda^0}_0$ must be either ≥ 1 or ≤ -1. Again, you cannot go continuously from ≥ 1 to ≤ -1, so no elements of $SO(3,1)$ have ${\Lambda^0}_0 < 0$. An element of $O(3,1)$ with ${\Lambda^0}_0 \geq 1$ is called *orthochronous*, and an element with ${\Lambda^0}_0 \leq -1$ is called *nonorthochronous*.

The canonical example of an improper (but orthochronous) element of $O(3,1)$ is the *parity transformation*,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{C.14}$$

which satisfies Eq. (C.6) but has determinant -1. The canonical example of a nonorthochronous (and also improper) transformation is the *time reversal transformation*,

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.15}$$

which also satisfies Eq. (C.6) but has ${T^0}_0 = -1$. It turns out that any element of $O(3,1)$ must be an element of $SO(3,1)$, times either the identity (proper orthochronous), P (improper orthochronous), T (improper nonorthochronous), or PT (proper nonorthochronous). The improper or nonorthochronous Lorentz transformations need not be symmetries of nature – in fact, in the standard model, they are not – but it is an axiom of field theory that the elements of $SO(3,1)$ must be symmetries.

C.2 Generators of the Lorentz group

As discussed in Section B.1, each element $\Lambda \in SO(3,1)$ must have associated with it a unitary operator $U(\Lambda)$ which implements it on the Hilbert space, and which represents the group operation

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2) \tag{C.16}$$

For an element infinitesimally close to the identity, it must be possible to expand these operators in a Lie algebra of generators,

$$U(\omega) = \mathbf{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} + O(\omega^2) \tag{C.17}$$

for some operators $\hat{J}^{\mu\nu}$, antisymmetric in μ, ν. Similarly, there are generators for translations,

$$U(\xi) = \mathbf{1} - \mathrm{i}\xi_\mu \hat{P}^\mu \tag{C.18}$$

The $\hat{P}^i$ are also called momentum operators, and the $\hat{J}^{ij}$ are called angular momentum operators.

The commutation relations between the operators $\hat{P}^\mu$, $\hat{J}^{\mu\nu}$ can be worked out by using Eq. (C.16). For instance, consider a translation by a small distance ξ^μ, either preceded or followed by a Lorentz transformation involving $\omega^\nu{}_\alpha$. We will evaluate the difference between the two orders of operation, in two ways.

First, if we translate first and then rotate, then the coordinate is transformed according to

$$\begin{aligned} x'^\mu &= x^\mu + \xi^\mu \\ x''^\mu &= (\delta^\mu_\nu + \omega^\mu{}_\nu)(x^\nu + \xi^\nu) \\ &= x^\mu + (\omega^\mu{}_\nu\, x^\nu) + (\xi^\mu + \omega^\mu{}_\nu\, \xi^\nu) \end{aligned} \tag{C.19}$$

where the first and second parenthesis represent a rotation and a translation. The result is the same rotation, and a translation by ξ^μ plus an extra piece involving ω and ξ. If the rotation is performed first, we get

$$\begin{aligned} x'^\mu &= x^\mu + \omega^\mu{}_\nu x^\nu \\ x''^\mu &= x^\mu + (\omega^\mu{}_\nu x^\nu) + (\xi^\mu) \end{aligned} \tag{C.20}$$

which is the rotation and the translation just by ξ. The unitary operators for these transformations are

$$\begin{aligned} U(\omega\xi) &= \mathbf{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} - \mathrm{i}\eta_{\mu\nu}(\xi^\mu + \omega^\mu{}_\alpha \xi^\alpha)\hat{P}^\nu \\ U(\xi\omega) &= \mathbf{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} - \mathrm{i}\eta_{\mu\nu}(\xi^\mu)\hat{P}^\nu \end{aligned} \tag{C.21}$$

The difference of the operators, to second order in the infinitesimals, is

$$U(\omega\xi) - U(\xi\omega) = -\mathrm{i}\eta_{\mu\nu}\omega^\mu{}_\alpha \xi^\alpha \hat{P}^\nu \tag{C.22}$$

(There is actually also a second order in ω piece, but it is the same for the two Us and therefore cancels in this difference.)

Alternately, we can say using Eq. (C.16) that

$$\begin{aligned} U(\omega\xi) = U(\omega)U(\xi) &= \left(\mathbf{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right)\left(\mathbf{1} - \mathrm{i}\eta_{\alpha\beta}\xi^\alpha \hat{P}^\beta\right) \\ U(\xi\omega) = U(\xi)U(\omega) &= \left(\mathbf{1} - \mathrm{i}\eta_{\alpha\beta}\xi^\alpha \hat{P}^\beta\right)\left(\mathbf{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right) \end{aligned}$$

$$U(\omega\xi) - U(\xi\omega) \quad = \quad -\frac{\mathrm{i}^2}{2}\omega_{\mu\nu}\eta_{\alpha\beta}\xi^\alpha\left[\hat{J}^{\mu\nu}\,,\,\hat{P}^\beta\right] \tag{C.23}$$

Now equating Eq. (C.22) and Eq. (C.23), we learn what the commutator of $\hat{P}$ with $\hat{J}$ must be:

$$-\frac{\mathrm{i}^2}{2}\omega_{\mu\nu}\xi_\alpha\left[\hat{J}^{\mu\nu}\,,\,\hat{P}^\alpha\right] = -\mathrm{i}\omega_{\mu\nu}\xi_\alpha\eta^{\nu\alpha}\hat{P}^\mu \tag{C.24}$$

This must hold for any antisymmetric $\omega_{\mu\nu}$ and any ξ_α, so (antisymmetrizing over the indices on ω) the operators must satisfy

$$\left[\hat{J}^{\mu\nu}\,,\,\hat{P}^\alpha\right] = \mathrm{i}\left(\eta^{\mu\alpha}\hat{P}^\nu - \eta^{\nu\alpha}\hat{P}^\mu\right) \tag{C.25}$$

By a completely analogous but more involved procedure one can also show

$$\left[\hat{J}^{\mu\nu}\,,\,\hat{J}^{\alpha\beta}\right] = \mathrm{i}\left(\eta^{\nu\beta}\hat{J}^{\mu\alpha} + \eta^{\mu\alpha}\hat{J}^{\nu\beta} - \eta^{\mu\beta}\hat{J}^{\nu\alpha} - \eta^{\nu\alpha}\hat{J}^{\mu\beta}\right) \tag{C.26}$$

and (this is simpler)

$$\left[\hat{P}^\mu\,,\,\hat{P}^\nu\right] = 0 \tag{C.27}$$

These commutation relations are called the Poincaré algebra.

To make contact with the more familiar generators of rotations and boosts, it is convenient to define

$$\hat{J}_i \quad \equiv \quad \frac{\epsilon_{ijk}}{2}\hat{J}_{jk} \tag{C.28}$$

$$\hat{K}_i \quad \equiv \quad \hat{J}^0{}_i \tag{C.29}$$

which are respectively the generator of rotations about the i axis and of boosts along the i axis, so a rotation by $\vec{\theta}$ is $\exp(-\mathrm{i}J_i\theta_i)$ and a boost by $\vec{v}$ is $\exp(-\mathrm{i}K_iv_i)$. They satisfy the commutation relations, following from Eq. (C.26),

$$\left[\hat{J}_i\,,\,\hat{J}_j\right] \quad = \quad \mathrm{i}\epsilon_{ijk}\hat{J}_k \tag{C.30}$$

$$\left[\hat{J}_i\,,\,\hat{K}_j\right] \quad = \quad \mathrm{i}\epsilon_{ijk}\hat{K}_k \tag{C.31}$$

$$\left[\hat{K}_i\,,\,\hat{K}_j\right] \quad = \quad -\mathrm{i}\epsilon_{ijk}\hat{J}_k \tag{C.32}$$

The first expression is the familiar commutator between rotations. The second means that, if a rotation is performed before a boost, the boost will be in a different direction than before the rotation is performed, which is intuitively clear. The third result is more surprising; the commutator of two boosts is a rotation. More importantly, the sign is opposite on the last result than on the previous two.

C.3 Representations of the Lorentz group

Just as for an internal symmetry, an $SO(3,1)$ transformation will carry a field to a linear combination of fields, so the fields must transform under representations of the group. The difference is that the transformed field will be at the Lorentz transformed point:

$$U(\omega)\varphi_a(x)U^*(\omega) = D^{-1}_{ab}(\omega)\varphi_b(\Lambda x) \tag{C.33}$$

with $D^{-1}(\omega) = D(-\omega)$ an ω dependent matrix in the space of fields. The fields can be chosen to block-diagonalize the matrix D into irreducible representations of the Lorentz group. For instance, in QED, the components of the gauge potential A^μ mix with each other under Lorentz transformations, but they never mix with the different spin components of the electron e_i; so there is one "block" of D which mixes the A^μ and an independent block mixing the e_i. Our goal now is to find the possible structures D can take.

Just as for internal symmetries, there are two very simple irreducible representations, which are also physically important. One is the trivial (singlet) representation,

$$U(\omega)\phi(x)U^*(\omega) = \phi(\Lambda x) \tag{C.34}$$

which for $SO(3,1)$ is called the *scalar representation*. Lorentz symmetry demands that the Lagrangian be a Lorentz scalar. The other is the *vector representation*, for which the field index is a four-vector index and the representation matrix is Λ itself:

$$U(\omega)A^\mu(x)U^*(\omega) = (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x) \tag{C.35}$$

A representation is determined by a set of matrices $\mathcal{J}^{\mu\nu}$ with the same commutation relations as the $\hat{J}^{\mu\nu}$. That is, the matrix D_{ab} must be of the form

$$D_{ab}(\omega) = \exp\left(\frac{\mathrm{i}}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}_{ab}\right) \tag{C.36}$$

with the exponentiation interpreted as matrix exponentiation with a, b the matrix indices, and $\mathcal{J}^{\mu\nu}$ satisfying

$$\left[\mathcal{J}^{\mu\nu}\, ,\, \mathcal{J}^{\alpha\beta}\right] = \mathrm{i}\left(\eta^{\nu\beta}\mathcal{J}^{\mu\alpha} + \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} - \eta^{\mu\beta}\mathcal{J}^{\nu\alpha} - \eta^{\nu\alpha}\mathcal{J}^{\mu\beta}\right) \tag{C.37}$$

The problem of finding representations is the problem of finding all sets of matrices with this algebra.

It is believed that only field theories containing finite numbers of fields are well defined. Therefore we need only look for finite-dimensional representations of $SO(3,1)$. The classification of the representations is made easier by the following convenient property of the group. Using Eq. (C.30)–Eq. (C.32), one can show that the operators

$$\hat{\mathcal{L}}_i \equiv \frac{\hat{J}_i + i\hat{K}_i}{2}, \qquad \hat{\mathcal{R}}_i \equiv \frac{\hat{J}_i - i\hat{K}_i}{2} \tag{C.38}$$

satisfy the commutation relations

$$\left[\hat{\mathcal{L}}_i\,,\,\hat{\mathcal{L}}_j\right] = \mathrm{i}\epsilon_{ijk}\hat{\mathcal{L}}_k \tag{C.39}$$

$$\left[\hat{\mathcal{R}}_i\,,\,\hat{\mathcal{R}}_j\right] = \mathrm{i}\epsilon_{ijk}\hat{\mathcal{R}}_k \tag{C.40}$$

$$\left[\hat{\mathcal{L}}_i\,,\,\hat{\mathcal{R}}_j\right] = 0 \tag{C.41}$$

Therefore the generators of $SO(3,1)$ can be split into two mutually commuting subsets, which each satisfy the same commutation relations as the group $SU(2)$. This group is familiar as the group of rotations and its representations are well known; they are the spin-zero representation, the spin-half representation, the spin-one representation, and so forth. A general irreducible representation can be described by its transformation properties under $\hat{\mathcal{L}}$ and under $\hat{\mathcal{R}}$, e.g., spin-$m/2$ under $\hat{\mathcal{L}}$ and spin-$n/2$ under $\hat{\mathcal{R}}$.

Only four representations will be of any interest to us in studying the standard model, because it turns out that only four representations can participate in renormalizable interactions in a theory satisfying the basic principles laid out in Section 1.2.

The first of these is the scalar representation already introduced. The scalar representation transforms as $(0,0)$, that is, as spin-zero under $\hat{\mathcal{L}}$ and spin-zero under $\hat{\mathcal{R}}$. The Lie algebra representations are $\hat{\mathcal{J}}^{\mu\nu} = 0$ and the transformation matrix $D = 1$ is the identity.

The second common representation is the vector representation, which transforms as $(\frac{1}{2}, \frac{1}{2})$. The Lie algebra is represented as $\mathcal{J}^{\mu\nu}_{\alpha\ \beta} = -\mathrm{i}(\eta^{\mu\alpha}\eta^{\nu}_{\beta} - \eta^{\mu}_{\beta}\eta^{\nu\alpha})$, and $D = \Lambda$, as already discussed. Note that for both of these representations, the matrix D is always real; therefore it is consistent to consider real-valued scalar or vector fields.

The other two interesting representations are called *spinor representations*, and consist of two fields which mix with each other under Lorentz transformations. Since these are probably less familiar to the reader and are in some ways more complicated than the scalar and vector representations, we will discuss them at length in the next section.

C.4 Spinors and the Dirac algebra

We now introduce the other two physically important representations of the Lorentz group, the left- and right-handed spinor representations. A field transforming in one of these can be rewritten in terms of the other, and it is convenient to combine them together using *Majorana notation*, which we will also introduce and which we use throughout this book.

C.4.1 Spinor representations

The simplest nontrivial matrices which satisfy Eq. (C.39) are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.42}$$

which satisfy the commutation relation

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = \mathrm{i}\epsilon_{ijk}\frac{\sigma_k}{2} \tag{C.43}$$

Therefore, if the matrices representing $\hat{\mathcal{L}}_i$ and $\hat{\mathcal{R}}_i$ are $\sigma_i/2$ and 0 respectively, we get a representation of the Lorentz algebra. Inverting Eq. (C.38), rotations and boosts are implemented by the matrices

$$\mathcal{J}_i = \frac{\sigma_i}{2}, \qquad \mathcal{K}_i = -\mathrm{i}\frac{\sigma_i}{2}, \qquad \text{(left-handed spinor)} \tag{C.44}$$

which it is easy to show satisfy Eq. (C.30) through Eq. (C.32).

Therefore, a pair of fields $\psi_a, a = 1, 2$ can transform under Lorentz transformations according to

$$U(-\omega)\psi_a U^*(-\omega) = D_{ab}(\omega)\psi_b, \quad D(\omega) = [\exp(-\mathrm{i}(r_i - \mathrm{i}b_i)\sigma_i/2)] \tag{C.45}$$

where r_i, b_i are the amount of rotation and boost performed, as introduced in Eq. (C.9). The two fields ψ_a are generally referred to as the components of a single *spinor field* with a the *spinor index*, which is almost always suppressed by writing ψ and D in matrix notation (ψ as a column vector, D as a matrix). Such a spinor field is called a *left-handed Weyl spinor* ψ_{L}.

Alternately, $\mathcal{R}_i$ could be represented by the Pauli matrices and $\mathcal{L}_i$ by 0s,

$$\mathcal{J}_i = \frac{\sigma_i}{2}, \qquad \mathcal{K}_i = \mathrm{i}\frac{\sigma_i}{2}, \qquad \text{(right-handed spinor)} \tag{C.46}$$

in which case a Lorentz transform acts on ψ via

$$U(-\omega)\psi_a U^*(-\omega) = D_{ab}(\omega)\psi_b, \quad D(\omega) = [\exp(-\mathrm{i}(r_i + \mathrm{i}b_i)\sigma_i/2)] \tag{C.47}$$

A field transforming this way is called a *right-handed Weyl spinor* ψ_{R}.

Since the matrices D we just constructed are in general complex, a spinor $\psi_{\rm L}$ or $\psi_{\rm R}$ must be a pair of complex fields. We can ask how the complex conjugate of $\psi_{\rm L}$ transforms. Because complex conjugation flips the i in front of $\mathcal{K}$ in Eq. (C.44), the answer is that it transforms as a right-handed Weyl spinor. More properly, defining the matrix

$$\varepsilon \equiv \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{satisfying} \qquad \varepsilon\sigma_i^* = -\sigma_i\varepsilon \tag{C.48}$$

we see that ε times the conjugate of $\psi_{\rm L}$ transforms according to

$$\begin{aligned} U(-\omega)\varepsilon\psi_{\rm L}^* U^*(-\omega) &= \varepsilon\left(\exp\left[-\mathrm{i}(r_i - \mathrm{i}b_i)\frac{\sigma_i}{2}\right]\psi_{\rm L}\right)^* \\ &= \varepsilon\exp\left[+\mathrm{i}(r_i + \mathrm{i}b_i)\frac{\sigma_i^*}{2}\right]\psi_{\rm L}^* \\ &= \exp\left[-\mathrm{i}(r_i + \mathrm{i}b_i)\frac{\sigma_i}{2}\right]\varepsilon\psi_{\rm L}^* \end{aligned} \tag{C.49}$$

which is precisely the transformation rule for a right-handed Weyl spinor. Similarly, $-\epsilon\psi_{\rm R}^*$ transforms as a left-handed Weyl spinor, and $-\epsilon(\epsilon\psi_{\rm L}^*)^* = \psi_{\rm L}$ transforms as a left-handed Weyl spinor again. Both the field and its complex conjugate will typically appear in the Lagrangian so it is important to have a notation which can deal with each. Whether we consider the left- or right-handed version as the field rather than the conjugated object is a matter of convention.

C.4.2 Weyl, Majorana, Dirac

There are two common notational ways of dealing with the fact that a field can be written either as a left- or a right-handed spinor.

One, called *Weyl notation*, expresses the fields as two component objects, and then specifies whether one is referring to $\psi_{\rm L}$ or to its right-handed conjugate $\varepsilon\psi_{\rm L}^*$ by using either an undotted or a dotted index: $\psi_\alpha = \psi_{\rm L}$ and $\psi^{\dot\alpha} = \varepsilon\psi_{\rm L}^*$. (Indices are raised and lowered using ε and dotted and undotted according to whether they are conjugated.) This notation is common in the supersymmetry and string theory literature.

An alternative which we will use, *Majorana notation*, writes a single *four* component field $\psi_{\rm M}$, defined as

$$\psi_{\rm M} = \begin{pmatrix} \psi_{\rm L} \\ \varepsilon\psi_{\rm L}^* \end{pmatrix} \tag{C.50}$$

that is, $\psi_{\rm M}$ redundantly records both the left-handed and the right-handed ways of writing the field. The individual pieces can be accessed separately by using the projection operators

$$P_{\mathrm{L}} \equiv \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_{\mathrm{R}} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{C.51}$$

The action of rotations and boosts on ψ_{M} are respectively,

$$\mathcal{J}_i = \begin{pmatrix} \frac{\sigma_i}{2} & 0 \\ 0 & \frac{\sigma_i}{2} \end{pmatrix}, \qquad \mathcal{K}_i = \begin{pmatrix} \frac{-i\sigma_i}{2} & 0 \\ 0 & \frac{i\sigma_i}{2} \end{pmatrix} \tag{C.52}$$

If a left-handed spinor transforms nontrivially under an internal symmetry group, then since the right-handed version involves complex conjugation, the right-handed version $\varepsilon\psi_{\mathrm{L}}^*$ transforms under the conjugate representation. In particular, if ψ_{L} has charge q under a $U(1)$ symmetry and is in the fundamental representation of an $SU(N)$ symmetry, then $\varepsilon\psi_{\mathrm{L}}^*$ has charge $-q$ and transforms under the antifundamental representation of $SU(N)$. One must keep this in mind when constructing Lagrangians out of Majorana spinors.

In QED and QCD, if we write the spinor fields as left-handed objects, the fields form pairs with conjugate symmetry transformation properties. For instance, in QED, there is a field $\mathcal{E}_{\mathrm{L}}$ which is charge -1 under $U_{\mathrm{em}}(1)$, called the left-handed electron, and a field $-\varepsilon E_{\mathrm{R}}^*$ which is charge 1 under $U_{\mathrm{em}}(1)$, called the left-handed positron. In this case it is most convenient to think of the latter as the conjugate of a right-handed field with charge -1, E_{R}, called the right-handed electron, and to combine them together in a single 4-component object called a *Dirac spinor*, $e = [\mathcal{E}_{\mathrm{L}} E_{\mathrm{R}}]^T$.

The Lorentz transformation properties of Majorana and Dirac spinors are the same. The two distinctions are that the upper and lower components of a Dirac spinor generally have the same transformation properties under internal symmetries, while for Majorana spinors they have conjugate transformation properties; and the upper and lower components of a Dirac spinor are independent, while for a Majorana spinor they are redundant notations for the same field.

C.4.3 Tensor products of spinors

Since the Lagrangian must be a Lorentz scalar, it must be a sum of terms even in spinorial fields. Therefore we need to know how products of two spinor fields transform. We will only consider the combination of a spinor field ψ_1 with the complex conjugate of another, $\psi_2^\dagger$. This is sufficient for Majorana spinors because $\psi_2^T\psi_1$ can be re-expressed in terms of $\psi_2^\dagger\psi_1$, and it suffices for Dirac spinors with internal symmetries because only such combinations are invariant under the internal symmetries.

The Hermitian conjugate of a spinor field ψ transforms as

$$U(-\omega)\psi^\dagger U^*(-\omega) = (D(\omega)\psi)^\dagger = \psi^\dagger D^\dagger(\omega) \tag{C.53}$$

The $\mathcal{J}_i$ are Hermitian, but the $\mathcal{K}_i$ are anti-Hermitian, so $D(\omega)$ is not in general unitary. Therefore $\psi^\dagger$ does not have the inverse transformation property of ψ. However, there is a Hermitian, unit determinant matrix β,

$$\beta \equiv \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \beta\mathcal{J}_i = \mathcal{J}_i\beta\,, \quad \beta\mathcal{K}_i = -\mathcal{K}_i\beta \tag{C.54}$$

which flips the sign of $\mathcal{K}$ but not $\mathcal{J}$ when commuted across $D^\dagger$, so $D^\dagger\beta = \beta D^{-1}$. Therefore, defining $\overline{\psi} = \psi^\dagger\beta$, called the *Dirac conjugate* of ψ,

$$U(-\omega)\overline{\psi}\,U^*(-\omega) = \psi^\dagger D^\dagger(\omega)\beta = \psi^\dagger\beta D^{-1}(\omega) = \overline{\psi}D^{-1}(\omega) \tag{C.55}$$

so $\overline{\psi}$ has the inverse transformation property of ψ.

Since ψ has four components, there are sixteen independent 4×4 matrices Γ which can be used to combine spinors, $\overline{\psi}_2\Gamma\psi_1$. These can all be gotten from four such matrices, called the *gamma matrices* γ^μ, given in Eq. (1.87). These satisfy anticommutation relations called the *Clifford algebra*,

$$\left\{\gamma^\mu\,,\,\gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbf{1} \tag{C.56}$$

The matrices $\mathcal{J}^{\mu\nu}$ can be expressed in terms of the gamma matrices:

$$\mathcal{J}^{\mu\nu} = \frac{-\mathrm{i}}{4}\left[\gamma^\mu\,,\,\gamma^\nu\right] \tag{C.57}$$

which together with Eq. (C.56) is enough to prove that $\mathcal{J}^{\mu\nu}$ satisfies the Lorentz algebra, Eq. (C.37). Further, these relations ensure that

$$\left[\mathcal{J}^{\mu\nu}\,,\,\gamma^\alpha\right] = i\left(\eta^{\mu\alpha}\gamma^\nu - \eta^{\nu\alpha}\gamma^\mu\right) \tag{C.58}$$

from which it follows that

$$D^{-1}(\omega)\gamma^\mu D(\omega) = \Lambda^\mu{}_\nu\gamma^\nu \tag{C.59}$$

Therefore, while the combination $\overline{\psi}_2\psi_1$ is a scalar

$$U(-\omega)\overline{\psi}_2\psi_1 U^{-1}(-\omega) = \overline{\psi}_2 D^{-1}(\omega)D(\omega)\psi = \overline{\psi}_2\psi_1 \text{ is a scalar} \tag{C.60}$$

the combination $\overline{\psi}_2\gamma^\mu\psi_1$ is a vector,

$$U(-\omega)\overline{\psi}_2\gamma^\mu\psi_1 U^{-1}(-\omega) = \overline{\psi}_2 D^{-1}(\omega)\gamma^\mu D(\omega)\psi_1 = \Lambda^\mu{}_\nu\psi_2\gamma^\nu\psi_1 \text{ is a vector} \tag{C.61}$$

Similarly, defining $\sigma^{\mu\nu} = 2\mathrm{i}\mathcal{J}^{\mu\nu}$, the combination

$$U(-\omega)\overline{\psi}_2\sigma^{\mu\nu}\psi_1 U^{-1}(-\omega) = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\overline{\psi}_2\sigma^{\alpha\beta}\psi_1 \tag{C.62}$$

is a rank-2 antisymmetric tensor.

Next, define

$$\gamma^5 = \gamma_5 \equiv \frac{\mathrm{i}}{24}\epsilon_{\mu\nu\alpha\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta} = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{C.63}$$

where the latter follows from the anticommutation of the distinct gamma matrices. We have that

$$\begin{aligned} U(-\omega)\overline{\psi}_2\gamma_5\psi_1 U^{-1}(-\omega) &= \frac{\mathrm{i}}{24}\epsilon_{\mu\nu\alpha\beta}\overline{\psi}_2 D^{-1}\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}D\psi_1 \\ &= \frac{\mathrm{i}}{24}\epsilon_{\mu\nu\alpha\beta}\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\rho}\Lambda^{\alpha}{}_{\kappa}\Lambda^{\beta}{}_{\zeta}\overline{\psi}_2\gamma^{\sigma}\gamma^{\rho}\gamma^{\kappa}\gamma^{\zeta}\psi_1 \\ &= (\det\Lambda)\frac{\mathrm{i}}{24}\epsilon_{\sigma\rho\kappa\zeta}\overline{\psi}_2\gamma^{\sigma}\gamma^{\rho}\gamma^{\kappa}\gamma^{\zeta}\psi_1 \\ &= (\det\Lambda)\overline{\psi}_2\gamma_5\psi_1 \end{aligned} \tag{C.64}$$

Therefore $\overline{\psi}_2\gamma_5\psi_1$ is a pseudoscalar, a scalar under $SO(3,1)$ which flips sign under parity transformations. Finally, the quantity $\overline{\psi}_2\gamma^{\mu}\gamma_5\psi$ transforms as a pseudovector,

$$U(-\omega)\overline{\psi}_2\gamma^{\mu}\gamma_5\psi_1 U^{-1}(-\omega) = \overline{\psi}_2 D^{-1}\gamma^{\mu}\gamma_5 D\psi_1 = (\det\Lambda)\Lambda^{\mu}{}_{\nu}\overline{\psi}_2\gamma^{\nu}\gamma_5\psi_1 \tag{C.65}$$

Since this gives $1+4+6+4+1=16$ independent contractions, the above are exhaustive; any other matrix sandwiched between $\overline{\psi}_2$ and ψ_1 must be a linear combination of $\mathbf{1}$, γ^{μ}, $\sigma^{\mu\nu}$, $\gamma^{\mu}\gamma_5$, and γ_5.

The choice of matrices made above is called the chiral basis and is convenient because the right- and left-handed components of ψ factorize. However, multiplying ψ by an arbitrary unitary matrix S and all matrices by $\Gamma \to S\Gamma S^{-1}$ leaves the theory unchanged. While the explicit expressions for the matrices are obviously changed, certain relations are not, and are therefore particularly valuable. In particular, the Clifford algebra, Eq. (C.56), the relations Eq. (C.57), Eq. (C.58), Eq. (C.59), the definition Eq. (C.63) of γ_5 in terms of the other γ matrices, and the relations between the projection operators and γ_5,

$$P_{\mathrm{L}} = \frac{1+\gamma_5}{2}, \qquad P_{\mathrm{R}} = \frac{1-\gamma_5}{2} \tag{C.66}$$

are basis independent and should therefore be sufficient to evaluate any invariant quantities.