Quantum Field Theory Homework 2

Due 14 May 2021

1 Commutation relations

In class we found that

$$\Big[\phi(\mathbf{x})\,,\,\pi(\mathbf{y})\Big] = i\delta^3(\mathbf{x}-\mathbf{y}) \tag{1}$$

and then defined

$$\tilde{\phi}(p_m) \equiv L^{-3/2}\int d^3\mathbf{x}e^{-i\mathbf{p}_m\cdot\mathbf{x}}\phi(\mathbf{x}) \tag{2}$$

and likewise for $\tilde{\pi}(p_m)$. Show that it really follows from these definitions that

$$\Big[\tilde{\phi}(p_n)\,,\,\tilde{\pi}(p_m)\Big] = i\delta_{p_n,-p_m}\,. \tag{3}$$

Next, show that the definition

$$a_{p_n} = \frac{\omega_p\tilde{\phi}(p_n) + i\tilde{\pi}(p_n)}{\sqrt{2\omega_p}} \qquad \text{leads to} \qquad a^\dagger_{p_n} = \frac{\omega_p\tilde{\phi}(-p_n) - i\tilde{\pi}(-p_n)}{\sqrt{2\omega_p}} \tag{4}$$

which together lead to

$$\Big[a_{p_n}\,,\,a^\dagger_{p_m}\Big] = 1\delta_{p_n,p_m}\,. \tag{5}$$

Finally, fill in the steps to show that

$$H = \frac{1}{2}\sum_{p_n}\tilde{\pi}(p_n)\tilde{\pi}(-p_n) + (p^2+m^2)\tilde{\phi}(p_n)\tilde{\phi}(-p_n) = \frac{1}{2}\sum_{p_n}\omega_p(a^\dagger_{p_n}a_{p_n} + a_{p_n}a^\dagger_{p_n})\,. \tag{6}$$

(That is, take the first expression for H to be shown, and derive the second expression from it, together with the definition of a and $a^\dagger$.)

2 Energy associated with the vacuum

This question answers something which came up at the end of the lecture where we "solved" free scalar field theory.

We saw that the Hamiltonian for a free field theory is

$$H = \sum_{\vec{p}=\frac{2\pi}{L}\vec{n}}\frac{\omega_p}{2}\Big(a^\dagger_{p_n}a_{p_n} + a_{p_n}a^\dagger_{p_n}\Big) \tag{7}$$

where $\Big[a_{p_n}\,,\,a^\dagger_{p_m}\Big] = \delta_{\vec{n},\vec{m}}$ (the Kroneker delta).

Show that the energy of the vacuum $\langle 0| H |0\rangle$ is

$$E_{\rm vac} \equiv \langle 0| H |0\rangle = \sum_{\vec{p}} \frac{\omega_p}{2} \,. \tag{8}$$

Suppose that, because space is somehow discrete on some scale a, there is a maximum size which the momentum p is allowed to take, call it $p_{\rm max}$.

Show that the energy of the vacuum is extensive – that is, that it grows as we increase the size of our "box" as L^3. This is to be expected. Then show that it also depends on $p_{\rm max}$ with proportionality $p_{\rm max}^4$, that is, $E_{\rm vac} \sim L^3 p_{\rm max}^4$. Do not try to evaluate the actual coefficient, which depends on exactly what we mean by "momenta cannot exceed some scale $p_{\rm max}$."

Show that, if we include a constant term C in the Lagrangian density, the Hamiltonian is shifted by $-L^3 C$.

[This energy associated with space is of no relevance in particle physics because it is not observable. Particle physicists who also worry about gravity *do* worry about it, though, because it should behave as a "cosmological constant," and we know observationally that the cosmological constant is very small, whereas presumably $p_{\rm max}^4$ is very large. In principle it might be that C is just the right value to balance the energy associated with all the SHO zero-points, but . . . .]

3 Euler-Lagrange in the quantum theory

Consider free scalar quantum field theory for one real field $\hat{\phi}$. Prove that

$$(\partial_\mu \partial^\mu + m^2)\hat{\phi}(x) = 0 \,. \tag{9}$$

The easiest way to do this is to write the expansion of $\hat{\phi}(x)$ in terms of creation and annihilation operators,

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left(e^{-ip\cdot x} \hat{a}_p + e^{ip\cdot x} \hat{a}_p^\dagger \right) \tag{10}$$

where $p^0 \equiv \sqrt{\vec{p}^2 + m^2}$. Apply the differential operators to this expression and show that the time derivatives and space derivatives add up to cancel the mass squared term. This shows that the operator $\hat{\phi}$ obeys the Euler-Lagrange equation.

4 Retarded propagator

Consider the retarded Green function

$$G_R(x) = -i\langle 0| \left[\phi(x) \,,\, \phi(0) \right] |0\rangle \Theta(x^0) \tag{11}$$

and show that, in the free scalar field theory, its momentum-space representation really does take the form

$$G_R(p) \equiv \int d^4x e^{ip\cdot x} G_R(x) = \frac{1}{p^2 - m^2 + i\epsilon p^0}\,, \tag{12}$$

with ϵ infinitesimal. There are a few ways to do this. One is to replace the $\Theta(x^0)$ function with the better-behaved

$$\Theta(x^0) \to \begin{cases} 0 & x^0 < 0 \\ e^{-\epsilon x^0} & x^0 > 0 \end{cases} \tag{13}$$

and to use this to help do the time integral. Alternatively one can use the following frequency-representation of the Heaviside function:

$$\Theta(x^0) = \int \frac{d\omega}{2\pi} e^{i\omega x^0} \frac{-i}{\omega - i\epsilon} \tag{14}$$

where the small ϵ limit is also implied. In either case, you use the expressions for $\phi(x)$ in terms of creation and annihilation operators (or the expressions for the correlation functions we already found) to evaluate the correlation function, and perform the time integration to get to frequency-momentum space using one of the above expressions for the step function.

You may actually find the sum of two factors which are each simple poles and which add up to $1/((p^0 + i\epsilon)^2 - \vec{p}^2 - m^2)$. Note that you can rewrite this into the desired form because you can always drop ϵ^2 compared to ϵ and you can also always rescale ϵ by a finite positive factor like 2 or 1/2. *Hint*: the main trick is to realize that p^0 is *not* fixed to be $\sqrt{\vec{p}^2 + m^2}$. But when you write an expression for ϕ in terms of creation and annihilation operators, $\phi(x) = \int d^3p'/(2\pi)^3 e^{-ip'_\mu x^\mu} a_{p'} + \ldots$ then $p'_0 = \sqrt{\vec{p}'^2 + m^2}$ *is* fixed. These variables are not the same thing and you need to keep them apart from each other.

5 Projection operators

Consider the free field theory of one scalar ϕ of mass m. Define the state

$$|p\rangle = a_p^\dagger |0\rangle\,. \tag{15}$$

(Recall that $\left[a_k\,,\, a_p^\dagger\right] = 2\omega_p (2\pi)^3 \delta^3(\vec{k} - \vec{p})$ and that $a_k|0\rangle = 0$.) Explain that the object

$$\hat{F}_{\text{range}} \equiv \int_{p\in\text{range}} \frac{d^3p}{(2\pi)^3 2\omega_p} |p\rangle\langle p| \tag{16}$$

is an operator. (Here ($p \in$ range) means that the integral is restricted to some well specified range of momenta p. For instance, the range could be $|\vec{p}| < q$ for some q, or $p^z > q_1$ but $q_2 < p^x < q_3$ and p^y anything, or ... Each choice for the range included in the integral gives rise to a distinct operator; we want to make general statements about such operators.)

An operator $\hat{F}$ is said to be a projection operator if it satisfies the property $(\hat{F})^2 = \hat{F}$. Show that $\hat{F}_{\text{range}}$, defined above, is a projection operator.