6
Structural aspects

We turn now to describe some important general features of quantum field theory so that one can obtain a better overall view of the theory. This will be done by continuing to use the scalar theory as an illustration. We shall first investigate the structure of the interacting propagator — the two-point Green's function. This structure will tell us how to construct single-particle states for an interacting theory. The single-particle state construction will then be extended to multiparticle states, and the relationship of n-point Green's functions to scattering amplitudes will be obtained. This relationship is called the "reduction formula". The interacting propagator can also describe unstable particles, and this will be explained including an outline of how such particles are produced and detected and how very short-lived particles appear as resonances in scattering amplitudes. The effective action will then be introduced, and it will be shown to be the generating functional of connected amplitudes. As a by-product, the cluster decomposition theorem will be described and the fact that the power of Planck's constant $\hbar$ counts the number of closed loops in a graph will be explained. Finally, the Legendre transform of the effective action will be examined. It is the generating function of single-particle irreducible, connected graphs. Its restriction to constant fields defines the effective potential which is a useful instrument for describing spontaneously broken symmetry. We already have a precursor to these topics in the discussion of spontaneous symmetry breakdown in many-particle systems in Chapter 2, Section 7, and that work should provide additional motivation for this later exposition.

6.1 Lehmann representation

We return to Minkowski space time and turn to discuss the structure of the Green's function of two interacting Heisenberg field operators which follows from simple assumptions — axioms — of a basic physical character. With sufficient work one could, in fact, derive the assumed axioms from the functional integral representation. However, since any physically sensible theory must have these properties, we shall follow the easy path of simply assuming the axioms.

The Green's function is a vacuum matrix element, and so we start with a consideration of the nature of the vacuum state. Since the energy-momentum operators P^μ commute, $[P^\mu, P^\nu] = 0$, they possess simultaneous eigenvalues. We assume that a *unique* vacuum state $|0\rangle$ exists* which is an eigenstate of these operators with zero eigenvalues,

$$P^\mu|0\rangle = 0\,. \tag{6.1.1}$$

It is a deep bit of physics that the momentum operator plays the dual role as the generator of space-time translations. This fact is reflected in its commutator with the generators of Lorentz transformations $J_{\mu\nu}$,

$$-\,i[J_{\mu\nu}\,,\,P_\lambda] = g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu\,. \tag{6.1.2}$$

Hence, in view of Eq. (6.1.1),

$$0 = [J_{\mu\nu}\,,\,P_\lambda]\,|0\rangle = -P_\lambda J_{\mu\nu}|0\rangle\,. \tag{6.1.3}$$

Since $|0\rangle$ is the unique state which is annihilated by P_λ, this implies that

$$J_{\mu\nu}|0\rangle = |0\rangle j_{\mu\nu}\,, \tag{6.1.4}$$

where the $j_{\mu\nu}$ are some set of numbers. These numbers must form a representation of the Lorentz group, they must obey the Lorentz group Lie algebra that is satisfied by the $J_{\mu\nu}$ operators,

$$i[J_{\mu\nu}\,,\,J_{\alpha\beta}] = g_{\nu\alpha}J_{\mu\beta} - \ldots\,. \tag{6.1.5}$$

This requirement follows by multiplying Eq. (6.1.4) by $J_{\alpha\beta}$ and subtracting the result where $J_{\alpha\beta}$ is interchanged with $J_{\mu\nu}$. It is impossible for the numbers $j_{\mu\nu}$ to satisfy this requirement except for the trivial representation $j_{\mu\nu} = 0$. Hence

$$J_{\mu\nu}|0\rangle = 0\,, \tag{6.1.6}$$

and we see that the vacuum state is necessarily a Lorentz invariant state. The Lorentz invariance of the vacuum state is *not* a separate postulate; it follows from the assumption that this state is the *unique* state of zero energy and momentum.

For simplicity, we still consider an Hermitian scalar field, $\phi(x) = \phi(x)^\dagger$, which we take to be the renormalized field. It should be emphasized,

* A theory with spontaneous symmetry breaking has a whole family of equivalent vacuum states. In such a case we assume that a particular vacuum state has been singled out.

however, that this is a scalar field in some arbitrary (but renormalizable) theory, not necessarily the simple $\lambda\phi^4$ which we have been considering. The unordered two-point function is given by

$$G^{(+)}(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle \,. \tag{6.1.7}$$

Writing

$$\phi(x) = e^{-iPx}\phi(0)e^{iPx} \,, \tag{6.1.8}$$

and using the fact that the vacuum state has no energy or momentum, we have

$$G^{(+)}(x-x') = \langle 0|\phi(0)e^{iP(x-x')}\phi(0)|0\rangle \,. \tag{6.1.9}$$

Let us now use the resolution of the identity provided by some complete set of orthonormal states

$$1 = \sum\nolimits_n |n\rangle\langle n| \,. \tag{6.1.10}$$

Here the summation is a complicated one which involves a summation over states with an arbitrary number of particles with an integration over all the particle momenta in each of these multiparticle states. We need not specify the precise nature of the states $|n\rangle$ save that they shall be required to be eigenstates of the (commuting) energy-momentum operators P^μ,

$$P^\mu|n\rangle = |n\rangle\, p_n^\mu \,. \tag{6.1.11}$$

Accordingly,

$$\begin{aligned} G^{(+)}(x-x') &= \sum\nolimits_n e^{ip_n(x-x')}\langle 0|\phi(0)|n\rangle\langle n|\phi(0)|0\rangle \\ &= \sum\nolimits_n e^{ip_n(x-x')}|\langle 0|\phi(0)|n\rangle|^2 \,. \end{aligned} \tag{6.1.12}$$

A constant can always be added to $\phi(x)$ to ensure that $\langle 0|\phi(x)|0\rangle = 0$, which we shall assume has been done. Thus there is no loss of generality in assuming that the sum in Eq. (6.1.12) excludes the vacuum state $n = 0$. The energies of all the intermediate states must be positive, $p_n^0 > 0$, for otherwise the vacuum state of zero energy would not be the ground state — it would not be stable. The invariant total mass squared, $-p_n^2$, of the n-th intermediate state is just the total squared energy of this state when a boost is made to put the total spatial momentum $\mathbf{p}_n$ of the state to zero. This total mass squared cannot be negative, and it can vanish only if the theory contains a massless particle. We shall assume that the latter is not the case and require that p_n^μ is a time-like vector, $-p_n^2 \geq M^2$, with $M > 0$. Within these constraints on p_n^μ,

$$\int_{M^2}^{\infty} ds \int (d^4p)\, \theta(p^0)\delta(p^2+s)\delta^{(4)}(p-p_n) = 1 \,. \tag{6.1.13}$$

Using this effective representation of the identity, we may make a "change of variables" by inserting it inside the sum and interchange sum and integral to write

$$G^{(+)}(x-x') = \int_{M^2}^{\infty} ds \int (d^4p)\, \theta(p^0)\delta(p^2+s)e^{ip(x-x')} \times \sum\nolimits_n \delta^{(4)}(p-p_n)|\langle 0|\phi(0)|n\rangle|^2 \,. \qquad (6.1.14)$$

In virtue of the Lorentz invariance of the theory,

$$\sum\nolimits_n \delta^{(4)}(p-p_n)|\langle 0|\phi(0)|n\rangle|^2 = \rho(-p^2)/(2\pi)^3 \qquad (6.1.15)$$

defines a Lorentz scalar, a function $\rho(-p^2)$ which depends only upon the invariant $-p^2$. This function appears as the sum of squared matrix elements and so it is real and non-negative,

$$\rho(-p^2)^* = \rho(-p^2) \geq 0 \,. \qquad (6.1.16)$$

We have now secured

$$\begin{aligned} G^{(+)}(x-x') &= \langle 0|\phi(x)\phi(x')|0\rangle \\ &= \int_{M^2}^{\infty} ds\, \rho(s)\, \Delta^{(+)}(x-x'\,;\,s)\,, \end{aligned} \qquad (6.1.17)$$

in which

$$\Delta^{(+)}(x-x'\,;\,s) = \int \frac{(d^4p)}{(2\pi)^3}\, \theta(p^0)\delta(p^2+s)e^{ip(x-x')} \,. \qquad (6.1.18)$$

This is the Lehmann representation.

Performing the p^0 integral in Eq. (6.1.18) with the aid of the δ-function gives

$$\Delta^{(+)}(x-x'\,;\,s) = \int \frac{(d^3\mathbf{p})}{(2\pi)^3}\, \frac{1}{2p^0}\, e^{ip(x-x')} \,, \qquad (6.1.19)$$

where

$$p^0 = E_s(\mathbf{p}) = \sqrt{\mathbf{p}^2 + s} \qquad (6.1.20)$$

is the energy of a particle of momentum $\mathbf{p}$ and mass $\sqrt{s}$. Recalling Eq. (3.2.29), this identifies $\Delta^{(+)}(x-x'\,;\,s)$ as the vacuum matrix element

$$\Delta^{(+)}(x-x'\,;\,s) = \langle 0|\phi(x)\phi(x')|0\rangle_s^{(0)} \qquad (6.1.21)$$

for a free field of mass s. Indeed, the single-particle part of the resolution of the identity placed between the two free ϕ fields in Eq. (6.1.21) produces the integral representation given in Eq. (6.1.19). We thus find that the two-field function of a general interacting scalar field can be expressed as a superposition of the corresponding free field functions of variable mass. If the theory were that of a free field of mass m, one would have $\rho(s) = \delta(s-m^2)$.

Interchanging $x \leftrightarrow x'$ in Eq. (6.1.17) (or equivalently taking the complex conjugate) gives the vacuum matrix of the two fields in the reverse order. Subtracting this from the original form yields the commutator representation

$$\langle 0|i[\phi(x),\, \phi(x')]|0\rangle = \int_{M^2}^{\infty} ds \rho(s) \Delta_c(x - x'\,;\, s)\,, \tag{6.1.22}$$

where

$$\Delta_c(x - x'\,;\, s) = i[\Delta^{(+)}(x - x'\,;\, s) - \Delta^{(+)}(x' - x\,;\, s)]\,. \tag{6.1.23}$$

The Fourier representation (6.1.18) shows that $\Delta^{(+)}(x - x'\,;\, s)$ is not changed by the interchange of the spatial coordinates, $\mathbf{x} \leftrightarrow \mathbf{x}'$, for this is equivalent to reversing the sign of the spatial momentum, $\mathbf{p} \to -\mathbf{p}$, which leaves the remainder of the integral invariant. Hence

$$\Delta_c(x - x'\,;\, s) = \int \frac{(d^4p)}{(2\pi)^3} i\epsilon(p^0)\delta(p^2 + s) e^{ip(x-x')}\,, \tag{6.1.24}$$

in which

$$\epsilon(p^0) = \theta(p^0) - \theta(-p^0) = \begin{cases} 1 & , \quad p^0 > 0 \\ -1 & , \quad p^0 < 0 \end{cases}\,. \tag{6.1.25}$$

The same procedure applied to the free field matrix element gives

$$\Delta_c(x - x'\,;\, s) = \langle 0|i[\phi(x),\, \phi(x')]|0\rangle_s^{(0)}\,. \tag{6.1.26}$$

The equal-time commutator vanishes,

$$\begin{aligned} \Delta_c(x - x'\,;\, s)\big|_{t=t'} &= \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \int dp^0 i\epsilon(p^0)\delta(p^2 + s) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \\ &= 0\,, \end{aligned} \tag{6.1.27}$$

since the integrand is odd in p^0. Now $\Delta_c(x - x'\,;\, s)$ is a Lorentz invariant. Any space-like interval $(\mathbf{x} - \mathbf{x}')^2 - (t - t')^2 > 0$ can be sent by a Lorentz transformation into the purely spatial interval $\bar{\mathbf{x}} - \bar{\mathbf{x}}' \neq 0,\ \bar{t} - \bar{t}' = 0$. Hence, Lorentz invariance together with Eq. (6.1.27) implies that the commutator function vanishes for any space-like separation of its coordinates,

$$(x - x')^2 > 0\ : \qquad \Delta_c(x - x'\,;\, s) = 0\,. \tag{6.1.28}$$

We conclude from the commutator representation (6.1.22) that for any space-like separation $(x - x')^2 > 0$,

$$\langle 0|[\phi(x),\, \phi(x')]|0\rangle = 0\,. \tag{6.1.29}$$

This is the quantum field theory expression of *causality*. The commutator of two fields in space-like relation, an interval that cannot be bridged by

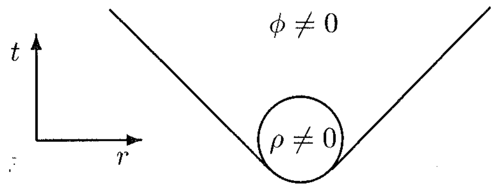

Fig. 6.1. Causality: A source produces an initial state expectation value of a field which is non-vanishing only in the future light cone of the source.

communication, even with a light signal, must vanish. Note that *scalar* field ϕ carries spin *zero*. Spin zero fields *commute* for space-like separation.

To make this connection with causality explicit, we consider the vacuum expectation value of the field in the presence of an external source function $\rho(x)$,

$$\langle 0-|\phi(x)|0-\rangle^{\rho} .$$

Note that this is *not* the "remote past–distant future" matrix element which is produced by the functional integral, the matrix element which enters into the description of a scattering experiment. Rather, it is the "diagonal" matrix element, the expectation value in the initial vacuum state of the field operator ϕ at the space-time point x. This matrix element gives the *average value* which the field ϕ will be observed to have at the spatial position $\mathbf{x}$ at the time x^0 if the initial state is the vacuum state. The expansion to first order in the external source involves the retarded commutator according to the solution of Problem 1:

$$\langle 0-|\phi(x)|0-\rangle^{\rho} = \int (d^4x')\, \theta(x^0 - x'^0) \langle 0|i[\phi(x),\, \phi(x')]|0\rangle \rho(x') + \cdots . \tag{6.1.30}$$

This result makes the relationship of the commutator function with causality clear: Since the commutator vanishes for space-like separations, the source produces a field response only within its future light cone. That is, if the source is restricted to be non-vanishing only in some finite space-time region — if it has a finite space-time "support", then the expectation value of ϕ will be non-vanishing only in the future light cone of the points of this support as shown in the sketch in Fig. 6.1.

Spin zero fields *cannot anti-commute* at space-like separation. To see this, we note that, just as in the commutator construction, the vacuum value of the anti-commutator has the representation

$$\langle 0|\{\phi(x)\,,\, \phi(x')\}|0\rangle = \int_{M^2}^{\infty} ds\, \rho(s) \Delta^{(1)}(x-x'\,;\, s)\,, \tag{6.1.31}$$

in which

$$\Delta^{(1)}(x\,;\, s) = \Delta^{(+)}(x\,;\, s) + \Delta^{(+)}(-x\,;\, s)\,. \tag{6.1.32}$$

It follows directly from Eq. (6.1.18) that $\Delta^{(+)}(x\,;\,s) = \Delta^{(+)}(-x\,;\,s)$ for $t = 0$, and Lorentz invariance extends this equality to all space-like x. Moreover, at space-like separation, the free-field unordered and time-ordered functions are essentially equal, $\Delta^{(+)} = -i\Delta_+$. Hence we may use the result (4.5.17) to conclude that for space-like separations

$$x^2 > 0 \; : \qquad \Delta^{(1)}(x\,;\,s) = \frac{1}{2\pi^2}\int_0^\infty d\zeta\, e^{-\zeta x^2 - s/4\zeta} > 0\,. \tag{6.1.33}$$

Hence for any space-like separation, $(x - x')^2 > 0$,

$$\langle 0|\{\phi(x)\,,\,\phi(x')\}|0\rangle = \int_{M^2}^\infty ds\,\rho(s)\Delta^{(1)}(x - x'\,;\,s) > 0\,. \tag{6.1.34}$$

Therefore, if the anti-commutator of two spin zero fields were to vanish for space-like separation, the integrand in Eq. (6.1.34) must vanish since $\rho(s)$ is non-negative. That is, this spectral weight must vanish identically,

$$\rho(s) = 0\,. \tag{6.1.35}$$

But Eq. (6.1.15) then implies that $\langle 0|\phi(0)|n\rangle = 0$, for all the states n, which further implies that the field ϕ annihilates the vacuum state,

$$\phi(x)|0\rangle = 0\,. \tag{6.1.36}$$

At this point, we need to quote a general theorem which we shall simply state without proof since this proof is lengthy and would take us too far astray: If $\phi(x)|0\rangle = 0$ then $\phi(x) = 0$. No field theory exists in which a local field annihilates the vacuum state. One concludes that a spin zero field *cannot* anticommute for any space-like separation.

What has been described is an elementary example of the general Spin–Statistics Theorem: At space-like separations integer spin fields commute while $\frac{1}{2}$-integer spin fields anticommute.

The Lehmann representation (6.1.17) expresses the vacuum matrix element of two field operators in the fully interacting theory in terms of a superposition of the corresponding matrix elements of free fields of mass s. We have just seen how this basic representation can be used to form representations of the commutator or anti-commutator functions — they are the same superposition of the corresponding free-field functions of mass s. This also carries over to the time-ordered product

$$\begin{aligned} G_+(x - x') &= \langle 0|iT\phi(x)\phi(x')|0\rangle \\ &= i\theta(x^0 - x'^0)\langle 0|\phi(x)\phi(x')|0\rangle + i\theta(x'^0 - x^0)\langle 0|\phi(x')\phi(x)|0\rangle\,, \end{aligned} \tag{6.1.37}$$

which appears as

$$G_+(x - x') = \int_{M^2}^\infty ds\,\rho(s)\Delta_+(x - x'\,;\,s)\,. \tag{6.1.38}$$

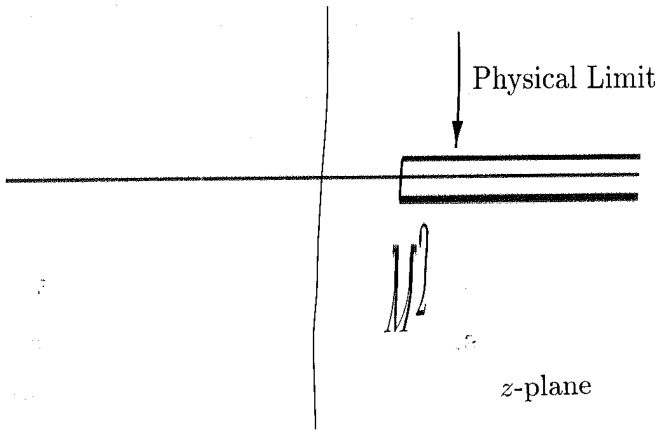

Fig. 6.2. Analytic structure of the Green's function.

The free-field propagator was discussed at length in Chapter 3, Section 2. It is given by

$$\begin{aligned}\Delta_+(x-x'\,;\,s) &= \langle 0|iT\phi(x)\phi(x')|0\rangle_s^{(0)}\\ &= i\theta(x^0-x'^0)\Delta^{(+)}(x-x';\,s) + i\theta(x'^0-x^0)\Delta^{(+)}(x'-x;\,s)\\ &= \int \frac{(d^4p)}{(2\pi)^4}\frac{e^{ip(x-x')}}{p^2+s-i\epsilon}\,. \end{aligned} \tag{6.1.39}$$

Thus the Fourier transform of the interacting Green's function has the dispersion relation form†

$$G_+(p) = \int_{M^2}^{\infty} ds\,\rho(s)\frac{1}{p^2+s-i\epsilon}\,. \tag{6.1.40}$$

The Green's function in momentum space $G_+(p)$ appears as the boundary value of analytic function $G(z)$, $z \to -p^2 + i\epsilon$ where

$$G(z) = \int_{M^2}^{\infty} ds\,\rho(s)\frac{1}{s-z} \tag{6.1.41}$$

defines a function analytic in the whole z plane save for a cut along the positive real axis which starts at $z = M^2$, as shown in Fig. 6.2.

† Although the representation (6.1.17) for the unordered function always exists in the distribution theory sense, the dispersion relation (6.1.40) for the Green's function may not exist unless subtractions are made. The point is that the products of the two distributions, the θ step functions and the $G^{(+)}$, may not be a well-defined distribution. This is the case in the perturbative expansion of the $\lambda\phi^4$ theory where $\rho(s)$ does not decrease as $s \to \infty$, and the integral in Eq. (6.1.40) does not converge. In such cases a subtraction must be made, with the dispersion relation written as

$$G_+(p) = A - (p^2+s_0)\int ds\frac{\rho(s)}{s-s_0}\frac{1}{p^2+s}\,.$$

Since we are concerned only with the formal aspects of the theory, we shall neglect such complications.

2 Single-particle states

Let us now assume that the field ϕ creates a single-particle state of mass m_{phy} from the vacuum state as well as creating other multiparticle states. Then, separating out the single-particle state with the energy-momentum relation

$$p^0 = E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_{\text{phy}}^2} \tag{6.1.42}$$

from the sum (6.1.15) that defines the spectral weight,

$$\begin{aligned}\rho(-p^2) = \int \frac{(d^3\mathbf{p}')}{(2\pi)^3}\frac{1}{2E(\mathbf{p}')}(2\pi)^3\delta^{(4)}(p-p')|\langle 0|\phi(0)|p'\rangle|^2 \\ + \sum_{\text{multiparticles}} (2\pi)^3\delta^{(4)}(p-p_n)|\langle 0|\phi|n\rangle|^2 .\end{aligned} \tag{6.1.43}$$

Here the same Lorentz invariant normalization for the single-particle state $|p'\rangle$ is used that has been previously employed in the free-field theory. Hence

$$|\langle 0|\phi(0)|p'\rangle|^2 = Z > 0 \tag{6.1.44}$$

defines a positive constant‡ Z. This is a *finite* wave-function renormalization constant which must appear since, although the field ϕ is the renormalized field, it is defined by a minimal subtraction scheme, and it does not couple the vacuum state to the single-particle state with unit strength. With $p^0 > 0$,

$$\begin{aligned}\int (d^3\mathbf{p}')\frac{1}{2E(\mathbf{p}')}\delta^{(4)}(p-p') &= \frac{1}{2p^0}\delta(p^0 - E(\mathbf{p})) \\ &= \delta(p^2 + m_{\text{phy}}^2) ,\end{aligned} \tag{6.1.45}$$

and so

$$\rho(-p^2) = Z\delta(-p^2 - m_{\text{phy}}^2) + \bar{\rho}(-p^2) , \tag{6.1.46}$$

where now $\bar{\rho}(-p^2)$ stands for the contribution of the multiparticle states with

$$\bar{\rho}(s) = 0 , \qquad s < M_{\text{th}}^2 . \tag{6.1.47}$$

The threshold mass M_{th} must be larger than the particle mass m_{phy}, $M_{\text{th}}^2 > m_{\text{phy}}^2$, for otherwise the single particle created by ϕ would decay into the particles associated with M_{th}; the particle would not be stable

‡ By Lorentz invariance, Z must be a function only of p'^2. But $-p'^2 = m_{\text{phy}}^2$, and so Z must be a constant.

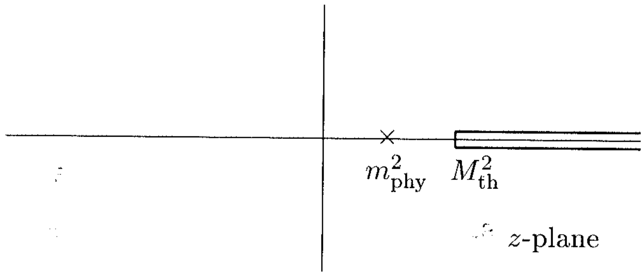

Fig. 6.3. Analytic structure of the Green's function with a single-particle pole.

and so it would not enter into the spectrum of physical states. Thus

$$G_+(p) = \frac{Z}{p^2 + m^2_{\rm phy}} + \int_{M^2_{\rm th}}^{\infty} ds \frac{\bar{\rho}(s)}{p^2 + s} . \qquad (6.1.48)$$

The physical mass of the particle is the position of pole of the two-point Green's function. Moreover, the positive residue of this pole is a finite wave function renormalization in addition to the minimal but infinite renormalization that relates the renormalized field ϕ to the bare field ϕ_0. The analytic structure is now described by the picture in Fig. 6.3.

6.2 Reduction formula

The Green's functions of a quantum field theory yield all the (n-particle) scattering amplitudes of the theory. To see how this goes, we first examine the construction of single-particle states $|p\rangle$ in terms of the field operator $\phi(x)$ acting on the vacuum state $|0\rangle$.

If $\phi(x)$ is a free field, the single-particle state has the construction (Chapter 3, Section 2)

$$\langle p'| = \int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle 0|\phi(x) , \qquad (6.2.1)$$

with these states having the normalization

$$\langle p'|p\rangle = 2p^0(2\pi)^3\delta(\mathbf{p}' - \mathbf{p}) . \qquad (6.2.2)$$

For this free field, the wave function is given by

$$\langle p'|\phi(x)|0\rangle = e^{-ip'x} . \qquad (6.2.3)$$

For an interacting field theory which has a single-particle state with the energy-momentum relation

$$p'^{\mu} = \left(\sqrt{\mathbf{p}'^2 + m^2_{\rm phy}} \, , \, \mathbf{p}'\right) , \qquad (6.2.4)$$

we are motivated to try a construction akin to that given in Eq. (6.2.1) for a free particle. This construction is tested by examining the resulting "wave function" which is now given by an operation on the interacting two-point Green's function. Using the Lehmann representation,

$$\begin{aligned}
&\int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle 0|\phi(x)\phi(x')|0\rangle \\
&= \int (d^3\mathbf{x}) e^{-i\mathbf{p}'\cdot\mathbf{x}+ip'^0 t} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \int ds \left[Z\delta(s - m_{\text{phy}}^2) + \bar{\rho}(s)\right] \\
&\quad \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \frac{1}{2E_s(\mathbf{p})} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-iE_s(p)(t-t')} \\
&= Ze^{-ip'x'} + \int_{M_{\text{th}}^2}^{\infty} ds \bar{\rho}(s) \frac{E_s(\mathbf{p}') + p'^0}{2E_s(\mathbf{p}')} e^{-iE_s(\mathbf{p}')(t-t')} e^{-i\mathbf{p}'\cdot\mathbf{x}'+ip'^0 t} .
\end{aligned} \tag{6.2.5}$$

The first term on the right-hand side of the last equality is just the free-particle wave function modified by the appearance of the finite wave-function renormalization constant Z. The second term represents the contribution of the continuum of states that is also produced when an interacting field acts on the vacuum state.

To understand the nature of the continuum contribution, it is worthwhile to first review the Riemann-Lebesgue Lemma. This lemma roughly states that if $f(\omega)$ is a "smooth" function which vanishes as $\omega \to \pm\infty$, then its Fourier transform vanishes in the infinite time limit,

$$t \to \infty \ : \qquad \int_{-\infty}^{\infty} d\omega f(\omega) e^{-i\omega t} \to 0 . \tag{6.2.6}$$

The point is that as $t \to \infty$, the exponential oscillates ever more rapidly and adjacent regions of the integration cancel. The proof follows by partial integration

$$\begin{aligned}
\int_{-\infty}^{\infty} d\omega f(\omega) e^{-i\omega t} &= \frac{i}{t} \int_{-\infty}^{\infty} d\omega f(\omega) \frac{\partial}{\partial\omega} e^{-i\omega t} \\
&= -\frac{i}{t} \int_{-\infty}^{\infty} d\omega \frac{df(\omega)}{d\omega} e^{-i\omega t} \\
&= \cdots\cdots \\
&= \left(\frac{-i}{t}\right)^N \int_{-\infty}^{\infty} d\omega \frac{d^N f(\omega)}{d\omega^N} e^{-i\omega t} .
\end{aligned} \tag{6.2.7}$$

This process can be repeated until a discontinuity or singularity is obtained in the N-th derivative, in which case one has proved that the Fourier transform vanishes at least as rapidly as $1/t^N$. If all the derivatives of $f(\omega)$ exist, then the Fourier transform vanishes faster than any power of $1/t$ as $t \to \infty$. In view of this lemma, we expect that

$$\lim_{|t-t'|\to\infty} \int_{M_{\text{th}}^2}^{\infty} ds\, \bar{\rho}(s)\, \frac{E_s + p'^0}{2E_s} e^{-iE_s(t-t')} = 0\,, \tag{6.2.8}$$

so that the continuum contribution vanishes in the infinite time limit. However the integrand is not smooth at the various n-particle thresholds — at the points $s = M_n^2$ where the states containing n particles start to make their contribution. Hence, the continuum contribution does not vanish faster than any power of $1/|t-t'|$.

To estimate the long-time limit of the continuum contribution, we note that the spectral weight $\bar{\rho}(s)$ has infinitely many thresholds corresponding to 2-particle, 3-particle, ... intermediate states. With s near the n-particle threshold, $s = M_n^2$, $\bar{\rho}(s)$ is proportional to the phase space for the production of n-particles (with additional factors of $\sqrt{s - M_n^2}$ appearing if the particles have spin). A straightforward calculation (see Problem 2) shows that near the n-particle threshold*

$$\bar{\rho}(s) \propto \left(\sqrt{s - M_n^2}\right)^{3n-5}\,. \tag{6.2.9}$$

The leading term in the long-time limit which arises from the threshold behavior of an n-particle intermediate state is obtained by the approximation

$$E_s(\mathbf{p}') = \sqrt{\mathbf{p}'^2 + s} \simeq E_{M_n}(\mathbf{p}') + \frac{s - M_n^2}{2E_{M_n}(\mathbf{p}')}\,, \tag{6.2.10}$$

with

$$E_{M_n}(\mathbf{p}') = \sqrt{\mathbf{p}'^2 + M_n^2}\,, \tag{6.2.11}$$

which is valid in the threshold region where s is near M_n. Thus

$$\begin{aligned}
&\int_{M_n^2}^{\infty} ds\, \bar{\rho}(s) \frac{E_s(\mathbf{p}') + p'^0}{2E_s(\mathbf{p}')} e^{-iE_s(\mathbf{p}')(t-t')} \\
&\sim \int_{M_n^2}^{\infty} ds (s - M_n^2)^{\frac{3n-5}{2}} \exp\left\{-\frac{i}{2}\frac{s-M_n^2}{E_{M_n}(\mathbf{p}')}(t-t')\right\} \exp\left\{-iE_{M_n}(\mathbf{p}')(t-t')\right\} \\
&\sim \left(\frac{1}{M|t-t'|}\right)^{\frac{3}{2}(n-1)} \exp\left\{-iE_{M_n}(\mathbf{p}')(t-t')\right\}\,,
\end{aligned} \tag{6.2.12}$$

where M is some characteristic elementary particle mass scale.

Taking, say, $M \sim 1$ GeV corresponds to $M^{-1} \sim 10^{-23}$ sec. Therefore for $(t - t') \gg M^{-1} \sim 10^{-23}$ sec the continuum contribution is negligible.

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* The simple square root behavior for a two particle threshold is evident from the calculation of the dispersion relation for the "bubble" graph found in Problem 2 of Chapter 4.

For example, with $t-t' \sim 10^{-13}$ sec, the contribution of an $n = 2$-particle threshold is on the order of 10^{-15}. Although the continuum contribution vanishes only as an inverse power for large time difference, the scale is set by the very short elementary particle time M^{-1}. Hence

$$\lim_{|t-t'|\to\infty} \int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle 0|\phi(x)\phi(x')|0\rangle = Ze^{-ip'x'}\,, \quad (6.2.13)$$

where, in physical terms, the "infinite limit" means time differences $|t-t'|$ on the order of 10^{-13} sec or so.

We can now see how to construct a single-particle state for an interacting field system, namely

$$\langle p| = \lim_{t\to\pm\infty} \int (d^3\mathbf{x}) e^{-ipx} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle 0|\frac{1}{\sqrt{Z}}\phi(x)\,, \quad (6.2.14)$$

for this procedure gives the properly normalized wave function

$$\langle p|\phi(x)|0\rangle = \sqrt{Z}e^{-ipx}\,. \quad (6.2.15)$$

The Hermitian adjoint yields

$$|p\rangle = \lim_{t\to\pm\infty} \int (d^3\mathbf{x}) e^{+ipx}(-i)(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\frac{1}{\sqrt{Z}}\phi(x)|0\rangle\,. \quad (6.2.16)$$

Using the wave function (6.2.15), the scalar product of the two states constructed by Eqs. (6.2.14) and (6.2.16) — but with opposite time limits [$t \to +\infty$ for the bra, $t \to -\infty$ for the ket (or the other way around)] so as to suppress the continuum contribution — is just that which has been used for free particles:

$$\langle p|p'\rangle = (2\pi)^3 2p^0 \delta^{(3)}(\mathbf{p}-\mathbf{p}')\,. \quad (6.2.17)$$

It should be emphasized that the infinite time limits used to construct the single-particle states picks out only the pole term in the Green's function — the continuum contributions are "washed out" in accordance with the Riemann-Lebesgue Lemma.

We shall use this method of state construction not only to obtain single-particle states but also to construct multiparticle states as well. This does not follow strictly from what we have done. We will not give a rigorous proof, but the result is so plausible that it is easy to accept. Suppose that there is an initial ϕ-particle wave packet which later collides with the wave packet of another particle or system of particles which we denote by ζ. Initially the two wave packets are separated by a great spatial distance and there is no interaction between them. Therefore, the initial state $|\zeta-\rangle$ containing the other system is essentially the vacuum state as far as the well-separated ϕ-particle wave packet is concerned. Hence the initial state with the additional ϕ-particle wave packet is constructed by replacing the vacuum state in Eq. (6.2.16) by $|\zeta-\rangle$ with the momentum p which appears

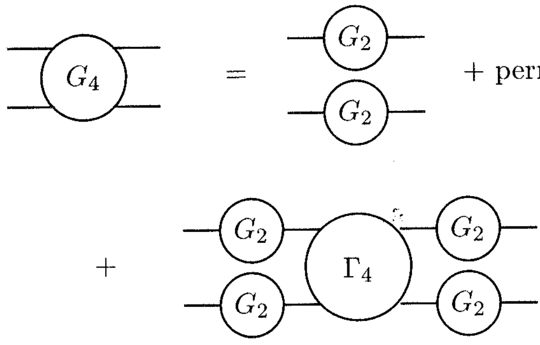

Fig. 6.4. Graphical description of the decomposition of the four-point Green's function into disconnected and single-particle irreducible parts.

in Eq. (6.2.16) integrated with the momentum-space wave function $f(p)$ which produces the well-separated ϕ-particle wave packet. In the limit in which the ϕ-particle wave packet becomes an incident plane-wave with momentum p, the initial state has the construction

$$|p\,,\zeta-\rangle = \lim_{t\to-\infty}\int (d^3\mathbf{x})e^{ipx}(-i)(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\frac{1}{\sqrt{Z}}\phi(x)|\zeta-\rangle\,. \qquad (6.2.18)$$

Similarly, the limit of the outgoing situation wave packet to a plane wave of momentum p' is given by

$$\langle p'\zeta'+| = \lim_{t\to+\infty}\int (d^3\mathbf{x})e^{-ip'x}i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\langle\zeta'+|\frac{1}{\sqrt{Z}}\phi(x)\,. \qquad (6.2.19)$$

Let us use this procedure to construct the transformation function $\langle p_1'p_2'+|p_1p_2-\rangle$ and thereby the two-particle scattering amplitude from the four-point Green's function

$$G(x_1,x_2,x_3,x_4) = \langle -T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle\,. \qquad (6.2.20)$$

To do this, we first decompose the Green's function into disconnected and irreducible parts as shown by the graphical structure in Fig. 6.4. The first graphs represent the unscattered, "straight through" propagation of the particles, but with the "fully dressed" interacting propagators. The final graph represents the processes that give rise to the scattering, but again with the interacting propagator factors removed to define an amplitude Γ which is single-particle irreducible.† These graphs correspond to the

† It is clear that the sum of all the Feynman graphs fall into the categories illustrated in Fig. 6.4. It is not obvious, however, that they combine to form precisely the propagator factors G_2 that are shown. This is explicitly proven in Sections 4 and 5 below.

formula

$$G(x_1, x_2, x_3, x_4) = G(x_1 - x_2)G(x_3 - x_4) + G(x_1 - x_3)G(x_2 - x_4) + G(x_1 - x_4)G(x_2 - x_3) + \int (d\bar{x}_1) \cdots (d\bar{x}_4) G(x_1 - \bar{x}_1) \cdots G(x_4 - \bar{x}_4) \Gamma(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4), \tag{6.2.21}$$

or, in momentum space,

$$(2\pi)^4 \delta(\Sigma p) G(p_1, p_2, p_3, p_4) = (2\pi)^4 \delta(p_1 + p_2) G(p_1)\, (2\pi)^4 \delta(p_3 + p_4) G(p_3) + \text{perms} + (2\pi)^4 \delta(\Sigma p) G(p_1) G(p_2) G(p_3) G(p_4) \Gamma(p_1, p_2, p_3, p_4). \tag{6.2.22}$$

Using Eqs. (6.2.19) and (6.2.18) to construct the states, the transformation function is given by

$$\langle p_1' p_2' + | p_1 p_2 - \rangle = \int_{+\infty} (d^3\mathbf{x}_1) e^{-ip_1' x_1} (\overrightarrow{\partial}_{10} - \overleftarrow{\partial}_{10}) \cdots \int_{-\infty} (d^3\mathbf{x}_4) e^{ip_2 x_4} (\overrightarrow{\partial}_{40} - \overleftarrow{\partial}_{40}) \frac{1}{Z^2} \langle T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle. \tag{6.2.23}$$

Note that the time-ordering in the matrix element orders the fields appropriately, with the field operators creating the outgoing state appearing on the left and the field operators creating the incoming state appearing on the right. Making use of the decomposition (6.2.21) we encounter terms of the form

$$\int_{+\infty} (d^3\mathbf{x}_1) e^{-ip_1' x_1} i (\overrightarrow{\partial}_{10} - \overleftarrow{\partial}_{10}) \frac{1}{\sqrt{Z}} G(x_1 - y) = i \langle p_1' | \phi(y) | 0 \rangle = i\sqrt{Z} e^{-ip_1' y}, \tag{6.2.24}$$

and

$$\int_{-\infty} (d^3\mathbf{x}_3) e^{ip_1 x_3} (-i) (\overrightarrow{\partial}_{30} - \overleftarrow{\partial}_{30}) \frac{1}{\sqrt{Z}} G(x_3 - z) = i \langle 0 | \phi(z) | p_1 \rangle = i\sqrt{Z} e^{ip_1 z}. \tag{6.2.25}$$

Using Eqs. (6.2.24) and (6.2.25) it is easy to check that the "straight through" propagation terms in the decomposition (6.2.21) connecting x_1 with x_2 and x_3 with x_4 do not contribute while the other "straight through" terms give single-particle transformation functions. Also using Eqs. (6.2.24) and (6.2.25) it is easy to see that the remaining scattering term involves the Fourier transform of the Γ amplitude. Thus one finds that

$$\langle p_1' p_2' + | p_1 p_2 - \rangle = \langle p_1' | p_1 \rangle \langle p_2' | p_2 \rangle + \langle p_1' | p_2 \rangle \langle p_2' | p_1 \rangle$$
$$- i(2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2) T \,, \qquad (6.2.26)$$

in which

$$T = -iZ^2 \Gamma(p_1' \,, p_2' \,, -p_1 \,, -p_2) \,. \qquad (6.2.27)$$

Note that as far as the scattering contribution described by T is concerned, the state construction just picks out the residue of the single-particle poles $(p^2 + m^2_{\text{phy}})^{-1}$. A factor of $\sqrt{Z}$ is removed for each external particle. This leaves a finite wave function renormalization which must be taken into account, the factor Z^2 in relating the scattering amplitude to the irreducible vertex Γ. We have already learned in Chapter 3, Section 4 how to construct the scattering cross section from the structure (6.2.26).

We have gone through this "derivation" in some detail particularly to explain how the "straight-through-propagation" factors work out. Let us now give a simple but heuristic discussion which leads to the "reduction mnemonic". First we note that a particle of momentum p' is added to a final state by writing

$$\langle \zeta' p' + | \xi - \rangle = \int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \frac{1}{\sqrt{Z}} \langle \zeta' + | \phi(x) | \xi - \rangle \,, \quad (6.2.28)$$

where the limit $t = x^0 \to +\infty$ is understood. A partial integration presents this as

$$\langle \zeta' p' + | \xi - \rangle = \int_{-\infty}^{\infty} dt \, \partial_0 \int (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \frac{1}{\sqrt{Z}} \langle \zeta' + | \phi(x) | \xi - \rangle$$
$$+ \int_{-\infty} (d^3\mathbf{x}) e^{-ip'x} i(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0) \frac{1}{\sqrt{Z}} \langle \zeta' + | \phi(x) | \xi - \rangle . \qquad (6.2.29)$$

The second integration at $t = -\infty$ gives no "scattering" contribution — this term has a rapid phase variation that produces a vanishing result unless it corresponds to destroying a particle of momentum p' in the initial state, which gives a "straight-through-propagation" contribution. The overall time derivative on the right-hand side of the first line of the equation combines to give the form $i(\overrightarrow{\partial_0^2} - \overleftarrow{\partial_0^2})$. To this can be added the combination $\overrightarrow{\nabla^2} - \overleftarrow{\nabla^2}$ which gives no contribution as an integration by parts establishes. Hence

$$\langle \zeta' p' + | \xi - \rangle = \int (d^4x) e^{-ip'x} i(-\overrightarrow{\partial^2} + \overleftarrow{\partial^2}) \frac{1}{\sqrt{Z}} \langle \zeta' + | \phi(x) | \xi - \rangle$$
$$= \int (d^4 x) e^{-ip'x} i(-\overrightarrow{\partial^2} + m^2_{\text{phy}}) \frac{1}{\sqrt{Z}} \langle \zeta' + | \phi(x) | \xi - \rangle . \quad (6.2.30)$$

One now takes p' slightly off mass shell, $-p'^2 \neq m_{\text{phy}}^2$. Then there are rapid phase variations at the space-time boundaries, and integrations by parts can be freely performed to give

$$\langle \zeta' p' + | \xi - \rangle = \frac{i}{\sqrt{Z}} (p'^2 + m_{\text{phy}}^2) \int (d^4x) e^{-ip'x} \langle \zeta' + | \phi(x) | \xi - \rangle \,. \tag{6.2.31}$$

The physical limit $p'^2 + m_{\text{phy}}^2 \to 0$ picks out the residue of the $(p'^2 + m_{\text{phy}}^2)^{-1}$ pole of the Fourier transform of the amplitude in Eq. (6.2.31) and produces the physical state matrix element. Similarly, an additional particle in an initial state is constructed as

$$\langle \zeta' + | \xi p - \rangle = \lim_{p^2 + m_{\text{phy}}^2 \to 0} \frac{i}{\sqrt{Z}} (p^2 + m_{\text{phy}}^2) \int (d^4x) e^{ipx} \langle \zeta' + | \phi(x) | \xi - \rangle \,. \tag{6.2.32}$$

As an example of the use of this reduction mnemonic, let us consider the previous example of the two-particle scattering amplitude. Applying the mnemonic to Eq. (6.2.22) we see again that that the scattering amplitude is identified as the residue in the $(p^2 + m_{\text{phy}}^2)^{-1}$ poles,

$$\begin{aligned} \langle p_1' p_2' + | p_1 p_2 - \rangle = {} & \text{straight through propagation terms} \\ & -(2\pi)^4 \delta(p_1' + p_2' - p_1 - p_2) Z^2 \Gamma(p_1', p_2', -p_1, -p_2) \,, \end{aligned} \tag{6.2.33}$$

in agreement with Eqs. (6.2.26) and (6.2.27).

6.3 Unstable particles

Let us first recall the analytic structure of the two-point Green's function or propagator when the theory contains a particle whose physical mass we will now denote simply by m. It has a pole at $-p^2 = m^2$ and a branch cut starting at $-p^2 = M_{\text{th}}^2$ as shown in Fig. 6.3. The Lehmann representation (6.1.40) implies that $G(p)$ has no singularities except for $-p^2$ real and positive. In particular a pole in $G(p)$ must lie on the real $-p^2$ axis. Suppose now that the parameters of the theory are changed. Then the pole and branch point may move towards one another. If further changes in the parameters are made the pole will move into the branch point. Then what? Since the pole cannot move onto the complex plane it must move into new Riemann sheets. As will be described shortly, the Green's function originally defined above the cut can be analytically continued into a second Riemann sheet below the cut. The final position of the pole in this second sheet is sketched in Fig. 6.5. Parameters giving poles on unphysical sheets produce an unstable particle if these poles are close to the real axis.