Quantum Field Theory Homework 3

Due 28 May 2021

1 Flux factor

In lecture, in discussing the longitudinal part of the wave packets for scattering, we reached the expression

$$\int \frac{dp'_{1z}dp''_{1z}dp'_{2z}dp''_{2z}}{(2\pi)^4(2E_1 2E_2)^2} \;\psi_1^*(p''_{1z})\psi_1(p'_{1z})\psi_2^*(p''_{2z})\psi_2(p'_{2z}) \\ \times(2\pi)^2\delta(p'_{1z}+p'_{2z}-p''_{1z}-p''_{2z})\delta(p_1^{0\prime}+p_2^{0\prime}-p_1^{0\prime\prime}-p_2^{0\prime\prime}) \tag{1}$$

with $p^0 \equiv E \equiv \sqrt{p^2+m^2}$, the p, m arguments of each E are implicit, and

$$\int \frac{dp_z}{(2\pi)2E} \;\psi_a^*(p_z)\psi_a(p_z) = 1 \quad \text{for } a = 1, 2. \tag{2}$$

I then claimed that (for wave packets tightly peaked in momentum) the integrals could be performed and give

$$\frac{1}{2E_1 2E_2|v_1 - v_2|} \tag{3}$$

with v_1 the group velocity along the z axis of particle 1. (Group velocity is defined in the usual way as dE/dp_z.)

Fill in the missing steps to complete this derivation. Hint: use the p_z delta function to perform the p''_{2z} integration. Remember that p^0 is a dependent variable, defined as $p^0 = \sqrt{p_z^2+m^2}$. When you do the p''_{2z} integral, forcing[1] $p''_{2z} = p'_{1z} + p'_{2z} - p''_{1z}$, this substitution must be made in $p_{2z}^{0\prime\prime}$. That means that using the remaining delta function to do the p''_{1z} integration will lead to a nontrivial Jacobian, which you have to take proper account of.

Next, show that the resulting factor, Eq. (3), is Lorentz invariant. First, show that it is unchanged by boosts along the beam axis. Then show that it equals

$$\frac{1}{2E_1 2E_2|v_1 - v_2|} = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}\,, \tag{4}$$

which is manifestly Lorentz invariant.

[1]There is a second solution to this delta function, for which $p'_1 \neq p''_1$ and $p'_2 \neq p''_2$. For instance, if $m_1 = m_2$, then $p''_1 = p'_2$ and $p''_2 = p'_1$ is also a solution. But the wave packets are tightly peaked and they approximately vanish on this solution, so you should ignore it.

2 Complex Gaußian Integrals

Consider the integral

$$\int_{-\infty}^{\infty} \exp\left[ix^2/2\right]\, dx \tag{5}$$

First show that it is not absolutely convergent. Then we should define it as

$$\lim_{R\to\infty} \int_{-R}^{R} \exp\left[ix^2/2\right]\, dx\,. \tag{6}$$

Next, show that the answer is $(1+i)\sqrt{\pi}$.

Now let's try to understand that result by making a few plots. Plot the real part of the integrand as a function of x for $x \in [-10, 10]$. Then plot the "incomplete" integral $\int_{-R}^{R} \ldots$ as a function of R for $R \in [0, 10]$. Repeat for the imaginary part of the integrand.

Use these plots to explain in words, how the integral manages to take the value that it takes.

3 Asymptotic series

Consider the "baby" or "toy" version of scalar ϕ^4 theory, where it is just a single integral;

$$Z = \int_{-\infty}^{\infty} d\phi\, \exp\left(\frac{-\phi^2}{2} + \frac{-\lambda\phi^4}{24}\right)\,. \tag{7}$$

This is what the path integral for scalar ϕ^4 theory would look like if there were only one point in spacetime (and after rotating the contour for ϕ so the i's go away).

Consider Z as a function of λ.

3.1 Values

Evaluate

- Z(0)
- Z(0.01)
- Z(0.1)
- Z(0.4)
- Z(1)
- Z(5)

numerically to 20 digits, for instance, with Mathematica.

3.2 Series expansion

Replace $\exp(-\lambda\phi^4/24)$ with its series expansion in λ (or equivalently, in ϕ). Find explicitly the λ^0 and λ^1 terms in the series. Evaluate the λ^0 and the sum of λ^0 and λ^1 terms (first and second partial sums) numerically, for each of the examples you did above. For which cases does the λ^1 term help improve the accuracy?

3.3 Asymptotic series

Find the complete series expansion in λ in closed form, that is, write

$$Z(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n \,, \tag{8}$$

and find an explicit expression for c_n. (Do this by expanding the exponent, exchanging orders of summation and integration, and doing the integral for each term in the series.)

Show that the radius of convergence (in λ) of this series is *zero*.

3.4 So what good is it?

The expansion

$$e^{-x} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots \tag{9}$$

has the following property for $x > 0$: the partial sums

$$f_n(x) \equiv \sum_{m=0}^{n} \frac{(-1)^m}{m!} x^m \tag{10}$$

are alternately strict over-estimates and strict under-estimates of the actual function; that is, for $x > 0$, $f_0(x) = 1 > e^{-x}$, $f_1(x) = (1-x) < e^{-x}$, $f_2(x) = (1-x+x^2/2) > e^{-x}$, and so forth with the $<, >$ alternating. (Extra credit: prove this.)

Use this property to *prove* that the partial sums found above, Eq. (8) with n cut off at $0, 1, 2, 3, \ldots$, are alternately over-estimates and under-estimates. Therefore, the true answer always lies between neighboring terms in the series of partial sums.

Use this property to find a *bound* for $Z(\lambda)$ at $\lambda = 1$, by evaluating alternating terms until they start to diverge. How tight is the bound?

Repeat for $Z(0.4)$ and $Z(0.1)$. Argue that the bound becomes tighter and tighter as λ gets smaller, so at small λ, while the series does not converge, it gives us very good information about the value of $Z(\lambda)$.

A series with this property–zero radius of convergence but the ability to give good information near the origin–is called an *Asymptotic Series*.

3.5 Negative λ

What happens when $\lambda < 0$?