Quantum Field Theory (Quanten Feldtheorie) Homework 1

Due 30 April 2021

1 4-vector notation and Maxwell equations

This problem should be "easy" if you are used to 4-vector notation; if not, its goal is to get you used to that notation.

Recall that the electric and magnetic fields can be derived in terms of two quantities, the scalar potential Φ and the vector-potential $\vec{A}$. It is also convenient to replace the magnetic field with the antisymmetric 2-tensor F_{ij} defined as

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i \qquad \text{(non-covariant index notation).} \tag{1}$$

In terms of this, $F_{12} = B_3$, $F_{23} = B_1$, and $F_{31} = B_2$, *e.g.*,

$$F_{ij} = \epsilon_{ijk}\, B_k \qquad \text{and} \qquad B_i = \frac{\epsilon_{ijk}}{2} F_{jk}\,. \tag{2}$$

Here as usual ϵ_{ijk} is the totally antisymmetric symbol (Levi-Civita tensor) with $\epsilon_{123} = 1$.

In terms of ∂_t, ∂_i, and these two potentials, write the standard (non-covariant) expressions for the electric and magnetic fields E_i and F_{ij} in terms of Φ and $\vec{A}$.

Now we move to 4-vector notation. Define $A^\mu = (\Phi, \vec{A})$ (where $\mu = 0, 1, 2, 3$ and the notation means that for $\mu = 0$ you choose the first object in the parenthesis and for $\mu = 1, 2, 3$ you choose the component of the second, *e.g.*, $A^0 = \Phi$ and $A^{1,2,3} = \vec{A}_{1,2,3}$. Careful; the 4-vector objects $A_{1,2,3}$ are *minus* the non-covariant components of $\vec{A}$.) Also define $\partial_\mu = (\partial_t, \partial_i)$. Introduce

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu\,, \tag{3}$$

and show how F^{0i} is related to the electric field and how F^{ij} is related to the magnetic tensor. Here F^{0i} means $F^{\mu\nu}$ for the case where $\mu = 0$ but $\nu \neq 0$–we will use Roman letters to mean that a Lorentz index μ is not zero.

Also introduce $j^\mu = (\rho, \vec{j})$ the 4-current. Show that the covariant equation

$$\partial_\mu F^{\nu\mu} = [\pm] j^\nu \tag{4}$$

is equivalent to both Gauß' law and Ampere's law. Figure out which is the correct sign on the current; is my $\pm$ a $+$ or a $-$?

Now define $\epsilon_{\mu\nu\alpha\beta}$ the 4D antisymmetric symbol which generalizes the 3-D Levi-Civita tensor: $\epsilon_{\nu\mu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}$ and similarly for any other permutation of the indices, and $\epsilon_{0123} = -1$.

[The minus sign is so that $\epsilon^{0123} = +1$; the sign flips because an odd number of the $g^{\mu\nu}$'s you need to raise the indices are negative.]

Show that

$$\epsilon_{\mu\nu\alpha\beta}\partial^\nu F^{\alpha\beta} = 0 \tag{5}$$

is an identity (is true regardless of what values A^μ take provided they are twice differentiable) and that this identity is equivalent both to Gauß' Law for magnetism and to Faraday's law.

You are now an expert with index notation.

2 Condition to be a Lorentz transformation

Here we clear up two simple pieces of the derivation of what is and is not a Lorentz transformation.

In class we saw that $\Lambda^\mu{}_\nu$ is a Lorentz transformation if and only if

$$x^\alpha \Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta x^\beta = x^\alpha g_{\alpha\beta} x^\beta \tag{6}$$

for *any* choice of 4-coordinate x^α. Show that this really does require that

$$\Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta = g_{\alpha\beta} \tag{7}$$

should hold. Hint: show that if (7) is NOT true, then there is some x^μ such that (6) is also NOT true. Then argue by contrapositive.

Next, consider

$$\Lambda^\mu{}_\nu = \exp \omega^\mu{}_\nu \tag{8}$$

where multiplication is defined by thinking of the first upper index as a column index and the second lower index as a row index, and using matrix multiplication, *eg*,

$$\exp \omega^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + \frac{1}{2}\omega^\mu{}_\alpha \omega^\alpha{}_\nu + \frac{1}{6}\omega^\mu{}_\alpha \omega^\alpha{}_\beta \omega^\beta{}_\nu + \frac{1}{24}\ldots \tag{9}$$

Show that, provided $\omega_{\mu\nu} = -\omega_{\nu\mu}$, that $\Lambda^\mu{}_\nu$ really is a Lorentz transform, that is, that it satisfies Eq. (7).

3 Why vector fields are Maxwell fields

Consider the classical field theory for a 4-vector field A_μ. Assume that the terms in the Lagrange density have at most two powers of the field and at most two powers of derivatives (in the spirit of a gradient expansion). Further, $\mathcal{L}$ must be a Lorentz scalar.

Recall that the Lagrangian density $\mathcal{L}$ can always be shifted by a total derivative,

$$\mathcal{L} \to \mathcal{L} + \partial_\mu[\text{stuff}^\mu] \tag{10}$$

without changing any physics. Show that there are 5 independent 2-field, 2-derivative terms which can be written down, but that using the freedom to shift by total derivatives, all but two of them can be eliminated, so the most general Lagrangian is

$$\mathcal{L}[A_\mu, \partial_\nu A_\mu] = C_1(\partial_\nu A_\mu)(\partial^\nu A^\mu) + C_2(\partial_\nu A_\mu)(\partial^\mu A^\nu) + C_3 A_\mu A^\mu\,, \tag{11}$$

with C_1, C_2, C_3 some constants.

Now determine the Hamiltonian associated with the Lagrangian $L = \int \mathcal{L}\, d^3x$. First determine Π_ν, the canonical momentum for the A^ν field:

$$\Pi_\nu \equiv \frac{\partial \mathcal{L}}{\partial \partial_0 A^\nu} \tag{12}$$

and then use that

$$H = \int d^3x \Big(\Pi_\nu \partial_0 A^\nu - \mathcal{L} \Big) \tag{13}$$

to determine the Hamiltonian density $\mathcal{H}$. Write this out–it may be convenient to do so in non-covariant notation. [You may find it easier to write out H in terms of derivatives of A^μ, rather than in terms of Π_μ. This is correct but is not the form H must be expressed in if you wanted to get Hamilton's equations from it.]

Now the punchline: show that $C_3 \neq 0$ leads to a Hamiltonian $\int d^3x \mathcal{H}$ which is unbounded above and below, meaning that in this case there is *some* value for the components of A^μ which will make H arbitrarily large and some different value which will make it arbitrarily negative. Therefore $C_3 = 0$ is required for the theory to make sense. Hint: it is easiest to see this if you write out $\mathcal{H}$ non-covariantly in terms of A^0 and $\vec{A}$ separately.

In addition, show that $C_2 \neq -C_1$ also leads to an unbounded H, so $C_2 = -C_1$ is also required. [This is harder.] For extra credit, verify that $C_2 = -C_1 > 0$ gives a Hamiltonian which is the integral of the sum of two squares and is therefore bounded from below by 0. (You will have to do some legwork to get H into this form!)

Now define

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{14}$$

Show that, for the special case $C_2 = -C_1$ and $C_3 = 0$ we have just found, the Lagrangian density can be written

$$\mathcal{L} = \frac{C_1}{2} F_{\mu\nu} F^{\mu\nu}\,. \tag{15}$$

Derive the Euler-Lagrange equations from this action. Show that it is the same as one of the equations from the last problem, with $j^\mu = 0$.

3.1 except that…

Show however that for this case,

1. The field A^0 does not have a canonical momentum. Therefore it acts as a Lagrange multiplier; varying with respect to A^0 gives a condition, rather than an equation of motion. What is this condition?

2. The Euler-Lagrange equations do not uniquely determine the evolution of the field. In particular, assume that $A_\mu(x,t)$ is a solution to the Euler-Lagrange equations you found. Show that $A_\mu(x,t) + \partial_\mu \Lambda(x,t)$ is also a solution, for *any* choice of $\Lambda(x,t)$.

The latter problem will rear its ugly head again some day when we want to do perturbation theory for electrodynamics.